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Multi-Valued Bipolar Neutrosophic Matrices: Operations and Application to Simplified-TOPSIS

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Abstract. This paper introduces the new concept of multi-valued bipolar neutrosophic matrix (MVBNM), which is an extension of the multi-valued neutrosophic matrix (MVNM) and simultaneously captures positive and negative membership degrees of truth, indeterminacy, and falsity, incorporating multi-valued quantities. We proposed the determinant, trace, and adjoint of the matrices and various operations, and proved basic algebraic properties through a set of propositions. In the practical application of MVBNM, the new linguistic variable corresponding to multi-valued bipolar neutrosophic numbers (MVBNNs) is introduced, and the proposed linguistic variable's application is numerically demonstrated by using the neutrosophic simplified TOPSIS approach. Finally, an example is given to illustrate the best apartment is given to show the applicability of the proposed decision-making method. A comparative analysis with the multi-valued neutrosophic matrices (MVNMs) is also provided.

1. Introduction

To handle the uncertainty and inconsistency, the concept of classical fuzzy set theory was introduced by Zadeh [1]. The membership values are taken from [0, 1]. The further extension of intuitionistic sets and neutrosophic sets was introduced by Atanassov [2] and Smarandache [3], respectively, where truth, indeterminacy, and falsity are considered as independent components in the neutrosophic set. These developments opened new pathways for representing complex and uncertain information, and the theories are applicable in real multi-criteria decision-making. The single-valued neutrosophic sets (SVNSs) [4], interval-valued neutrosophic sets (IVNSs) [5], rough neutrosophic sets (RNSs) [6], bipolar neutrosophic sets [3], neutrosophic soft sets (NSSs) [7] and multi-valued neutrosophic sets [8] are another special case of neutrosophic sets. In 2023, there was a paper published for combining the multi-valued neutrosophic sets and m-polar neutrosophic

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sets and formulating the multi-valued m-polar neutrosophic Bonferroni mean (MVmNBM) and multi-valued m-polar neutrosophic weighted Bonferroni mean (MVmNWBM) operators [9], and many hybrid neutrosophic sets [9–13]. This framework is widely used in multicriteria decision making (MCDM), pattern recognition, image processing, and control systems [4, 14, 15]. The Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is the most favorable MCDM method in a neutrosophic environment due to its computational efficiency and ease of interpretation [16] [17]. Corresponding progress in the hesitant fuzzy set (HFS) framework, as presented by Torra [18], agrees with the membership of each element being a set of potential values rather than a single degree, expressing hesitation in expert judgment. Rodríguez et al. (2014) [19] provided a review of HFSs, their mathematical properties, and potential research directions. They also introduced the concept of hesitant fuzzy linguistic term sets (HFLTSs) [20], which extends HFS theory into linguistic decision-making. There are numerous extensions of fuzzy sets that were expanded by combining them with hesitant fuzzy sets, such as hesitant fuzzy soft sets (HFSS) [21], interval-valued fuzzy sets (IVHFS) [22], interval-valued dual hesitant fuzzy sets (IVDHFS) [23], interval-valued hesitant fuzzy soft sets (IVHFSS) [24], dual hesitant fuzzy sets (DHFS) [25], dual hesitant fuzzy soft sets (DHFSS) [26]. All these extensions of HFSs have been extensively studied for MCDM [27–29]. Extended the concepts of HFSs in the framework of neutrosophic sets are single-valued neutrosophic hesitant fuzzy sets (SVNHFSs) [30], interval-valued neutrosophic hesitant fuzzy sets (IVNHFSs) [31]. These frameworks are applied in multi-attribute decision-making (MADM) and group decision-making (GDM). This concept was further extended to bipolar hesitant fuzzy sets in the neutrosophic framework to combine the hesitation, bipolarity, and neutrosophic in a single structure [32], thereby handling the vague, inconsistent, and incomplete information.

Expand the groundwork of MVNSs [8] demonstrated to represent the three membership functions in the collection of possible values instead of a single value. This structure, beneficial for effective decision-making, relies on multiple experts and extensive data. Several aggregation operators, such as the multi-valued neutrosophic weighted average (MVNWA), power weighted geometric mean (MVNWG), and power partitioned Hamy mean (WMNPHAM), have been developed [8, 33]. The bipolar neutrosophic set represents the bipolarity of the information introduced by [34]. Some researchers have done studies on the applications of these representations [35–37].

The matrix-based fuzzy models, which were introduced by Thomason in 1977 [38], are the study of the convergence of powers of fuzzy matrices. Kandasamy and Smarandache introduced the fuzzy relational maps and neutrosophic relational maps in 2004 [39]. Square neutrosophic matrices are introduced in this paper. The concept, some operations, and application of MCDM of a single-valued neutrosophic matrix (SVNM) proposed by Karaalaan [40]. Martina and Deepa introduced the multi-valued neutrosophic matrices (MVNMs) and some operations of MVNMs and applied them in a neutrosophic simplified decision method [41]. Similar works happened in multi-valued neutrosophic sets in the improved PROMETHEE methods and their applications in multi-attribute decision-making [42] [43]. The connection between the neutrosophic hesitant matrix (NFM) and MVNM is provided by [44]. The application of multi-valued rough neutrosophic sets (MVRNS) and matrix (MVRNM) in multi criteria decision-making by 2023 [45]. The bipolar neutrosophic matrix is an extension of the neutrosophic matrix, which was first introduced [34]. Based on this paper, a Python tool for implementing bipolar neutrosophic matrices was developed [46].

In this paper, the multi-valued bipolar neutrosophic matrix (MVBNM) is defined by combining the multi-valued neutrosophic matrix (MVNM) and the bipolar neutrosophic matrix (BNM). It is an extension of the multi-valued neutrosophic matrix (MVNM). Then we define the determinant, trace, adjoint, and other fundamental matrix operations under the MVBNM environment and demonstrate several propositions that describe their basic properties. Moreover, it introduced the linguistic variable for multi-valued bipolar neutrosophic numbers (MVBNNs) to ease decision-making. The proposed structure is mathematically established through a neutrosophic simplified-TOPSIS method, and the apartment problem is used as a case study. The comparative study between the proposed MVBNM and the existing approach based on the neutrosophic simplified TOPSIS method highlights how much better the former is at representing the bipolarity of the information and managing indeterminacy simultaneously.

The outline of the paper is as follows. Section 2 presents the necessary definitions and background of the study for developing the MVBNM. Section 3 introduces the MVBNMs and defines the determinant, traces, and adjoint of the matrix. In the subsection of this section, operations are defined and also provide the propositions of those operations. In Sections 4 and 5, we proposed the simplified TOPSIS method and the linguistic variable for MVBNNs. The given method is used to numerically demonstrate the MCDM problem in Section 6. Section 7 is the comparative analysis and discussion of the proposed method in MVBNM and MVNM environments. Finally, Section 8 concludes the paper.

2. Background

This section provides the basic definitions and concepts of the development of the MVBNMs.

Definition 2.1. Neutrosophic Set (NS) *[3]*
Let $\mathbb{X}$ be a space of points, with elements in $\mathbb{X}$ denoted by x. A neutrosophic set $\mathbb{B}$ in $\mathbb{X}$ is defined as
$\mathbb{B} = \{\langle x; T_{\mathbb{B}}(x), I_{\mathbb{B}}(x), F_{\mathbb{B}}(x)\rangle \mid x \in \mathbb{X}\}$.
Where $T_{\mathbb{B}}(x), I_{\mathbb{B}}(x), F_{\mathbb{B}}(x) : \mathbb{X} \to [0,1]$ are the truth, indeterminacy, and falsity membership functions, respectively, that satisfy the condition that

$$0 \le T_{\mathbb{B}}(x) + I_{\mathbb{B}}(x) + F_{\mathbb{B}}(x) \le 3, \quad \forall x \in \mathbb{X}.$$

Definition 2.2. Multi-valued Neutrosophic Set (MVNS) *[8].*
Let $\mathbb{X}$ be a space of points, with elements in $\mathbb{X}$ denoted by x. A multi-valued neutrosophic set $\mathbb{B}$ in $\mathbb{X}$ is defined as

$$\mathbb{B} = \{\langle\, x;\ \widehat{T}_{\mathbb{B}}(x), \widehat{I}_{\mathbb{B}}(x), \widehat{F}_{\mathbb{B}}(x)\rangle \mid x \in \mathbb{X}\,\}$$

Where $\widehat{T}_{\mathbb{B}}(x), \widehat{I}_{\mathbb{B}}(x), \widehat{F}_{\mathbb{B}}(x) : \mathbb{X} \to [0,1]$ *the truth, indeterminacy, and falsity membership functions, respectively, the supremum values of each component satisfy the constraint*

$$0 \le \widehat{T}_{\mathbb{B}}(x) + \widehat{I}_{\mathbb{B}}(x) + \widehat{F}_{\mathbb{B}}(x) \le 3, \quad \forall x \in \mathbb{X}.$$

The ordered triple $(\widehat{T}_{\mathbb{B}}, \widehat{I}_{\mathbb{B}}, \widehat{F}_{\mathbb{B}})$ *is called the multi-valued neutrosophic number (MVNN). The MVNSs are an extension of NSs.*

Definition 2.3. Bipolar neutrosophic set(BNS) *[3].*
Let $\mathbb{X}$ *be a space of points, with a generic element* $\mathbb{X}$ *denoted by* x*. A bipolar neutrosophic set (BNS)* $\mathbb{B}$ *in* $\mathbb{X}$ *is defined as*

$$\mathbb{B} = \{ x; T^{\rho e}_{\mathbb{B}}(x), I^{\rho e}_{\mathbb{B}}(x), F^{\rho e}_{\mathbb{B}}(x), T^{\nu e}_{\mathbb{B}}(x), I^{\nu e}_{\mathbb{B}}(x), F^{\nu e}_{\mathbb{B}}(x) \mid x \in \mathbb{X} \}$$

Where, $T^{\rho e}_{\mathbb{B}}(x), I^{\rho e}_{\mathbb{B}}(x), F^{\rho e}_{\mathbb{B}}(x) : \mathbb{X} \to [0,1]$ *are the positive truth, indeterminacy, and falsity membership functions, respectively. Thus, each* $x \in \mathbb{X}$ *satisfies the condition that* $0 \le T^{\rho e}_{\mathbb{B}}(x) + I^{\rho e}_{\mathbb{B}}(x) + F^{\rho e}_{\mathbb{B}}(x) \le 3$. $T^{\nu e}_{\mathbb{B}}(x), I^{\nu e}_{\mathbb{B}}(x), F^{\nu e}_{\mathbb{B}}(x) : \mathbb{X} \to [-1,0]$ *are negative truth, indeterminacy, and falsity membership functions, respectively. Thus, each* $x \in \mathbb{X}$ *satisfies the condition that* $-3 \le T^{\nu e}_{\mathbb{B}}(x) + I^{\nu e}_{\mathbb{B}}(x) + F^{\nu e}_{\mathbb{B}}(x) \le 0$*. The sextuple* $\{T^{\rho e}_{\mathbb{B}}, I^{\rho e}_{\mathbb{B}}, F^{\rho e}_{\mathbb{B}}, T^{\nu e}_{\mathbb{B}}, I^{\nu e}_{\mathbb{B}}, F^{\nu e}_{\mathbb{B}}\}$ *is called a bipolar neutrosophic number (BNN). If we consider only the* $T^{\rho e}_{\mathbb{B}}, I^{\rho e}_{\mathbb{B}}, F^{\rho e}_{\mathbb{B}}$ *values then we can reduce this into an NS.*

Definition 2.4. Operations of Bipolar Neutrosophic Numbers (BNNs) *[34].*
Let $\mathbb{A} = \langle T^{\rho e}_{\mathbb{A}}, I^{\rho e}_{\mathbb{A}}, F^{\rho e}_{\mathbb{A}}, T^{\nu e}_{\mathbb{A}}, I^{\nu e}_{\mathbb{A}}, F^{\nu e}_{\mathbb{A}}\rangle$*, and* $\mathbb{B} = \langle T^{\rho e}_{\mathbb{B}}, I^{\rho e}_{\mathbb{B}}, F^{\rho e}_{\mathbb{B}}, T^{\nu e}_{\mathbb{B}}, I^{\nu e}_{\mathbb{B}}, F^{\nu e}_{\mathbb{B}}\rangle$ *be two MVNNs,* $\lambda > 0$*, then the following arithmetic operations:*

(i) Addition:

$$\mathbb{A} + \mathbb{B} = \begin{array}{l} \langle\{\theta^{\rho e}_{\mathbb{A}} + \theta^{\rho e}_{\mathbb{B}} - \theta^{\rho e}_{\mathbb{A}} \cdot \theta^{\rho e}_{\mathbb{B}}\}, \{\phi^{\rho e}_{\mathbb{A}} \cdot \phi^{\rho e}_{\mathbb{B}}\}, \{\psi^{\rho e}_{\mathbb{A}} \cdot \psi^{\rho e}_{\mathbb{B}}\}, \{-\theta^{\nu e}_{\mathbb{A}} \cdot \theta^{\nu e}_{\mathbb{B}}\}, \\ \{-(-\phi^{\nu e}_{\mathbb{A}} - \phi^{\nu e}_{\mathbb{B}} - \phi^{\nu e}_{\mathbb{A}} \cdot \phi^{\nu e}_{\mathbb{B}})\}, \{-(-\psi^{\nu e}_{\mathbb{A}} - \psi^{\nu e}_{\mathbb{B}} - \psi^{\nu e}_{\mathbb{A}} \cdot \psi^{\nu e}_{\mathbb{B}})\}\rangle. \end{array}$$

(ii) Multiplication:

$$\mathbb{A} \times \mathbb{B} = \begin{array}{l} \langle\{\theta^{\rho e}_{\mathbb{A}} \cdot \theta^{\rho e}_{\mathbb{B}}\}, \{\phi^{\rho e}_{\mathbb{A}} + \phi^{\rho e}_{\mathbb{B}} - \phi^{\rho e}_{\mathbb{A}} \cdot \phi^{\rho e}_{\mathbb{B}}\}, \{\psi^{\rho e}_{\mathbb{A}} + \psi^{\rho e}_{\mathbb{B}} - \psi^{\rho e}_{\mathbb{A}} \cdot \psi^{\rho e}_{\mathbb{B}}\}, \\ \{-(-\theta^{\nu e}_{\mathbb{A}} - \theta^{\nu e}_{\mathbb{B}} - \theta^{\nu e}_{\mathbb{A}} \cdot \theta^{\nu e}_{\mathbb{B}})\}, \{-\phi^{\nu e}_{\mathbb{A}} \cdot \phi^{\nu e}_{\mathbb{B}}\}, \{-\psi^{\nu e}_{\mathbb{A}} \cdot \psi^{\nu e}_{\mathbb{B}}\}\rangle \end{array}$$

(iii) Scalar multiplication

$$\lambda\mathbb{A} = \begin{array}{l} \langle\{1 - (1 - \theta^{\rho e}_{\mathbb{A}})^{\lambda}\}, \{(\phi^{\rho e}_{\mathbb{A}})^{\lambda}\}, \{(\psi^{\rho e}_{\mathbb{A}})^{\lambda}\}, \{-((-\theta^{\nu e}_{\mathbb{A}})^{\lambda})\}, \\ \left\{-\left(1 - (1 - (-\phi^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\}, \left\{-\left(1 - (1 - (-\psi^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\}\rangle \end{array}$$

(iv) Power:

$$\mathbb{A}^{\lambda} = \begin{array}{l} \langle\{(\theta^{\rho e}_{\mathbb{A}})^{\lambda}\}, \{1 - (1 - \phi^{\rho e}_{\mathbb{A}})^{\lambda}\}, \{1 - (1 - \psi^{\rho e}_{\mathbb{A}})^{\lambda}\}, \\ \left\{-\left(1 - (1 - (-\theta^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\}, \{-((-\phi^{\nu e}_{\mathbb{A}})^{\lambda})\}, \{-((-\psi^{\nu e}_{\mathbb{A}})^{\lambda})\}\rangle \end{array}$$

Definition 2.5. Bipolar Neutrosophic Matrix(BNM) *[34].*
An BNM $\mathbb{M}'$ *of order* $m \times n$ *is defined as* $\mathbb{M}' = (m'_{ij})_{m \times n}$*, where each element* m'_{ij} *is a BNM expressed as*

$$\mathbb{M}' = [X_{ij}, \langle T^{\rho e}_{ij\mathbb{M}}, I^{\rho e}_{ij\mathbb{M}}, F^{\rho e}_{ij\mathbb{M}}, T^{\nu e}_{ij\mathbb{M}}, I^{\nu e}_{ij\mathbb{M}}, F^{\nu e}_{ij\mathbb{M}}\rangle]_{m \times n}.$$

Such that, $T^{\rho e}_{ij\mathbb{M}}, I^{\rho e}_{ij\mathbb{M}}, F^{\rho e}_{ij\mathbb{M}} \in [0,1]$, *and* $T^{ve}_{ij\mathbb{M}}, I^{ve}_{ij\mathbb{M}}, F^{ve}_{ij\mathbb{M}} \in [-1,0]$ *and satisfy the condition for* $i = 1,2,3,......,n$, *and* $j = 1,2,3,......,m$.

Multi-valued bipolar neutrosophic matrices are an extension of the bipolar neutrosophic matrices.

3. Proposed Framework

In this section, we proposed its various operations and properties.

Definition 3.1. Multi-Valued Bipolar Neutrosophic Set (MVBNS).
Let $\mathbb{X}$ *be a space of points, with elements* $\mathbb{X}$ *denoted by* x. *A* Multi-valued bipolar neutrosophic set (MVBNS) $\mathbb{B}$ *in* $\mathbb{X}$ *is defined as*

$$\mathbb{B} = \{\, x;\, \widehat{T}^{\rho e}_{\mathbb{B}}(x), \widehat{I}^{\rho e}_{\mathbb{B}}(x), \widehat{F}^{\rho e}_{\mathbb{B}}(x), \widehat{T}^{ve}_{\mathbb{B}}(x), \widehat{I}^{ve}_{\mathbb{B}}(x), \widehat{F}^{ve}_{\mathbb{B}}(x) \mid x \in \mathbb{X} \,\}$$

Where, $\widehat{T}^{\rho e}_{\mathbb{B}}(x), \widehat{I}^{\rho e}_{\mathbb{B}}(x), \widehat{F}^{\rho e}_{\mathbb{B}}(x) : \mathbb{X} \to [0,1]$ *are the positive multi-valued truth, indeterminacy, and falsity membership functions, and* $\widehat{T}^{ve}_{\mathbb{B}}(x), \widehat{I}^{ve}_{\mathbb{B}}(x), \widehat{F}^{ve}_{\mathbb{B}}(x) : \mathbb{X} \to [-1,0]$ *are the negative multi-valued truth, indeterminacy, and falsity membership functions, respectively. These value sets are such that for each* $x \in \mathbb{X}$, *the supremum values of each component satisfy the constraint* $0 \le \sup(\widehat{T}^{\rho e}_{\mathbb{B}}(x)) + \sup(\widehat{I}^{\rho e}_{\mathbb{B}}(x)) + \sup(\widehat{F}^{\rho e}_{\mathbb{B}}(x)) \le 3$, *and the infimum of each component satisfies the constraint* $-3 \le \inf(\widehat{T}^{ve}_{\mathbb{B}}(x)) + \inf(\widehat{I}^{ve}_{\mathbb{B}}(x)) + \inf(\widehat{F}^{ve}_{\mathbb{B}}(x)) \le 0$.

If each of $\widehat{T}^{\rho e}_{\mathbb{B}}, \widehat{I}^{\rho e}_{\mathbb{B}}, \widehat{F}^{\rho e}_{\mathbb{B}}, \widehat{T}^{ve}_{\mathbb{B}}, \widehat{I}^{ve}_{\mathbb{B}}, \widehat{F}^{ve}_{\mathbb{B}}$ *has only one value, then it reduces to a single-valued bipolar neutrosophic set (SVBNS), and if they are represented as interval values, then it reduces to an interval-valued bipolar neutrosophic set (IVBNS). If we consider only* $\widehat{T}^{\rho e}_{ij\mathbb{B}}, \widehat{I}^{\rho e}_{ij\mathbb{B}}, \widehat{F}^{\rho e}_{ij\mathbb{B}}$ *then, it will reduce to MVNS.*

Definition 3.2. Multi-Valued Bipolar Neutrosophic Number (MVBNN).
The multi-valued bipolar neutrosophic set $\mathbb{B}$ *in* $\mathbb{X}$, *the ordered sextuple* $(\widehat{T}^{\rho e}_{\mathbb{B}}, \widehat{I}^{\rho e}_{\mathbb{B}}, \widehat{F}^{\rho e}_{\mathbb{B}}, \widehat{T}^{ve}_{\mathbb{B}}, \widehat{I}^{ve}_{\mathbb{B}}, \widehat{F}^{ve}_{\mathbb{B}})$ *is called a multi-valued bipolar neutrosophic number.*

Definition 3.3. *Let* $\mathbb{A} = \langle \widehat{T}^{\rho e}_{\mathbb{A}}, \widehat{I}^{\rho e}_{\mathbb{A}}, \widehat{F}^{\rho e}_{\mathbb{A}}, \widehat{T}^{ve}_{\mathbb{A}}, \widehat{I}^{ve}_{\mathbb{A}}, \widehat{F}^{ve}_{\mathbb{A}} \rangle$, *and* $\mathbb{B} = \langle \widehat{T}^{\rho e}_{\mathbb{B}}, \widehat{I}^{\rho e}_{\mathbb{B}}, \widehat{F}^{\rho e}_{\mathbb{B}}, \widehat{T}^{ve}_{\mathbb{B}}, \widehat{I}^{ve}_{\mathbb{B}}, \widehat{F}^{ve}_{\mathbb{B}} \rangle$ *be two MVBNNs,* $\lambda > 0$, *then the following arithmetic operations:*

(i) Addition:

$$\mathbb{A} + \mathbb{B} = \left\langle \begin{array}{l} \bigcup_{\widehat{\theta}^{\rho e}_{\mathbb{A}} \in \widehat{T}^{\rho e}_{\mathbb{A}},\, \widehat{\theta}^{\rho e}_{\mathbb{B}} \in \widehat{T}^{\rho e}_{\mathbb{B}}} \{\widehat{\theta}^{\rho e}_{\mathbb{A}} + \widehat{\theta}^{\rho e}_{\mathbb{B}} - \widehat{\theta}^{\rho e}_{\mathbb{A}} \cdot \widehat{\theta}^{\rho e}_{\mathbb{B}}\}, \\ \bigcup_{\widehat{\phi}^{\rho e}_{\mathbb{A}} \in \widehat{I}^{\rho e}_{\mathbb{A}},\, \widehat{\phi}^{\rho e}_{\mathbb{B}} \in \widehat{I}^{\rho e}_{\mathbb{B}}} \{\widehat{\phi}^{\rho e}_{\mathbb{A}} \cdot \widehat{\phi}^{\rho e}_{\mathbb{B}}\}, \\ \bigcup_{\widehat{\psi}^{\rho e}_{\mathbb{A}} \in \widehat{F}^{\rho e}_{\mathbb{A}},\, \widehat{\psi}^{\rho e}_{\mathbb{B}} \in \widehat{F}^{\rho e}_{\mathbb{B}}} \{\widehat{\psi}^{\rho e}_{\mathbb{A}} \cdot \widehat{\psi}^{\rho e}_{\mathbb{B}}\}, \\ \bigcup_{\widehat{\theta}^{ve}_{\mathbb{A}} \in \widehat{T}^{ve}_{\mathbb{A}},\, \widehat{\theta}^{ve}_{\mathbb{B}} \in \widehat{T}^{ve}_{\mathbb{B}}} \{-\widehat{\theta}^{ve}_{\mathbb{A}} \cdot \widehat{\theta}^{ve}_{\mathbb{B}}\}, \\ \bigcup_{\widehat{\phi}^{ve}_{\mathbb{A}} \in \widehat{I}^{ve}_{\mathbb{A}},\, \widehat{\phi}^{ve}_{\mathbb{B}} \in \widehat{I}^{ve}_{\mathbb{B}}} \{-(-\widehat{\phi}^{ve}_{\mathbb{A}} - \widehat{\phi}^{ve}_{\mathbb{B}} - \widehat{\phi}^{ve}_{\mathbb{A}} \cdot \widehat{\phi}^{ve}_{\mathbb{B}})\}, \\ \bigcup_{\widehat{\psi}^{ve}_{\mathbb{A}} \in \widehat{F}^{ve}_{\mathbb{A}},\, \widehat{\psi}^{ve}_{\mathbb{B}} \in \widehat{F}^{ve}_{\mathbb{B}}} \{-(-\widehat{\psi}^{ve}_{\mathbb{A}} - \widehat{\psi}^{ve}_{\mathbb{B}} - \widehat{\psi}^{ve}_{\mathbb{A}} \cdot \widehat{\psi}^{ve}_{\mathbb{B}})\} \end{array} \right\rangle$$

(ii) Multiplication:

$$\mathbb{A}\times\mathbb{B}=\left\langle\begin{array}{l}\bigcup_{\widehat{\theta}^{\rho e}_{\mathbb{A}}\in\widehat{T}^{\rho e}_{\mathbb{A}},\widehat{\theta}^{\rho e}_{\mathbb{B}}\in\widehat{T}^{\rho e}_{\mathbb{B}}}\{\widehat{\theta}^{\rho e}_{\mathbb{A}}\cdot\widehat{\theta}^{\rho e}_{\mathbb{B}}\},\\ \bigcup_{\widehat{\phi}^{\rho e}_{\mathbb{A}}\in\widehat{I}^{\rho e}_{\mathbb{A}},\widehat{\phi}^{\rho e}_{\mathbb{B}}\in\widehat{I}^{\rho e}_{\mathbb{B}}}\{\widehat{\phi}^{\rho e}_{\mathbb{A}}+\widehat{\phi}^{\rho e}_{\mathbb{B}}-\widehat{\phi}^{\rho e}_{\mathbb{A}}\cdot\widehat{\phi}^{\rho e}_{\mathbb{B}}\},\\ \bigcup_{\widehat{\psi}^{\rho e}_{\mathbb{A}}\in\widehat{F}^{\rho e}_{\mathbb{A}},\widehat{\psi}^{\rho e}_{\mathbb{B}}\in\widehat{F}^{\rho e}_{\mathbb{B}}}\{\widehat{\psi}^{\rho e}_{\mathbb{A}}+\widehat{\psi}^{\rho e}_{\mathbb{B}}-\widehat{\psi}^{\rho e}_{\mathbb{A}}\cdot\widehat{\psi}^{\rho e}_{\mathbb{B}}\},\\ \bigcup_{\widehat{\theta}^{\nu e}_{\mathbb{A}}\in\widehat{T}^{\nu e}_{\mathbb{A}},\widehat{\theta}^{\nu e}_{\mathbb{B}}\in\widehat{T}^{\nu e}_{\mathbb{B}}}\{-(-\widehat{\theta}^{\nu e}_{\mathbb{A}}-\widehat{\theta}^{\nu e}_{\mathbb{B}}-\widehat{\theta}^{\nu e}_{\mathbb{A}}\cdot\widehat{\theta}^{\nu e}_{\mathbb{B}})\},\\ \bigcup_{\widehat{\phi}^{\nu e}_{\mathbb{A}}\in\widehat{I}^{\nu e}_{\mathbb{A}},\widehat{\phi}^{\nu e}_{\mathbb{B}}\in\widehat{I}^{\nu e}_{\mathbb{B}}}\{-\widehat{\phi}^{\nu e}_{\mathbb{A}}\cdot\widehat{\phi}^{\nu e}_{\mathbb{B}}\},\\ \bigcup_{\widehat{\psi}^{\nu e}_{\mathbb{A}}\in\widehat{F}^{\nu e}_{\mathbb{A}},\widehat{\psi}^{\nu e}_{\mathbb{B}}\in\widehat{F}^{\nu e}_{\mathbb{B}}}\{-\widehat{\psi}^{\nu e}_{\mathbb{A}}\cdot\widehat{\psi}^{\nu e}_{\mathbb{B}}\}\end{array}\right\rangle$$

(iii) Scalar multiplication

$$\lambda\mathbb{A}=\left\langle\begin{array}{l}\bigcup_{\widehat{\theta}^{\rho e}_{\mathbb{A}}\in\widehat{T}^{\rho e}_{\mathbb{A}}}\{1-(1-\widehat{\theta}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\phi}^{\rho e}_{\mathbb{A}}\in\widehat{I}^{\rho e}_{\mathbb{A}}}\{(\widehat{\phi}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\psi}^{\rho e}_{\mathbb{A}}\in\widehat{F}^{\rho e}_{\mathbb{A}}}\{(\widehat{\psi}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\theta}^{\nu e}_{\mathbb{A}}\in\widehat{T}^{\nu e}_{\mathbb{A}}}\{-((-\widehat{\theta}^{\nu e}_{\mathbb{A}})^{\lambda})\},\\ \bigcup_{\widehat{\phi}^{\nu e}_{\mathbb{A}}\in\widehat{I}^{\nu e}_{\mathbb{A}}}\left\{-\left(1-(1-(-\widehat{\phi}^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\},\\ \bigcup_{\widehat{\psi}^{\nu e}_{\mathbb{A}}\in\widehat{F}^{\nu e}_{\mathbb{A}}}\left\{-\left(1-(1-(-\widehat{\psi}^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\}\end{array}\right\rangle$$

(iv) Power:

$$\mathbb{A}^{\lambda}=\left\langle\begin{array}{l}\bigcup_{\widehat{\theta}^{\rho e}_{\mathbb{A}}\in\widehat{T}^{\rho e}_{\mathbb{A}}}\{(\widehat{\theta}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\phi}^{\rho e}_{\mathbb{A}}\in\widehat{I}^{\rho e}_{\mathbb{A}}}\{1-(1-\widehat{\phi}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\psi}^{\rho e}_{\mathbb{A}}\in\widehat{F}^{\rho e}_{\mathbb{A}}}\{1-(1-\widehat{\psi}^{\rho e}_{\mathbb{A}})^{\lambda}\},\\ \bigcup_{\widehat{\theta}^{\nu e}_{\mathbb{A}}\in\widehat{T}^{\nu e}_{\mathbb{A}}}\left\{-\left(1-(1-(-\widehat{\theta}^{\nu e}_{\mathbb{A}}))^{\lambda}\right)\right\},\\ \bigcup_{\widehat{\phi}^{\nu e}_{\mathbb{A}}\in\widehat{I}^{\nu e}_{\mathbb{A}}}\{-((-\widehat{\phi}^{\nu e}_{\mathbb{A}})^{\lambda})\},\\ \bigcup_{\widehat{\psi}^{\nu e}_{\mathbb{A}}\in\widehat{F}^{\nu e}_{\mathbb{A}}}\{-((-\widehat{\psi}^{\nu e}_{\mathbb{A}})^{\lambda})\}\end{array}\right\rangle$$

Definition 3.4. Multi-Valued Bipolar Neutrosophic Matrix (MVBNM).
An MVBNM $\mathbb{M}$ of order $m \times n$ is defined as $\mathbb{M} = (\widehat{m}_{ij})_{m\times n}$, where each element $\widehat{m}_{ij}$ is a MVBNN expressed as

$$\mathbb{M} = [X_{ij}, \langle \widehat{T}^{\rho e}_{ij\mathbb{M}}, \widehat{I}^{\rho e}_{ij\mathbb{M}}, \widehat{F}^{\rho e}_{ij\mathbb{M}}, \widehat{T}^{\nu e}_{ij\mathbb{M}}, \widehat{I}^{\nu e}_{ij\mathbb{M}}, \widehat{F}^{\nu e}_{ij\mathbb{M}} \rangle]_{m\times n}.$$

Such that, $\widehat{T}^{\rho e}_{ij\mathbb{M}}, \widehat{I}^{\rho e}_{ij\mathbb{M}}, \widehat{F}^{\rho e}_{ij\mathbb{M}} \in [0,1]$, and $\widehat{T}^{\nu e}_{ij\mathbb{M}}, \widehat{I}^{\nu e}_{ij\mathbb{M}}, \widehat{F}^{\nu e}_{ij\mathbb{M}} \in [-1,0]$ and satisfy the condition for $i = 1,2,3,......,n$, and $j = 1,2,3,......,m$.

$$0 \le \sup(\widehat{T}^{\rho e}_{\mathbb{M}}(x)) + \sup(\widehat{I}^{\rho e}_{\mathbb{M}}(x)) + \sup(\widehat{F}^{\rho e}_{\mathbb{M}}(x)) \le 3, and$$
$$-3 \le \inf(\widehat{T}^{\nu e}_{\mathbb{M}}(x)) + \inf(\widehat{I}^{\nu e}_{\mathbb{M}}(x)) + \inf(\widehat{F}^{\nu e}_{\mathbb{M}}(x)) \le 0$$

If each of $\widehat{T}^{\rho e}_{ij\mathbb{M}}, \widehat{I}^{\rho e}_{ij\mathbb{M}}, \widehat{F}^{\rho e}_{ij\mathbb{M}}, \widehat{T}^{\nu e}_{ij\mathbb{M}}, \widehat{I}^{\nu e}_{ij\mathbb{M}}, \widehat{F}^{\nu e}_{ij\mathbb{M}}$ has only one value, then it reduces to a single-valued bipolar neutrosophic matrix (SVBNM), and if they are represented as interval values, then it reduces to interval-valued bipolar neutrosophic matrix (IVBNM). If we consider only $\widehat{T}^{\rho e}_{ij\mathbb{M}}, \widehat{I}^{\rho e}_{ij\mathbb{M}}, \widehat{F}^{\rho e}_{ij\mathbb{M}}$ then, it will reduce to MVNM.

Example 3.1. *Let it $\mathbb{M}$ be a 2×2 MVBNM.*

$$\mathbb{M} = \begin{bmatrix} \langle\{0.5\},\{0.3,0.2\},\{0.6\}\,\{-0.1,-0.3\},\{-0.6\},\{-0.4\}\rangle & \langle\{0.3,0.6\},\{0.3,0.5\},\{0.1\},\{-0.2\},\{-0.3\},\{-0.7\}\rangle \\ \langle\{0.3\},\{0.6\},\{0.8\},\{-0.5\},\{-0.2,-0.3\},\{-0.3\}\rangle & \langle\{0.1,0.2\},\{0.7\},\{0.5\},\{-0.5\},\{-0.8\},\{-0.3\}\rangle \end{bmatrix}_{2\times 2}$$

Here, all elements $\mathbb{M}$ are MVBNSs. If each element $\mathbb{M}$ has only positive values, it is reduced to MVNM. Then it $\mathbb{M}_m$ is said to be MVNM.

$$\mathbb{M}_m = \begin{bmatrix} \langle\{0.5\},\{0.3,0.2\},\{0.6\}\rangle & \langle\{0.3,0.6\},\{0.3,0.5\},\{0.1\}\rangle \\ \langle\{0.3\},\{0.6\},\{0.8\}\rangle & \langle\{0.1,0.2\},\{0.7\},\{0.5\}\rangle \end{bmatrix}_{2\times 2}.$$

Definition 3.5 (Trace). *The trace of a square $\mathbb{M} = (\widehat{m}_{ij})_{n\times n}$ MVBNM is the product of the diagonal elements.*

$$\mathrm{Tr}(\mathbb{M}) = \prod_{i=1}^{n} \widehat{m}_{ii}.$$

Definition 3.6 (Determinant of MVBNM). *Let $\mathbb{M} = (\widehat{m}_{ij})_{n\times n}$ be MVBNM, where each $\widehat{m}_{ij}$ be the MVBNNs. The determinant is denoted by* $\det(\mathbb{M})$ *or* $|\mathbb{M}|$.

$$\det(\mathbb{M}) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} \langle \widehat{m}_{i\sigma(i)} \rangle$$

where $\langle \widehat{m}_{i\sigma(i)} \rangle = \langle \widehat{m}^{\rho e}_{i\sigma(i)\theta}, \widehat{m}^{\rho e}_{i\sigma(i)\phi}, \widehat{m}^{\rho e}_{i\sigma(i)\psi}, \widehat{m}^{\nu e}_{i\sigma(i)\theta}, \widehat{m}^{\nu e}_{i\sigma(i)\phi}, \widehat{m}^{\nu e}_{i\sigma(i)\psi} \rangle$ and S_n represent the symmetric group of all permutations of the index set $\{1,2,3,.....\}$.

Definition 3.7 (Adjoint of MVBNM). *Let the adjoint of a square MVBNM $\mathbb{M} = (\widehat{m}_{ij})_{n\times n}$ is denoted by $adj(\mathbb{M}) = (\widehat{m}_{ji})_{n\times n}$, Where $\widehat{m}_{ji}$ represents the transpose of the matrix $\mathbb{M}$. The adjoint is expressed as*

$$adj(\mathbb{M}) = \sum_{\sigma\in S_n, S_m} \prod_{t\in n_j}^{n} \langle \widehat{m}_{t\sigma(t)} \rangle$$

Where $\langle \widehat{m}_{t\sigma(t)} \rangle = \langle \widehat{m}^{\rho e}_{t\sigma(t)\theta}, \widehat{m}^{\rho e}_{t\sigma(t)\phi}, \widehat{m}^{\rho e}_{t\sigma(t)\psi}, \widehat{m}^{\nu e}_{t\sigma(t)\theta}, \widehat{m}^{\nu e}_{t\sigma(t)\phi}, \widehat{m}^{\nu e}_{t\sigma(t)\psi} \rangle$ and S_{ninj} is the set of all possible permutations of n_j over n_i.

Example 3.2.
Let,

$$\mathbb{M} = \begin{bmatrix} \langle\{0.7\},\{0.1\},\{0.2\},\ \{-0.2\},\{-0.1\},\{-0.7,-0.8\}\rangle & \langle\{0.4\},\{0.2\},\{0.5\},\{-0.1,-0.2\},\{-0.7\},\{-0.4\}\rangle \\ \langle\{0.2\},\{0.2,0.3\},\{0.5,0.6\},\{-0.5\},\{-0.8\},\{-0.1,-0.3\}\rangle & \langle\{0.4,0.5\},\{0.5,0.6\},\{0.3\},\{-0.6\},\{-0.4\},\{-0.7\}\rangle \end{bmatrix}_{2\times 2}$$

(i) Trace: $\mathrm{Tr}(\mathbb{M}) = \langle\{0.4,0.5\},\{0.5,0.6\},\{0.3\},\{-0.2\},\{-0.4\},\{-0.7\}\rangle$.

(ii) Determinant: $\det(\mathbb{M}) = \langle\{0.4,0.5\},\{0.2,0.3\},\{0.3\},\{-0.2\},\{-0.4\},\{-0.4\}\rangle$.

3.1. **Operations on MVBNMs.** Let $\mathbb{M} = [\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{\nu e}_{ij\theta},\widehat{m}^{\nu e}_{ij\phi},\widehat{m}^{\nu e}_{ij\psi}\rangle]_{m\times n}$ and $\mathbb{N} = [\langle\widehat{n}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\psi},\widehat{n}^{\nu e}_{ij\theta},\widehat{n}^{\nu e}_{ij\phi},\widehat{n}^{\nu e}_{ij\psi}\rangle]_{m\times n}$ be two MVBNMs. Then

(i) Complement:
$\mathbb{M}' = [\langle\widehat{m}^{\rho e}_{ij\psi},\ 1-\widehat{m}^{\rho e}_{ij\phi},\ \widehat{m}^{\rho e}_{ij\theta},\ \widehat{m}^{\nu e}_{ij\psi},\ -1-\widehat{m}^{\nu e}_{ij\phi},\ \widehat{m}^{\nu e}_{ij\theta}\rangle]_{m\times n}$.

(ii) Transpose:
$\mathbb{M}^T = [\langle\widehat{m}^{\rho e}_{ji\theta},\widehat{m}^{\rho e}_{ji\phi},\widehat{m}^{\rho e}_{ji\psi},\widehat{m}^{\nu e}_{ji\theta},\widehat{m}^{\nu e}_{ji\phi},\widehat{m}^{\nu e}_{ji\psi}\rangle]_{n\times m}$.

(iii) Addition:

$$\mathbb{M}+\mathbb{N} = \begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\psi},\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\theta},\widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\phi},\widehat{n}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\psi},\widehat{n}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

(iv) Multiplication:

$$\mathbb{M}\times\mathbb{N} = \begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\psi},\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\theta},\widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\phi},\widehat{n}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\psi},\widehat{n}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

(v) Direct sum.

$$\mathbb{M}\oplus\mathbb{N}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\theta}+\widehat{n}^{\rho e}_{ij\theta}-\widehat{m}^{\rho e}_{ij\theta}\cdot\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\phi}\cdot\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\psi}\cdot\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}}\{-\widehat{m}^{\nu e}_{ij\theta}\cdot\widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}}\{-(-\widehat{m}^{\nu e}_{ij\phi}-\widehat{n}^{\nu e}_{ij\phi}-\widehat{m}^{\nu e}_{ij\phi}\cdot\widehat{n}^{\nu e}_{ij\phi})\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}}\{-(-\widehat{m}^{\nu e}_{ij\psi}-\widehat{n}^{\nu e}_{ij\psi}-\widehat{m}^{\nu e}_{ij\psi}\cdot\widehat{n}^{\nu e}_{ij\psi})\} \end{pmatrix}_{m\times n}$$

(vi) Element-wise multiplication.

$$\mathbb{M}\otimes\mathbb{N}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\theta}\cdot\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\phi}+\widehat{n}^{\rho e}_{ij\phi}-\widehat{m}^{\rho e}_{ij\phi}\cdot\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}}\{\widehat{m}^{\rho e}_{ij\psi}+\widehat{n}^{\rho e}_{ij\psi}-\widehat{m}^{\rho e}_{ij\psi}\cdot\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}}\{-(-\widehat{m}^{\nu e}_{ij\theta}-\widehat{n}^{\nu e}_{ij\theta}-\widehat{m}^{\nu e}_{ij\theta}\cdot\widehat{n}^{\nu e}_{ij\theta})\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}}\{-\widehat{m}^{\nu e}_{ij\phi}\cdot\widehat{n}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}}\{-\widehat{m}^{\nu e}_{ij\psi}\cdot\widehat{n}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

(vii) Average:

$$\mathbb{M}\oslash\mathbb{N}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}}\dfrac{\widehat{m}^{\rho e}_{ij\theta}+\widehat{n}^{\rho e}_{ij\theta}}{\ell_{\widehat{m}^{\rho e}_{ij\theta}}+\ell_{\widehat{n}^{\rho e}_{ij\theta}}} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}}\dfrac{\widehat{m}^{\rho e}_{ij\phi}+\widehat{n}^{\rho e}_{ij\phi}}{\ell_{\widehat{m}^{\rho e}_{ij\phi}}+\ell_{\widehat{n}^{\rho e}_{ij\phi}}} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}}\dfrac{\widehat{m}^{\rho e}_{ij\psi}+\widehat{n}^{\rho e}_{ij\psi}}{\ell_{\widehat{m}^{\rho e}_{ij\psi}}+\ell_{\widehat{n}^{\rho e}_{ij\psi}}} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}}\dfrac{\widehat{m}^{\nu e}_{ij\theta}+\widehat{n}^{\nu e}_{ij\theta}}{\ell_{\widehat{m}^{\nu e}_{ij\theta}}+\ell_{\widehat{n}^{\nu e}_{ij\theta}}} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}}\dfrac{\widehat{m}^{\nu e}_{ij\phi}+\widehat{n}^{\nu e}_{ij\phi}}{\ell_{\widehat{m}^{\nu e}_{ij\phi}}+\ell_{\widehat{n}^{\nu e}_{ij\phi}}} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}}\dfrac{\widehat{m}^{\nu e}_{ij\psi}+\widehat{n}^{\nu e}_{ij\psi}}{\ell_{\widehat{m}^{\nu e}_{ij\psi}}+\ell_{\widehat{n}^{\nu e}_{ij\psi}}} \end{pmatrix}_{m\times n}$$

Where, $\ell_{\widehat{m}^{\rho e}_{ij\theta}}, \ell_{\widehat{m}^{\rho e}_{ij\phi}}, \ell_{\widehat{m}^{\rho e}_{ij\psi}}, \ell_{\widehat{m}^{\nu e}_{ij\theta}}, \ell_{\widehat{m}^{\nu e}_{ij\phi}}, \ell_{\widehat{m}^{\nu e}_{ij\psi}}$ are denotes the number of elements in $\widehat{m}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\psi}$, $\widehat{m}^{\nu e}_{ij\theta}, \widehat{m}^{\nu e}_{ij\phi}, \widehat{m}^{\nu e}_{ij\psi}$ respectively.

3.2. **Properties of MVBNMs.**

Proposition 3.1. *Let* $\mathbb{L}, \mathbb{M}, \mathbb{N}$ *be three* $m \times n$ *MVBNMs. In addition:*

(i) $\mathbb{M} + \mathbb{N} = \mathbb{N} + \mathbb{M}$ *(commutativity).*
(ii) $(\mathbb{L} + \mathbb{M}) + \mathbb{N} = \mathbb{L} + (\mathbb{M} + \mathbb{N})$ *(associativity).*
(iii) $\mathbb{M} + \mathbb{I} = \mathbb{I} + \mathbb{M} = \mathbb{M}$ *(additive identity).*

Proof. Let $\mathbb{M} = [\langle \widehat{m}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\psi}, \widehat{m}^{\nu e}_{ij\theta}, \widehat{m}^{\nu e}_{ij\phi}, \widehat{m}^{\nu e}_{ij\psi} \rangle]$, $\mathbb{N} = [\langle \widehat{n}^{\rho e}_{ij\theta}, \widehat{n}^{\rho e}_{ij\phi}, \widehat{n}^{\rho e}_{ij\psi}, \widehat{n}^{\nu e}_{ij\theta}, \widehat{n}^{\nu e}_{ij\phi}, \widehat{n}^{\nu e}_{ij\psi} \rangle]$ and $\mathbb{L} = [\langle \widehat{l}^{\rho e}_{ij\theta}, \widehat{l}^{\rho e}_{ij\phi}, \widehat{l}^{\rho e}_{ij\psi}, \widehat{l}^{\nu e}_{ij\theta}, \widehat{l}^{\nu e}_{ij\phi}, \widehat{l}^{\nu e}_{ij\psi} \rangle]$

(i)

$$\mathbb{M} + \mathbb{N} = \begin{pmatrix} \bigcup\limits_{\widehat{m}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijm}, \widehat{n}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\theta}, \widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijm}, \widehat{n}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\phi}, \widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijm}, \widehat{n}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\psi}, \widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{m}^{\nu e}_{ij\theta} \in \widehat{T}^{\nu e}_{ijm}, \widehat{n}^{\nu e}_{ij\theta} \in \widehat{T}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\theta}, \widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{\nu e}_{ij\phi} \in \widehat{I}^{\nu e}_{ijm}, \widehat{n}^{\nu e}_{ij\phi} \in \widehat{I}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\phi}, \widehat{n}^{\nu e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{\nu e}_{ij\psi} \in \widehat{F}^{\nu e}_{ijm}, \widehat{n}^{\nu e}_{ij\psi} \in \widehat{F}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\psi}, \widehat{n}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m \times n}$$

$$\mathbb{N} + \mathbb{M} = \begin{pmatrix} \bigcup\limits_{\widehat{n}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijm}} \max\{\widehat{n}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijm}} \min\{\widehat{n}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijm}} \min\{\widehat{n}^{\rho e}_{ij\psi}, \widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{n}^{\nu e}_{ij\theta} \in \widehat{T}^{\nu e}_{ijn}, \widehat{m}^{\nu e}_{ij\theta} \in \widehat{T}^{\nu e}_{ijm}} \min\{\widehat{n}^{\nu e}_{ij\theta}, \widehat{m}^{\nu e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{n}^{\nu e}_{ij\phi} \in \widehat{I}^{\nu e}_{ijn}, \widehat{m}^{\nu e}_{ij\phi} \in \widehat{I}^{\nu e}_{ijm}} \max\{\widehat{n}^{\nu e}_{ij\phi}, \widehat{m}^{\nu e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{n}^{\nu e}_{ij\psi} \in \widehat{F}^{\nu e}_{ijn}, \widehat{m}^{\nu e}_{ij\psi} \in \widehat{F}^{\nu e}_{ijm}} \max\{\widehat{n}^{\nu e}_{ij\psi}, \widehat{m}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m \times n}$$

Therefore, $\mathbb{M} + \mathbb{N} = \mathbb{N} + \mathbb{M}$.

(ii) Similarly, we can prove that the associativity property $(\mathbb{L} + \mathbb{M}) + \mathbb{N} = \mathbb{L} + (\mathbb{M} + \mathbb{N})$.

(iii) Let $\mathbb{I}$ be the additive identity; here $\mathbb{I}$ is a zero of MVBM, which is denoted by $\mathbb{I}_{\langle 0,1,1,0,-1,-1\rangle}$.

$$\mathbb{M}+\mathbb{I}=\begin{pmatrix} \bigcup\limits_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \max\{\widehat{m}^{\rho e}_{ij\theta},0\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \min\{\widehat{m}^{\rho e}_{ij\phi},1\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \min\{\widehat{m}^{\rho e}_{ij\psi},1\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijm}} \min\{\widehat{m}^{ve}_{ij\theta},0\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijm}} \max\{\widehat{m}^{ve}_{ij\phi},-1\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijm}} \max\{\widehat{m}^{ve}_{ij\psi},-1\} \end{pmatrix}_{m\times n}$$

$$=\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{ve}_{ij\theta},\widehat{m}^{ve}_{ij\phi},\widehat{m}^{ve}_{ij\psi}\rangle=\mathbb{M}$$

$$\mathbb{I}+\mathbb{M}=\begin{pmatrix} \bigcup\limits_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \max\{0,\widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \min\{1,\widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \min\{1,\widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijm}} \min\{0,\widehat{m}^{ve}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijm}} \max\{-1,\widehat{m}^{ve}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijm}} \max\{-1,\widehat{m}^{ve}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

$$=\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{ve}_{ij\theta},\widehat{m}^{ve}_{ij\phi},\widehat{m}^{ve}_{ij\psi}\rangle=\mathbb{M}$$

Therefore, $\mathbb{M}+\mathbb{I}=\mathbb{I}+\mathbb{M}=\mathbb{M}$.

□

Proposition 3.2. *Let $\mathbb{L},\mathbb{M},\mathbb{N}$ be three $m\times n$ MVBNMs. Define the following under multiplication.*

(i) $\mathbb{M}\times\mathbb{N}=\mathbb{N}\times\mathbb{M}$ *(commutativity).*
(ii) $(\mathbb{L}\times\mathbb{M})\times\mathbb{N}=\mathbb{L}\times(\mathbb{M}\times\mathbb{N})$ *(associativity).*
(iii) $\mathbb{M}\times\mathbb{I}=\mathbb{I}\times\mathbb{M}=\mathbb{M}$ *(multiplicative identity).*

Proof. Let $\mathbb{M}=[\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{ve}_{ij\theta},\widehat{m}^{ve}_{ij\phi},\widehat{m}^{ve}_{ij\psi}\rangle]$, $\mathbb{N}=[\langle\widehat{n}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\psi},\widehat{n}^{ve}_{ij\theta},\widehat{n}^{ve}_{ij\phi},\widehat{n}^{ve}_{ij\psi}\rangle]$ and $\mathbb{L}=[\langle\widehat{l}^{\rho e}_{ij\theta},\widehat{l}^{\rho e}_{ij\phi},\widehat{l}^{\rho e}_{ij\psi},\widehat{l}^{ve}_{ij\theta},\widehat{l}^{ve}_{ij\phi},\widehat{l}^{ve}_{ij\psi}\rangle]$

(i)

$$\mathbb{M}\times\mathbb{N}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}} \min\{\widehat{m}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}} \max\{\widehat{m}^{\rho e}_{ij\psi},\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}} \max\{\widehat{m}^{\nu e}_{ij\theta},\widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\phi},\widehat{n}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}} \min\{\widehat{m}^{\nu e}_{ij\psi},\widehat{n}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

$$\mathbb{N}\times\mathbb{M}=\begin{pmatrix} \bigcup_{\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn},\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \min\{\widehat{n}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn},\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \max\{\widehat{n}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn},\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \max\{\widehat{n}^{\rho e}_{ij\psi},\widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn},\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm}} \max\{\widehat{n}^{\nu e}_{ij\theta},\widehat{m}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn},\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm}} \min\{\widehat{n}^{\nu e}_{ij\phi},\widehat{m}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn},\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm}} \min\{\widehat{n}^{\nu e}_{ij\psi},\widehat{m}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

Therefore, $\mathbb{M}\times\mathbb{N}=\mathbb{N}\times\mathbb{M}$.

(ii) Similarly, We can prove that associativity property $(\mathbb{L}\times\mathbb{M})\times\mathbb{N}=\mathbb{L}\times(\mathbb{M}\times\mathbb{N})$.

(iii) Let $\mathbb{I}$ be the multiplicative identity; here, $\mathbb{I}$ is the unit MVBNM denoted by $\mathbb{I}_{\langle 1,0,0,-1,0,0\rangle}$.

$$\mathbb{M}\times\mathbb{I}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \min\{\widehat{m}^{\rho e}_{ij\theta},1\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \max\{\widehat{m}^{\rho e}_{ij\phi},0\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \max\{\widehat{m}^{\rho e}_{ij\psi},0\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm}} \max\{\widehat{m}^{\nu e}_{ij\theta},-1\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm}} \min\{\widehat{m}^{\nu e}_{ij\phi},0\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm}} \min\{\widehat{m}^{\nu e}_{ij\psi},0\} \end{pmatrix}_{m\times n}$$

$$=\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{\nu e}_{ij\theta},\widehat{m}^{\nu e}_{ij\phi},\widehat{m}^{\nu e}_{ij\psi}\rangle=\mathbb{M}$$

$$\mathbb{I}\times\mathbb{M}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \min\{1,\widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \max\{0,\widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \max\{0,\widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm}} \max\{-1,\widehat{m}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm}} \min\{0,\widehat{m}^{\nu e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm}} \min\{0,\widehat{m}^{\nu e}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

$$=\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{\nu e}_{ij\theta},\widehat{m}^{\nu e}_{ij\phi},\widehat{m}^{\nu e}_{ij\psi}\rangle=\mathbb{M}$$

Therefore, $\mathbb{M}\times\mathbb{I}=\mathbb{I}\times\mathbb{M}=\mathbb{M}$..

□

Proposition 3.3. *Let $\mathbb{L},\mathbb{M},\mathbb{N}$ be the $m\times n$ MVBNMs. Define the following with the operation direct sum.*

(i) $\mathbb{M}\oplus\mathbb{N}=\mathbb{N}\oplus\mathbb{M}$ *(commutativity).*

(ii) $(\mathbb{L}\oplus\mathbb{M})\oplus\mathbb{N}=\mathbb{L}\oplus(\mathbb{M}\oplus\mathbb{N})$ *(associativity).*

Proof. Let $\mathbb{M}=[\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{\nu e}_{ij\theta},\widehat{m}^{\nu e}_{ij\phi},\widehat{m}^{\nu e}_{ij\psi}\rangle]$, $\mathbb{N}=[\langle\widehat{n}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\psi},\widehat{n}^{\nu e}_{ij\theta},\widehat{n}^{\nu e}_{ij\phi},\widehat{n}^{\nu e}_{ij\psi}\rangle]$ and $\mathbb{L}=[\langle\widehat{l}^{\rho e}_{ij\theta},\widehat{l}^{\rho e}_{ij\phi},\widehat{l}^{\rho e}_{ij\psi},\widehat{l}^{\nu e}_{ij\theta},\widehat{l}^{\nu e}_{ij\phi},\widehat{l}^{\nu e}_{ij\psi}\rangle]$.

(i)

$$\mathbb{M}\oplus\mathbb{N}=\begin{pmatrix} \bigcup_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\theta}+\widehat{n}^{\rho e}_{ij\theta}-\widehat{m}^{\rho e}_{ij\theta}\cdot\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\phi}\cdot\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\psi}\cdot\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\theta}\in\widehat{T}^{\nu e}_{ijn}} \{-\widehat{m}^{\nu e}_{ij\theta}\cdot\widehat{n}^{\nu e}_{ij\theta}\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\phi}\in\widehat{I}^{\nu e}_{ijn}} \{-(-\widehat{m}^{\nu e}_{ij\phi}-\widehat{n}^{\nu e}_{ij\phi}-\widehat{m}^{\nu e}_{ij\phi}\cdot\widehat{n}^{\nu e}_{ij\phi})\} \\ \bigcup_{\widehat{m}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijm},\widehat{n}^{\nu e}_{ij\psi}\in\widehat{F}^{\nu e}_{ijn}} \{-(-\widehat{m}^{\nu e}_{ij\psi}-\widehat{n}^{\nu e}_{ij\psi}-\widehat{m}^{\nu e}_{ij\psi}\cdot\widehat{n}^{\nu e}_{ij\psi})\} \end{pmatrix}_{m\times n}$$

$$\mathbb{N}\oplus\mathbb{M}=\begin{pmatrix} \bigcup\limits_{\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn},\,\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\theta}+\widehat{m}^{\rho e}_{ij\theta}-\widehat{n}^{\rho e}_{ij\theta}\cdot\widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn},\,\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\phi}\cdot\widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn},\,\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\psi}\cdot\widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijn},\,\widehat{m}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijm}} \{-\widehat{n}^{ve}_{ij\theta}\cdot\widehat{m}^{ve}_{ij\theta}\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijn},\,\widehat{m}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijm}} \{-(-\widehat{n}^{ve}_{ij\phi}-\widehat{m}^{ve}_{ij\phi}-\widehat{n}^{ve}_{ij\phi}\cdot\widehat{m}^{ve}_{ij\phi})\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijn},\,\widehat{m}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijm}} \{-(-\widehat{n}^{ve}_{ij\psi}-\widehat{m}^{ve}_{ij\psi}-\widehat{n}^{ve}_{ij\psi}\cdot\widehat{m}^{ve}_{ij\psi})\} \end{pmatrix}_{m\times n}$$

Therefore, $\mathbb{M}\oplus\mathbb{N}=\mathbb{N}\oplus\mathbb{M}$.

(ii) Similarly, we can prove that $(\mathbb{L}\oplus\mathbb{M})\oplus\mathbb{N}=\mathbb{L}\oplus(\mathbb{M}\oplus\mathbb{N})$.

□

Proposition 3.4. *Let $\mathbb{L},\mathbb{M},\mathbb{N}$ be the $m\times n$ MVBNMs. Define the following with the operation element-wise multiplication.*

(i) $\mathbb{M}\otimes\mathbb{N}=\mathbb{N}\otimes\mathbb{M}$ *(commutativity).*

(ii) $(\mathbb{L}\otimes\mathbb{M})\otimes\mathbb{N}=\mathbb{L}\otimes(\mathbb{M}\otimes\mathbb{N})$*(associativity).*

Proof. Let $\mathbb{M}=[\langle\widehat{m}^{\rho e}_{ij\theta},\widehat{m}^{\rho e}_{ij\phi},\widehat{m}^{\rho e}_{ij\psi},\widehat{m}^{ve}_{ij\theta},\widehat{m}^{ve}_{ij\phi},\widehat{m}^{ve}_{ij\psi}\rangle]$, $\mathbb{N}=[\langle\widehat{n}^{\rho e}_{ij\theta},\widehat{n}^{\rho e}_{ij\phi},\widehat{n}^{\rho e}_{ij\psi},\widehat{n}^{ve}_{ij\theta},\widehat{n}^{ve}_{ij\phi},\widehat{n}^{ve}_{ij\psi}\rangle]$ and $\mathbb{L}=[\langle\widehat{l}^{\rho e}_{ij\theta},\widehat{l}^{\rho e}_{ij\phi},\widehat{l}^{\rho e}_{ij\psi},\widehat{l}^{ve}_{ij\theta},\widehat{l}^{ve}_{ij\phi},\widehat{l}^{ve}_{ij\psi}\rangle]$.

(i)

$$\mathbb{M}\otimes\mathbb{N}=\begin{pmatrix} \bigcup\limits_{\widehat{m}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijm},\,\widehat{n}^{\rho e}_{ij\theta}\in\widehat{T}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\theta}\cdot\widehat{n}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijm},\,\widehat{n}^{\rho e}_{ij\phi}\in\widehat{I}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\phi}+\widehat{n}^{\rho e}_{ij\phi}-\widehat{m}^{\rho e}_{ij\phi}\cdot\widehat{n}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijm},\,\widehat{n}^{\rho e}_{ij\psi}\in\widehat{F}^{\rho e}_{ijn}} \{\widehat{m}^{\rho e}_{ij\psi}+\widehat{n}^{\rho e}_{ij\psi}-\widehat{m}^{\rho e}_{ij\psi}\cdot\widehat{n}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijm},\,\widehat{n}^{ve}_{ij\theta}\in\widehat{T}^{ve}_{ijn}} \{-(-\widehat{m}^{ve}_{ij\theta}-\widehat{n}^{ve}_{ij\theta}-\widehat{m}^{ve}_{ij\theta}\cdot\widehat{n}^{ve}_{ij\theta})\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijm},\,\widehat{n}^{ve}_{ij\phi}\in\widehat{I}^{ve}_{ijn}} \{-\widehat{m}^{ve}_{ij\phi}\cdot\widehat{n}^{ve}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijm},\,\widehat{n}^{ve}_{ij\psi}\in\widehat{F}^{ve}_{ijn}} \{-\widehat{m}^{ve}_{ij\psi}\cdot\widehat{n}^{ve}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

$$\mathbb{N} \otimes \mathbb{M} = \begin{pmatrix} \bigcup\limits_{\widehat{n}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\theta} \cdot \widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\phi} + \widehat{m}^{\rho e}_{ij\phi} - \widehat{n}^{\rho e}_{ij\phi} \cdot \widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{n}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijn}, \widehat{m}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijm}} \{\widehat{n}^{\rho e}_{ij\psi} + \widehat{m}^{\rho e}_{ij\psi} - \widehat{n}^{\rho e}_{ij\psi} \cdot \widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\theta} \in \widehat{T}^{ve}_{ijn}, \widehat{m}^{ve}_{ij\theta} \in \widehat{T}^{ve}_{ijm}} \{-(-\widehat{n}^{ve}_{ij\theta} - \widehat{m}^{ve}_{ij\theta} - \widehat{n}^{ve}_{ij\theta} \cdot \widehat{m}^{ve}_{ij\theta})\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\phi} \in \widehat{I}^{ve}_{ijn}, \widehat{m}^{ve}_{ij\phi} \in \widehat{I}^{ve}_{ijm}} \{-\widehat{n}^{ve}_{ij\phi} \cdot \widehat{m}^{ve}_{ij\phi}\} \\ \bigcup\limits_{\widehat{n}^{ve}_{ij\psi} \in \widehat{F}^{ve}_{ijn}, \widehat{m}^{ve}_{ij\psi} \in \widehat{F}^{ve}_{ijm}} \{-\widehat{n}^{ve}_{ij\psi} \cdot \widehat{m}^{ve}_{ij\psi}\} \end{pmatrix}_{m \times n}$$

Therefore, $\mathbb{M} \otimes \mathbb{N} = \mathbb{N} \otimes \mathbb{M}$.

(ii) Similarly, we can prove that $(\mathbb{L} \otimes \mathbb{M}) \otimes \mathbb{N} = \mathbb{L} \otimes (\mathbb{M} \otimes \mathbb{N})$.

□

Proposition 3.5. *Let $\mathbb{L}, \mathbb{M}, \mathbb{N}$ be the $m \times n$ MVBNMs. Define the following with the distributive property.*

(i) $\mathbb{L} \times (\mathbb{M} + \mathbb{N}) \neq (\mathbb{L} \times \mathbb{M}) + (\mathbb{L} \times \mathbb{N})$, *and* $(\mathbb{L} + \mathbb{M}) \times \mathbb{N} \neq (\mathbb{L} \times \mathbb{N}) + (\mathbb{M} \times \mathbb{N})$ *(multiplication distributes over addition).*

(ii) $\mathbb{L} \otimes (\mathbb{M} \oplus \mathbb{N}) \neq (\mathbb{L} \otimes \mathbb{M}) \oplus (\mathbb{L} \otimes \mathbb{N})$, *and* $(\mathbb{L} \oplus \mathbb{M}) \otimes \mathbb{N} \neq (\mathbb{L} \otimes \mathbb{N}) \oplus (\mathbb{M} \otimes \mathbb{N})$ *(element-wise multiplication distributes over direct sum).*

Proof.

$$\mathbb{L} = \begin{bmatrix} \langle\{0.5\}, \{0.3, 0.2\}, \{0.6\} \{-0.1, -0.3\}, \{-0.6\}, \{-0.4\}\rangle & \langle\{0.3, 0.6\}, \{0.3, 0.5\}, \{0.1\}, \{-0.2\}, \{-0.3\}, \{-0.7\}\rangle \\ \langle\{0.3\}, \{0.6\}, \{0.8\}, \{-0.5\}, \{-0.2, -0.3\}, \{-0.3\}\rangle & \langle\{0.1, 0.2\}, \{0.7\}, \{0.5\}, \{-0.5\}, \{-0.8\}, \{-0.3\}\rangle \end{bmatrix}_{2\times 2}$$

$$\mathbb{M} = \begin{bmatrix} \langle\{0.7\}, \{0.1\}, \{0.2\}, \{-0.2\}, \{-0.1\}, \{-0.7, -0.8\}\rangle & \langle\{0.4\}, \{0.2\}, \{0.5\}, \{-0.1, -0.2\}, \{-0.7\}, \{-0.4\}\rangle \\ \langle\{0.2\}, \{0.2, 0.3\}, \{0.5, 0.6\}, \{-0.5\}, \{-0.8\}, \{-0.1, -0.3\}\rangle & \langle\{0.4, 0.5\}, \{0.5, 0.6\}, \{0.3\}, \{-0.6\}, \{-0.4\}, \{-0.7\}\rangle \end{bmatrix}_{2\times 2}$$

$$\mathbb{N} = \begin{bmatrix} \langle\{0.5\}, \{0.1, 0.3\}, \{0.2\} \{-0.2, -0.3\}, \{-0.7\}, \{-0.4, -0.6\}\rangle & \langle\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}, \{-0.2, -0.3\}, \{-0.2\}, \{-0.7\}\rangle \\ \langle\{0.4\}, \{0.3\}, \{0.1, 0.2\}, \{-0.1, -0.2\}, \{-0.7\}, \{-0.3\}\rangle & \langle\{0.5, 0.6\}, \{0.3\}, \{0.2\}, \{-0.4\}, \{-0.6\}, \{-0.5\}\rangle \end{bmatrix}_{2\times 2}$$

(i)

$$\mathbb{L} \times (\mathbb{M} + \mathbb{N}) = \begin{bmatrix} \langle\{0.5\}, \{0.3, 0.2\}, \{0.6\} \{-0.1, -0.3\}, \{-0.6\}, \{-0.4, -0.6\}\rangle & \langle\{0.3, 0.6\}, \{0.1, 0.2\}, \{0.2, 0.3\}, \{-0.2\}, \{-0.3\}, \{-0.7\}\rangle \\ \langle\{0.3\}, \{0.6\}, \{0.8\}, \{-0.5\}, \{-0.7\}, \{-0.3\}\rangle & \langle\{0.1, 0.2\}, \{0.7\}, \{0.5\}, \{-0.5\}, \{-0.8\}, \{-0.5\}\rangle \end{bmatrix}_{2\times 2}$$

$$(\mathbb{L} \times \mathbb{M}) + (\mathbb{L} \times \mathbb{N}) = \begin{bmatrix} \langle\{0.5\}, \{0.3, 0.2\}, \{0.6\} \{-0.1, -0.2, -0.3\}, \{-0.7\}, \{-0.3\}\rangle & \langle\{0.3, 0.6\}, \{0.3, 0.5\}, \{0.2, 0.3\}, \{-0.2\}, \{-0.3\}, \{-0.7\}\rangle \\ \langle\{0.3\}, \{0.6\}, \{0.8\}, \{-0.5\}, \{-0.7\}, \{-0.3\}\rangle & \langle\{0.1, 0.2\}, \{0.7\}, \{0.5\}, \{-0.5\}, \{-0.8\}, \{-0.5\}\rangle \end{bmatrix}_{2\times 2}$$

That is, $\mathbb{L} \times (\mathbb{M} + \mathbb{N}) \neq (\mathbb{L} \times \mathbb{M}) + (\mathbb{L} \times \mathbb{N})$ and similarly, we can prove that $(\mathbb{L} + \mathbb{M}) \times \mathbb{N} \neq (\mathbb{L} \times \mathbb{N}) + (\mathbb{M} \times \mathbb{N})$. The operations are defined with min-max sets of unions that are not linear, so this property does not always hold. Bipolar neutrosophic matrices are algebraically more stable than MVBNMs.

(ii) Similarly, we can prove that $\mathbb{L} \otimes (\mathbb{M} \oplus \mathbb{N}) \neq (\mathbb{L} \otimes \mathbb{M}) \oplus (\mathbb{L} \otimes \mathbb{N})$, and $(\mathbb{L} \oplus \mathbb{M}) \otimes \mathbb{N} \neq (\mathbb{L} \otimes \mathbb{N}) \oplus (\mathbb{M} \otimes \mathbb{N})$.

□

Proposition 3.6. *Let $\mathbb{M}$ be an $n \times n$ ordered square MVBNM. Then satisfies the following properties:*

(i) $\mathbb{M} + \mathbb{M} = \mathbb{M}$

(ii) $\text{Tr}(\mathbb{M}) = \text{Tr}(\mathbb{M}^T)$

(iii) $\text{Tr}(\mathbb{M}) + \text{Tr}(\mathbb{N}) \neq \text{Tr}(\mathbb{M} + \mathbb{N})$

(iv) $\det(\mathbb{M}) + \det(\mathbb{N}) \neq \det(\mathbb{M} + \mathbb{N})$

(v) $\det(\mathbb{M}) \times \det(\mathbb{N}) \neq \det(\mathbb{M} \times \mathbb{N})$

Proof. Let $\mathbb{M}$ be a MVBNM, $\mathbb{M} = [\langle \widehat{m}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\psi}, \widehat{m}^{ve}_{ij\theta}, \widehat{m}^{ve}_{ij\phi}, \widehat{m}^{ve}_{ij\psi} \rangle]_{n\times n}$.

(i)

$$\mathbb{M} + \mathbb{M} = \begin{pmatrix} \bigcup\limits_{\widehat{m}^{\rho e}_{ij\theta} \in \widehat{T}^{\rho e}_{ijm}} \max\{\widehat{m}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\phi} \in \widehat{I}^{\rho e}_{ijm}} \min\{\widehat{m}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{\rho e}_{ij\psi} \in \widehat{F}^{\rho e}_{ijm}} \min\{\widehat{m}^{\rho e}_{ij\psi}, \widehat{m}^{\rho e}_{ij\psi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\theta} \in \widehat{T}^{ve}_{ijm}} \min\{\widehat{m}^{ve}_{ij\theta}, \widehat{m}^{ve}_{ij\theta}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\phi} \in \widehat{I}^{ve}_{ijm}} \max\{\widehat{m}^{ve}_{ij\phi}, \widehat{m}^{ve}_{ij\phi}\} \\ \bigcup\limits_{\widehat{m}^{ve}_{ij\psi} \in \widehat{F}^{ve}_{ijm}} \max\{\widehat{m}^{ve}_{ij\psi}, \widehat{m}^{ve}_{ij\psi}\} \end{pmatrix}_{m\times n}$$

$$= [\langle \widehat{m}^{\rho e}_{ij\theta}, \widehat{m}^{\rho e}_{ij\phi}, \widehat{m}^{\rho e}_{ij\psi}, \widehat{m}^{ve}_{ij\theta}, \widehat{m}^{ve}_{ij\phi}, \widehat{m}^{ve}_{ij\psi} \rangle]_{m\times n} = \mathbb{M}.$$

This property satisfies any order MVBNMs.

(ii) $\text{Tr}(\mathbb{M}) = \prod_{i=1}^{n} m_{ii} = m_{11} + m_{22} + m_{33} + ... + m_{nn}$.
$\mathbb{M}^T = m_{ji}$.
That is, $\text{Tr}(\mathbb{M}^T) = \prod_{j=1}^{n} m_{jj} = m_{11} + m_{22} + m_{33} + ... + m_{nn}$.
Therefore, $\text{Tr}(\mathbb{M}) = \text{Tr}(\mathbb{M}^T)$.

(iii) Take the same MVBNMs $\mathbb{M}$ and $\mathbb{N}$ from proposition 3.5.
$\text{Tr}(\mathbb{M}) = \langle \{0.4, 0.5\}, \{0.5, 0.6\}, \{0.3\}, \{-0.2\}, \{-0.4\}, \{-0.7\} \rangle$.
$\text{Tr}(\mathbb{N}) = \langle \{0.5\}, \{0.3\}, \{0.2\}, \{-0.2, -0.3\}, \{-0.7\}, \{-0.5, -0.6\} \rangle$.
$\text{Tr}(\mathbb{M}) + \text{Tr}(\mathbb{N}) = \langle \{0.5\}, \{0.3\}, \{0.2\}, \{-0.2, -0.3\}, \{-0.4\}, \{-0.5, -0.6\} \rangle$.

$$\mathbb{M} + \mathbb{N} = \begin{bmatrix} \langle \{0.7\}, \{0.1\}, \{0.2\} \{-0.2, -0.3\}, \{-0.1\}, \{-0.4, -0.6\} \rangle & \langle \{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}, \{-0.2, -0.3\}, \{-0.2\}, \{-0.4\} \rangle \\ \langle \{0.4\}, \{0.2, 0.3\}, \{0.1, 0.2\}, \{-0.5\}, \{-0.7\}, \{-0.1, -0.3\} \rangle & \langle \{0.5, 0.6\}, \{0.3\}, \{0.2\}, \{-0.6\}, \{-0.4\}, \{-0.5\} \rangle \end{bmatrix}_{2\times 2}$$

$\text{Tr}(\mathbb{M} + \mathbb{N}) = \langle \{0.5, 0.6\}, \{0.3\}, \{0.2\}, \{-0.2, -0.3\}, \{-0.4\}, \{-0.4, -0.5\} \rangle$.
Therefore, $\text{Tr}(\mathbb{M}) + \text{Tr}(\mathbb{N}) \neq \text{Tr}(\mathbb{M} + \mathbb{N})$.

(iv) $\det(\mathbb{M}) = \langle \{0.4, 0.5\}, \{0.2, 0.3\}, \{0.3\}, \{-0.2\}, \{-0.4\}, \{-0.4\} \rangle$.
$\det(\mathbb{N}) = \langle \{0.5\}, \{0.3\}, \{0.2\}, \{-0.3, -0.2\}, \{-0.7\}, \{-0.5, -0.6\} \rangle$.
$\det(\mathbb{M}) + \det(\mathbb{N}) = \langle \{0.4, 0.5\}, \{0.2, 0.3\}, \{0.3\}, \{-0.2, -0.3\}, \{-0.3\}, \{-0.3\} \rangle$.

$\det(\mathbb{M}+\mathbb{N}) = \langle\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.2, 0.3\}, \{-0.2, -0.3\}, \{-0.4\}, \{-0.2, -0.3\}\rangle$.

Therefore, $\det(\mathbb{M}) + \det(\mathbb{N}) \neq \det(\mathbb{M}+\mathbb{N})$.

(v) Similarly, we can prove that $\det(\mathbb{M}) \times \det(\mathbb{N}) \neq \det(\mathbb{M}\times\mathbb{N})$.

□

4. Simplified-TOPSIS Method for MBVNNs.

In this section, we proposed an algorithm for the simplified neutrosophic TOPSIS in the case of MVBNNs.

Step 1: ***Determine alternatives and criteria***.

Experts evaluate the alternatives according to each criterion using linguistic variables.

Step 2: ***Compute weights of decision makers***.

Convert linguistic variables into MVBNNs.

Definition 4.1. ***Weights of decision makers.***

Assume the weights of decision makers based on their influence and importance in the decision-making process. The rating of the i^{th} decision maker is defined as $E_i = \left\langle \widehat{T}^{\rho e}_{i\theta(x)}, \widehat{I}^{\rho e}_{i\phi(x)}, \widehat{F}^{\rho e}_{i\psi}(x), \widehat{T}^{\nu e}_{i\theta}(x), \widehat{I}^{\nu e}_{i(x)\phi}, \widehat{F}^{\nu e}_{i\psi}(x) \right\rangle$. *The weight of i^{th} decision maker expressed as,*

$$\omega_i = \bigcup_{\substack{\widehat{T}^{\rho e}_{i\theta}, \widehat{I}^{\rho e}_{i\phi}, \widehat{F}^{\rho e}_{i\psi}, \\ \widehat{T}^{\nu e}_{i\theta}, \widehat{I}^{\nu e}_{i\phi}, \widehat{F}^{\nu e}_{i\psi}}} \frac{1 - \left(\left((1-\widehat{T}^{\rho e}_{i\theta}(x))^2 + (\widehat{I}^{\rho e}_{i\phi}(x))^2 + (\widehat{F}^{\rho e}_{i\psi}(x))^2 + (\widehat{T}^{\nu e}_{i\theta}(x))^2 + (\widehat{I}^{\nu e}_{i\phi}(x))^2 + (1+\widehat{F}^{\nu e}_{i\psi}(x))^2\right) \times \frac{1}{6}\right)^{\frac{1}{2}} \times \frac{1}{n}}{\sum_{i=1}^{m} \left(1 - \left(\left((1-\widehat{T}^{\rho e}_{i\theta}(x))^2 + (\widehat{I}^{\rho e}_{i\phi}(x))^2 + (\widehat{F}^{\rho e}_{i\psi}(x))^2 + (\widehat{T}^{\nu e}_{i\theta}(x))^2 + (\widehat{I}^{\nu e}_{i\phi}(x))^2 + (1+\widehat{F}^{\nu e}_{i\psi}(x))^2\right) \times \frac{1}{6}\right)^{\frac{1}{2}} \times \frac{1}{n}\right)} \tag{4.1}$$

so that $\omega_i > 0$ and $\sum_{i=1}^{m} \omega_i = 1$. Where n is the number of combinations and m is the number of decision makers.

Step 3: ***Build MVBNS decision matrix and criteria weights***

Compose the MVBNN decision matrix $\Delta = (\delta_{ij}) = (\widehat{T}^{\rho e}_{ij\theta}, \widehat{I}^{\rho e}_{ij\phi}, \widehat{F}^{\rho e}_{ij\psi}, \widehat{T}^{\nu e}_{ij\theta}, \widehat{I}^{\nu e}_{ij\phi}, \widehat{F}^{\nu e}_{ij\psi})$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

The positive membership values $\widehat{T}^{\rho e}_{ij\theta}, \widehat{I}^{\rho e}_{ij\phi}, \widehat{F}^{\rho e}_{ij\psi}$ denote the truth, indeterminacy, and falsity of alternative i with respect to criterion j in MVBNNs, and the negative membership values $\widehat{T}^{\nu e}_{ij\theta}, \widehat{I}^{\nu e}_{ij\phi}, \widehat{F}^{\nu e}_{ij\psi}$ denote the truth, indeterminacy, and falsity of alternative i with respect to criterion j in MVBNNs.

Step 4: ***Construct MVBNN weighted decision matrix.***

The calculation of the weighted MVBNN using the product rule of MVBNNs, such that, $\Delta^{\omega} = (\delta^{\omega}_{ij}) = \omega_j \otimes \delta_{ij} = \left(\widehat{T}^{\rho e^{\omega}}_{ij\theta}, \widehat{I}^{\rho e^{\omega}}_{ij\phi}, \widehat{F}^{\rho e^{\omega}}_{ij\psi}, \widehat{T}^{\nu e^{\omega}}_{ij\theta}, \widehat{I}^{\nu e^{\omega}}_{ij\phi}, \widehat{F}^{\nu e^{\omega}}_{ij\psi}\right)$

$$\omega_j \otimes \delta_{ij} = \begin{pmatrix} \bigcup_{\widehat{p}^{\rho e}_{j\theta}, \widehat{T}^{\rho e}_{ij\theta}} \left\{ \widehat{p}^{\rho e}_{j\theta} \cdot \widehat{T}^{\rho e}_{ij\theta} \right\} \\ \bigcup_{\widehat{q}^{\rho e}_{j\phi}, \widehat{I}^{\rho e}_{ij\phi}} \left\{ \widehat{q}^{\rho e}_{j\phi} + \widehat{I}^{\rho e}_{ij\phi} - \widehat{q}^{\rho e}_{j\phi} \cdot \widehat{I}^{\rho e}_{ij\phi} \right\} \\ \bigcup_{\widehat{r}^{\rho e}_{j\psi}, \widehat{F}^{\rho e}_{ij\psi}} \left\{ \widehat{r}^{\rho e}_{j\psi} + \widehat{F}^{\rho e}_{ij\psi} - \widehat{r}^{\rho e}_{j\psi} \cdot \widehat{F}^{\rho e}_{ij\psi} \right\} \\ \bigcup_{\widehat{p}^{\nu e}_{j\theta}, \widehat{T}^{\nu e}_{ij\theta}} \left\{ -(-\widehat{p}^{\nu e}_{j\theta} - \widehat{T}^{\nu e}_{ij\theta} - \widehat{p}^{\nu e}_{j\theta} \cdot \widehat{T}^{\nu e}_{ij\theta}) \right\} \\ \bigcup_{\widehat{q}^{\nu e}_{j\phi} \cdot \widehat{I}^{\nu e}_{ij\phi}} \left\{ -\widehat{q}^{\nu e}_{j\phi} \cdot \widehat{I}^{\nu e}_{ij\phi} \right\} \\ \bigcup_{\widehat{r}^{\nu e}_{j\psi}, \widehat{F}^{\nu e}_{ij\psi}} \left\{ -\widehat{r}^{\nu e}_{j\psi} \cdot \widehat{F}^{\nu e}_{ij\psi} \right\} \end{pmatrix}$$

Here, ω_j is the criterion weight of multi-valued bipolar neutrosophic numbers, which is denoted as $\omega_j = \left(\widehat{p}^{\rho e}_{ij\theta}, \widehat{q}^{\rho e}_{ij\phi}, \widehat{r}^{\rho e}_{ij\psi}, \widehat{p}^{\nu e}_{ij\theta}, \widehat{q}^{\nu e}_{ij\phi}, \widehat{r}^{\nu e}_{ij\psi} \right)$.

Step 5: ***Identify the positive and negative ideal solutions.***

The maximum (positive) neutrosophic ideal solution for each criterion is defined as,

$\delta^{*\omega}_{ij} = (\widehat{T}^{*\rho e^{\omega}}_{ij\theta}, \widehat{I}^{*\rho e^{\omega}}_{ij\phi}, \widehat{F}^{*\rho e^{\omega}}_{ij\psi}, \widehat{T}^{*\nu e^{\omega}}_{ij\theta}, \widehat{I}^{*\nu e^{\omega}}_{ij\phi}, \widehat{F}^{*\nu e^{\omega}}_{ij\psi})$

$\widehat{T}^{*\rho e^{\omega}}_{ij\theta} = \{\max(\widehat{T}^{\rho e}_{ij\theta})\}, \quad \widehat{I}^{*\rho e^{\omega}}_{ij\phi} = \{\min(\widehat{I}^{\rho e}_{ij\phi})\}, \quad \widehat{F}^{*\rho e^{\omega}}_{ij\psi} = \{\min(\widehat{F}^{\rho e}_{ij\psi})\},$

$\widehat{T}^{*\nu e^{\omega}}_{ij\theta} = \{\min(\widehat{T}^{\nu e}_{ij\theta})\}, \quad \widehat{I}^{*\nu e^{\omega}}_{ij\phi} = \{\max(\widehat{I}^{\nu e}_{ij\phi})\}, \quad \widehat{F}^{*\nu e^{\omega}}_{ij\psi} = \{\max(\widehat{F}^{\nu e}_{ij\psi})\}.$

The minimum (negative) neutrosophic ideal solution for each criterion defined as,

$\delta^{\omega}_{*ij} = (\widehat{T}^{\rho e^{\omega}}_{*ij\theta}, \widehat{I}^{\rho e^{\omega}}_{*ij\phi}, \widehat{F}^{\rho e^{\omega}}_{*ij\psi}, \widehat{T}^{\nu e^{\omega}}_{*ij\theta}, \widehat{I}^{\nu e^{\omega}}_{*ij\phi}, \widehat{F}^{\nu e^{\omega}}_{*ij\psi})$

$\widehat{T}^{\rho e^{\omega}}_{*ij\theta} = \{\min(\widehat{T}^{\rho e}_{ij\theta})\}, \quad \widehat{I}^{\rho e^{\omega}}_{*ij\phi} = \{\max(\widehat{I}^{\rho e}_{ij\phi})\}, \quad \widehat{F}^{\rho e^{\omega}}_{*ij\psi} = \{\max(\widehat{F}^{\rho e}_{ij\psi})\},$

$\widehat{T}^{\nu e^{\omega}}_{*ij\theta} = \{\max(\widehat{T}^{\nu e}_{ij\theta})\}, \quad \widehat{I}^{\nu e^{\omega}}_{*ij\phi} = \{\min(\widehat{I}^{\nu e}_{ij\phi})\}, \quad \widehat{F}^{\nu e^{\omega}}_{*ij\psi} = \{\min(\widehat{F}^{\nu e}_{ij\psi})\}.$

Where $j = 1, 2, ..., n$ and θ, ϕ, ψ are truth, indeterminacy, and false membership degrees with multiple values.

Step 6: ***Compute separation measures.***

Calculate the multi-valued bipolar neutrosophic separation measure from the maximum and minimum ideal solutions.

Definition 4.2. ***Separation measure of MVBNNs.***
The separation measure of two MVBNNs $\widehat{S}_{m1} = (\widehat{m}^{\rho e}_{1\theta}, \widehat{m}^{\rho e}_{1\phi}, \widehat{m}^{\rho e}_{1\psi}, \widehat{m}^{\nu e}_{1\theta}, \widehat{m}^{\nu e}_{1\phi}, \widehat{m}^{\nu e}_{1\psi})$ *and* $\widehat{S}_{m2} = (\widehat{m}^{\rho e}_{2\theta}, \widehat{m}^{\rho e}_{2\phi}, \widehat{m}^{\rho e}_{2\psi}, \widehat{m}^{\nu e}_{2\theta}, \widehat{m}^{\nu e}_{2\phi}, \widehat{m}^{\nu e}_{2\psi})$ *defined as follows:*

$$\Delta(\widehat{S}_{m1}, \widehat{S}_{m2}) = \bigcup_{\theta,\phi,\psi} \left(|\widehat{m}^{\rho e}_{1\theta} - \widehat{m}^{\rho e}_{2\theta}| + |\widehat{m}^{\rho e}_{1\phi} - \widehat{m}^{\rho e}_{2\phi}| + |\widehat{m}^{\rho e}_{1\psi} - \widehat{m}^{\rho e}_{1\psi}| + |\widehat{m}^{\nu e}_{1\theta} - \widehat{m}^{\nu e}_{2\theta}| + |\widehat{m}^{\nu e}_{1\phi} - \widehat{m}^{\nu e}_{2\phi}| + |\widehat{m}^{\nu e}_{1\psi} - \widehat{m}^{\nu e}_{2\psi}| \right)$$

The separation measure from the positive ideal μ^*:

$$\Delta^*(\delta_{ij}^{\omega}, \delta_{ij}^{*\omega}) = \bigcup_{\theta,\phi,\psi} \Big(|\widehat{m}_{1\theta}^{\rho e} - \widehat{m}_{2\theta}^{*\rho e}| + |\widehat{m}_{1\phi}^{\rho e} - \widehat{m}_{2\phi}^{*\rho e}| + |\widehat{m}_{1\psi}^{\rho e} - \widehat{m}_{2\psi}^{*\rho e}| + |\widehat{m}_{1\theta}^{\nu e} - \widehat{m}_{2\theta}^{*\nu e}| + |\widehat{m}_{1\phi}^{\nu e} - \widehat{m}_{2\phi}^{*\nu e}| + |\widehat{m}_{1\psi}^{\nu e} - \widehat{m}_{2\psi}^{*\nu e}| \Big) \quad (4.2)$$

$$\mu^* = \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta^*\Big(\delta_{ij}^{\omega}, \delta_{ij}^{*\omega}\Big), \quad (4.3)$$

The separation measure from the negative ideal solution:

$$\Delta_*(\delta_{ij}^{\omega}, \delta_{*ij}^{\omega}) = \bigcup_{\theta,\phi,\psi} \Big(|\widehat{m}_{1\theta}^{\rho e} - \widehat{m}_{*2\theta}^{\rho e}| + |\widehat{m}_{1\phi}^{\rho e} - \widehat{m}_{*2\phi}^{\rho e}| + |\widehat{m}_{1\psi}^{\rho e} - \widehat{m}_{*2\psi}^{\rho e}| + |\widehat{m}_{1\theta}^{\nu e} - \widehat{m}_{*2\theta}^{\nu e}| + |\widehat{m}_{1\phi}^{\nu e} - \widehat{m}_{*2\phi}^{\nu e}| + |\widehat{m}_{1\psi}^{\nu e} - \widehat{m}_{*2\psi}^{\nu e}| \Big) \quad (4.4)$$

$$\mu_* = \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_*\Big(\delta_{ij}^{\omega}, \delta_{*ij}^{\omega}\Big), \quad (4.5)$$

Step 7. ***Ranking coefficient***

$$R_K = \frac{\mu_*}{\mu^* + \mu_*}. \quad (4.6)$$

The alternatives are ranked in descending order R.

5. Linguistic Variable for MVBNNs

This section introduces the linguistic variable framework for multi-valued bipolar neutrosophic numbers. Each linguistic variable is represented by a 6-tuple of membership values $\langle \{\widehat{T}^{\rho e}\}, \{\widehat{I}^{\rho e}\}, \{\widehat{F}^{\rho e}\}, \{\widehat{T}^{\nu e}\}, \{\widehat{I}^{\nu e}\}, \{\widehat{F}^{\nu e}\} \rangle$. The framework uses nine linguistic variables based on the performance level, ranging from "Extremely Low (EL)" representing the lowest level at $\langle \{0.05, 0.10\}, \{0.85, 0.90\}, \{0.90, 0.95\} \rangle$ for positive components and $\langle \{-0.95, -0.90\}, \{-0.90, -0.85\}, \{-0.10, -0.05\} \rangle$ for negative components to "Extremely High (EH)" representing the highest level at$\langle \{0.88, 0.95\}, \{0.05, 0.10\}, \{0.05, 0.10\} \rangle$ for positive components and $\langle \{-0.12, -0.05\}, \{-0.10, -0.05\}, \{-0.95, -0.90\} \rangle$ for negative components. The multi-valued bipolar neutrosophic number is taken as interval values. The membership values follow the structure that $\widehat{T}^{\rho e}$ increases from $\{0.05, 0.10\}$ to $\{0.88, 0.95\}$ and $\widehat{I}^{\rho e}$ and $\widehat{F}^{\rho e}$ decreases from $\{0.85, 0.90\}$ to $\{0.05, 0.10\}$ and $\{0.90, 0.95\}$ to $\{0.05, 0.10\}$ respectively. Similarly, in negative membership components. The multi-valued bipolar neutrosophic set extension of the multi-valued neutrosophic set.

Table 1. Linguistic variables of MVBNNs.

Linguistic variables	Positive Part $\langle\{\widehat{T}^{pe}\}, \{\widehat{I}^{pe}\}, \{\widehat{F}^{pe}\}\rangle$	Negative Part $\langle\{\widehat{T}^{ve}\}, \{\widehat{I}^{ve}\}, \{\widehat{F}^{ve}\}\rangle$
Extremely Low (EL)	$\langle\{0.05, 0.10\}, \{0.85, 0.90\}, \{0.90, 0.95\}\rangle$	$\langle\{-0.95, -0.90\}, \{-0.90, -0.85\}, \{-0.10, -0.05\}\rangle$
Very Low (VL)	$\langle\{0.15, 0.20\}, \{0.75, 0.80\}, \{0.80, 0.85\}\rangle$	$\langle\{-0.85, -0.80\}, \{-0.80, -0.75\}, \{-0.20, -0.15\}\rangle$
Low (L)	$\langle\{0.25, 0.30\}, \{0.65, 0.70\}, \{0.70, 0.75\}\rangle$	$\langle\{-0.75, -0.70, \}, \{-0.70, -0.65\}, \{-0.30, -0.25\}\rangle$
Moderately Low (ML)	$\langle\{0.35, 0.40\}, \{0.55, 0.60\}, \{0.60, 0.65\}\rangle$	$\langle\{-0.65, -0.60\}, \{-0.60, -0.55\}, \{-0.40, -0.35\}\rangle$
Medium (M)	$\langle\{0.45, 0.50\}, \{0.45, 0.50\}, \{0.50, 0.55\}\rangle$	$\langle\{-0.55, -0.50\}, \{-0.50, -0.45\}, \{-0.50, -0.45\}\rangle$
Moderately High (MH)	$\langle\{0.55, 0.60\}, \{0.35, 0.40\}, \{0.40, 0.45\}\rangle$	$\langle\{-0.45, -0.40\}, \{-0.40, -0.35\}, \{-0.60, -0.55\}\rangle$
High (H)	$\langle\{0.65, 0.70\}, \{0.25, 0.30\}, \{0.30, 0.35\}\rangle$	$\langle\{-0.35, -0.30\}, \{-0.30, -0.25\}, \{-0.70, -0.65\}\rangle$
Very High (VH)	$\langle\{0.75, 0.80\}, \{0.15, 0.20\}, \{0.20, 0.25\}\rangle$	$\langle\{-0.25, -0.20\}, \{-0.20, -0.15\}, \{-0.80, -0.75\}\rangle$
Extremely High (MH)	$\langle\{0.88, 0.95\}, \{0.05, 0.10\}, \{0.05, 0.10\}\rangle$	$\langle\{-0.12, -0.05\}, \{-0.10, -0.05\}, \{-0.95, -0.90\}\rangle$

6. Mathematical Demonstration

There is a group of professors who want to buy an apartment in a particular area. The problem is to select the most suitable apartment in a particular area from a set of options. There are three experts (E_1, E_2, E_3) to make the decision for the best choice in four alternatives A_1, A_2, A_3, A_4 based on the criteria: the cost(C_1), the location(C_2), the safety(C_3), and the amenities (C_4). The experts are using linguistic term in their evaluation of alternatives, which are then converted to MVBNNs. These criteria will change depending on the problem. Each criterion is evaluated by the experts for all alternatives.

Step 1: Each expert provides an assessment of the alternative, and the weight of each criterion represented in terms of linguistic variables is shown in Table 2. Based on Table 1, these evaluations are systematically converted into numerical values using MVBNNs.

Table 2. Decision matrix with linguistic variables.

Alternatives	Evaluators	Cost C_1	Location C_2	Safety C_3	Amenities C_4
A_1	E_1	MH	H	MH	H
	E_2	L	ML	M	VH
	E_3	MH	VH	H	H
A_2	E_1	H	VH	VH	MH
	E_2	ML	H	VL	H
	E_3	M	EH	EH	H
A_3	E_1	H	M	VH	EH
	E_2	VH	ML	H	VH
	E_3	H	VH	EH	EH
A_4	E_1	EL	ML	ML	L
	E_2	L	L	L	ML
	E_3	ML	H	M	MH
Weights	E_1	MH	H	EH	H
	E_2	H	MH	H	M
	E_3	EH	VH	H	EL

Step 2: The weights for the three experts are based on the experience and knowledge of the decision makers, represented as E_1 = VH, E_2 = H, and E_3 = EH. Calculate the weights of each decision maker using Equation (4.1).

$$\omega_1 = \frac{1 - \frac{\sqrt{\frac{(1-0.75)^2+(0.15)^2+(0.20)^2+(-0.25)^2+(-0.20)^2+(1-(-(-0.80)))^2}{6}} + \sqrt{\frac{(1-0.80)^2+(0.20)^2+(0.25)^2+(-0.20)^2+(-0.15)^2+(1-(-(-0.75)))^2}{6}}}{2}}{3-(0.211+0.084+0.310)}$$

$$= 0.329$$

Similarly, we get the weights of each decision maker

$\omega_1 = 0.329$

$\omega_2 = 0.288$

$\omega_3 = 0.383$

Step 3: To construct the decision matrix, convert the linguistic variable into MVBNNs as shown in Table 3.

$$\begin{aligned}
A_1C_1 = \langle \{ & 1-((1-0.55)^{0.329}\times(1-0.25)^{0.288}\times(1-0.55)^{0.383}), 1-((1-0.60)^{0.329}\times(1-0.30)^{0.288}\times(1-0.60)^{0.383})\}, \\
& \{(0.35)^{0.329}\times(0.65)^{0.288}\times(0.35)^{0.383},\ (0.40)^{0.329}\times(0.70)^{0.288}\times(0.40)^{0.383}\}, \\
& \{(0.40)^{0.329}\times(0.70)^{0.288}\times(0.40)^{0.383},\ (0.45)^{0.329}\times(0.75)^{0.288}\times(0.45)^{0.383}\}, \\
& \{-((0.45)^{0.329}\times(0.75)^{0.288}\times(0.45)^{0.383}),\ -((0.40)^{0.329}\times(0.70)^{0.288}\times(0.40)^{0.383})\}, \\
& \{-((1-0.40)^{0.329}\times(1-0.70)^{0.288}\times(1-0.40)^{0.383}),\ -((1-0.35)^{0.329}\times(1-0.65)^{0.288}\times(1-0.35)^{0.383})\}, \\
& \{-((1-0.60)^{0.329}\times(1-0.30)^{0.288}\times(1-0.60)^{0.383}),\ -((1-0.55)^{0.329}\times(1-0.25)^{0.288}\times(1-0.55)^{0.383})\}\Big\}\rangle \\
= & \langle\{0.479, 0.530\},\ \{0.418, 0.470\},\ \{0.470, 0.521\},\ \{-0.521, -0.470\},\ \{-0.509, -0.456\},\ \{-0.530, -0.479\}\rangle.
\end{aligned}$$

Step 4: The weighted decision matrix Table 4 was calculated by the criterion weights of the decision matrix from the values of Table 3. For example, the weight of A_1C_1 is obtained as

$$\begin{aligned}
A_1C_1 = \langle\Big\{ & \{(0.737\times0.675), (0.737\times0.734)\}, \\
& \{(0.151+0.418-0.151\times0.418), (0.217+0.470-0.217\times0.470)\}, \\
& \{(0.166+0.470-0.166\times0.470), (0.235+0.521-0.235\times0.521)\}, \\
& \{-(0.252+0.521-0.252\times0.521), -(0.166+0.470-0.166\times0.470)\}, \\
& \{-(0.267\times0.509), -(0.217\times0.456)\}, \\
& \{-(0.834\times0.530), -(0.765\times0.479)\}\Big\}\rangle \\
= \langle\Big\{ & \{0.358, 0.442\},\ \{0.506, 0.585\},\ \{0.558, 0.634\},\ \{-0.642, -0.558\},\ \{-0.136, -0.099\},\ \{-0.442, -0.366\}\Big\}\rangle.
\end{aligned}$$

Step 5: This step is to detect the most favorable and least favorable ideal solutions from the Table 4. These results are in Table 5.

TABLE 3. MVBNN decision matrix δ_{ij}.

	C_1
A_1	$\langle\{0.479, 0.530\}, \{0.418, 0.470\}, \{0.470, 0.521\}, \{-0.521, -0.470\}, \{-0.509, -0.456\}, \{-0.530, -0.479\}\rangle$
A_2	$\langle\{0.503, 0.555\}, \{0.393, 0.445\}, \{0.445, 0.497\}, \{-0.497, -0.445\}, \{-0.476, -0.425\}, \{-0.555, -0.503\}\rangle$
A_3	$\langle\{0.682, 0.733\}, \{0.216, 0.267\}, \{0.267, 0.318\}, \{-0.318, -0.267\}, \{-0.273, -0.222\}, \{-0.733, -0.682\}\rangle$
A_4	$\langle\{0.233, 0.283\}, \{0.666, 0.717\}, \{0.717, 0.767\}, \{-0.767, -0.717\}, \{-0.767, -0.708\}, \{-0.283, -0.233\}\rangle$
$\omega_1 C_1$	$\langle\{0.748, 0.834\}, \{0.151, 0.217\}, \{0.166, 0.235\}, \{-0.252, -0.166\}, \{-0.267, -0.217\}, \{-0.834, -0.765\}\rangle$
	C2
A_1	$\langle\{0.632, 0.686\}, \{0.258, 0.314\}, \{0.314, 0.368\}, \{-0.368, -0.314\}, \{-0.373, -0.321\}, \{-0.686, -0.632\}\rangle$
A_2	$\langle\{0.792, 0.868\}, \{0.114, 0.172\}, \{0.132, 0.194\}, \{-0.208, -0.132\}, \{-0.195, -0.144\}, \{-0.868, -0.806\}\rangle$
A_3	$\langle\{0.573, 0.629\}, \{0.313, 0.371\}, \{0.371, 0.427\}, \{-0.427, -0.371\}, \{-0.439, -0.387\}, \{-0.629, -0.573\}\rangle$
A_4	$\langle\{0.466, 0.519\}, \{0.427, 0.481\}, \{0.481, 0.534\}, \{-0.534, -0.481\}, \{-0.544, -0.491\}, \{-0.519, -0.466\}\rangle$
$\omega_2 C_2$	$\langle\{0.669, 0.721\}, \{0.226, 0.279\}, \{0.279, 0.331\}\{-0.331, -0.279\}\{-0.295, -0.245\}\{-0.721, -0.669\}\rangle$
	C3
A_1	$\langle\{0.567, 0.618\}, \{0.331, 0.382\}, \{0.382, 0.433\}, \{-0.433, -0.382\}, \{-0.396, -0.346\}, \{-0.618, -0.567\}\rangle$
A_2	$\langle\{0.732, 0.825\}, \{0.157, 0.229\}, \{0.175, 0.250\}, \{-0.268, -0.175\}, \{-0.439, -0.376\}, \{-0.825, -0.750\}\rangle$
A_3	$\langle\{0.792, 0.868\}, \{0.114, 0.172\}, \{0.132, 0.194\}, \{-0.208, -0.132\}, \{[-0.195, -0.144\}, \{-0.868, -0.806\}\rangle$
A_4	$\langle\{0.365, 0.415\}, \{0.534, 0.585\}, \{0.585, 0.635\}, \{-0.635, -0.585\}, \{-0.599, -0.548\}, \{-0.415, -0.365\}\rangle$
$\omega_3 C_3$	$\langle\{0.754, 0.834\}, \{0.147, 0.209\}, \{0.166, 0.232\}, \{-0.246, -0.166\}, \{-0.240, -0.189\}, \{-0.834, -0.768\}\rangle$
	C4
A_1	$\langle\{0.682, 0.733\}, \{0.216, 0.267\}, \{0.267, 0.318\}, \{-0.318, -0.267\}, \{-0.273, -0.222\}, \{-0.733, -0.682\}\rangle$
A_2	$\langle\{0.620, 0.670\}, \{0.279, 0.330\}, \{0.330, 0.380\}, \{-0.380, -0.330\}, \{-0.335, -0.284\}, \{-0.670, -0.620\}\rangle$
A_3	$\langle\{0.852, 0.925\}, \{0.069, 0.122\}, \{0.075, 0.130\}, \{-0.148, -0.075\}, \{-0.130, -0.080\}, \{-0.925, -0.870\}\rangle$
A_4	$\langle\{0.408, 0.460\}, \{0.489, 0.540\}, \{0.540, 0.592\}, \{-0.592, -0.540\}, \{-0.575, -0.523\}, \{-0.460, -0.408\}\rangle$
$\omega_4 C_4$	$\langle\{0.416, 0.470\}, \{0.473, 0.529\}, \{0.529, 0.584\}, \{-0.584, -0.529\}, \{-0.698, -0.630\}, \{-0.471, -0.416\}\rangle$

TABLE 4. Multi-valued bipolar neutrosophic weighted decision matrix (δ_{ij}^{ω}).

	C_1
A_1	$\langle\{0.358, 0.442\}, \{0.506, 0.585\}, \{0.558, 0.634\}, \{-0.642, -0.558\}, \{-0.136, -0.099\}, \{-0.442, -0.366\}\rangle$
A_2	$\langle\{0.376, 0.462\}, \{0.484, 0.566\}, \{0.538, 0.616\}, \{-0.624, -0.538\}, \{-0.127, -0.092\}, \{-0.462, -0.384\}\rangle$
A_3	$\langle\{0.5100.611\}, \{0.334, 0.426\}, \{0.389, 0.478\}, \{-0.490, -0.389\}, \{-0.073, -0.048\}, \{-0.611, -0.521\}\rangle$
A_4	$\langle\{0.174, 0.236\}, \{0.716, 0.778\}, \{0.764, 0.822\}, \{-0.826, -0.764\}, \{-0.205, -0.154\}, \{-0.236, -0.178\}\rangle$
	C_2
A_1	$\langle\{0.423, 0.495\}, \{0.426, 0.505\}, \{0.505, 0.577\}, \{-0.577, -0.505\}, \{-0.110, -0.079\}, \{-0.495, -0.423\}\rangle$
A_2	$\langle\{0.530, 0.626\}, \{0.315, 0.403\}, \{0.374, 0.461\}, \{-0.470, -0.374\}, \{-0.057, -0.035\}, \{-0.626, -0.539\}\rangle$
A_3	$\langle\{0.384, 0.453\}, \{0.469, 0.547\}, \{0.547, 0.616\}, \{-0.616, -0.547\}, \{-0.130, -0.095\}, \{-0.453 - 0.384\}\rangle$
A_4	$\langle\{0.312, 0.374\}, \{0.557, 0.626\}, \{0.626, 0.688\}, \{-0.688, -0.626\}, \{-0.161, -0.120\}, \{-0.374 - 0.312\}\rangle$
	C_3
A_1	$\langle\{0.427, 0.515\}, \{0.429, 0.511\}, \{0.485, 0.564\}, \{-0.573, -0.485\}, \{-0.095, -0.065\}, \{-0.515, -0.436]\}\rangle$
A_2	$\langle\{0.551, 0.687\}, \{0.281, 0.390\}, \{0.313, 0.424\}, \{-0.449, -0.313\}, \{-0.105, -0.071\}, \{-0.687, -0.576\}\rangle$
A_3	$\langle\{0.597, 0.723\}, \{0.245, 0.345\}, \{0.277, 0.381\}, \{-0.403, -0.277\}, \{-0.047, -0.027\}, \{-0.723, -0.619\}\rangle$
A_4	$\langle\{0.275, 0.346\}, \{0.603, 0.672\}, \{0.654, 0.720\}, \{-0.725, -0.654\}, \{-0.144, -0.104\}, \{-0.346, -0.280\}\rangle$
	C_4
A_1	$\langle\{0.284, 0.345\}, \{0.587, 0.655\}, \{0.655, 0.716\}, \{-0.716, -0.655\}, \{-0.190, -0.140\}, \{-0.345, -0.284\}\rangle$
A_2	$\langle\{0.258, 0.315\}, \{0.620, 0.685\}, \{0.685, 0.742\}, \{-0.742, -0.685\}, \{-0.234, -0.179\}, \{-0.315, -0.258\}\rangle$
A_3	$\langle\{0.354, 0.436\}, \{0.509, 0.587\}, \{0.564, 0.638\}, \{-0.646, -0.564\}, \{-0.091, -0.050\}, \{-0.436, -0.362\}\rangle$
A_4	$\langle\{0.170, 0.216\}, \{0.731, 0.784\}, \{0.784, 0.830\}, \{-0.830, -0.784\}, \{-0.402, -0.330\}, \{-0.216, -0.170\}\rangle$

TABLE 5. Multi-valued bipolar maximum (Δ^*) and minimum (Δ_*) ideal solutions.

	δ^*_{ij}
C_1	$\langle\{0.510, 0.611\}, \{0.334, 0.426\}, \{0.389, 0.478\}, \{-0.826, -0.764\}, \{-0.073, -0.048\}, \{-0.236, -0.178\}\}\rangle$
C_2	$\langle\{0.530, 0.626\}, \{0.315, 0.403\}, \{0.374, 0.461\}, \{-0.688, -0.626\}, \{-0.0574, -0.035\}, \{-0.374, -0.312\}\rangle$
C_3	$\langle\{0.597, 0.723\}, \{0.245, 0.345\}, \{0.277, 0.381\}, \{-0.725, -0.654\}, \{-0.047, -0.027\}, \{-0.346, -0.280\}\rangle$
C_4	$\langle\{0.354, 0.436\}, \{0.509, 0.587\}, \{0.564, 0.638\}, \{-0.830, -0.784\}, \{-0.091, -0.050\}, \{-0.216, -0.170\}\rangle$
	δ_{*ij}
C_1	$\langle\{0.174, 0.236\}, \{0.716, 0.778\}, \{0.764, 0.822\}, \{-0.490, -0.389\}, \{-0.205, -0.154\}, \{-0.611, -0.522\}\rangle$
C_2	$\langle\{0.312, 0.374\}, \{0.557, 0.626\}, \{0.626, 0.688\}, \{-0.470, -0.374\}, \{-0.161, -0.120\}, \{-0.626, -0.539\}\rangle$
C_3	$\langle\{(\{0.275, 0.346\}, \{0.603, 0.672\}, \{0.654, 0.720\}, \{-0.403, -0.277\}, \{-0.144, -0.104\}, \{-0.723, -0.619\}\}\rangle$
C_4	$\langle\{\{0.170, 0.216\}, \{0.731, 0.784\}, \{0.784, 0.830\}, \{-0.646, -0.564\}, \{-0.402, -0.329\}, \{-0.436, -0.362\}\rangle$

Step 6: In this step, calculate the values using Equation (4.2), for instance, the calculation is obtained as

$$\begin{aligned}\Delta^*(A_1C_1, \delta^*_{ij}) =& |0.358-0.510| + |0.442-0.611| + |0.506-0.334| + |0.585-0.426| + \\ & |0.558-0.389| + |0.634-0.478| + |-0.642-(-0.826)| + \\ & |-0.558-(-0.764)| + |-0.136-(-0.073)| + |-0.099-(-0.048)| + \\ & |-0.442-(-0.236)| + |-0.366-(-0.178)| \\ =& 1.875\end{aligned}$$

Similarly, find the separation value A_1 under each criterion.
$\Delta^*(A_1C_2, \delta^*_{ij}) = 1.259$.
$\Delta^*(A_1C_3, \delta^*_{ij}) = 1.853$.
$\Delta^*(A_1C_4, \delta^*_{ij}) = 1.150$.
The separation from the maximum ideal solution is calculated by Equation (4.3).
$\mu^*(A_1) = 1.875 + 1.259 + 1.853 + 1.150 = 6.136$

The same procedure for other alternatives. The separation measure from the minimum ideal solution is calculated by a similar process using equation (4.5). It was shown in Table 6.

TABLE 6. Maximum and Minimum Separation Measure.

Alternative	μ^*	μ_*
A_1	6.137	6.575
A_2	5.663	7.049
A_3	5.039	7.673
A_4	8.101	4.611

Step 7: Determine the rank of each alternative based on their ranking coefficient measure by Equation (4.6) to get final rank, represented in Table 7, $R_K(A_1) = \frac{6.575}{6.137+6.575} = 0.517$.

Table 7. Ranking coefficient measure.

Alternative	R_K	*Order*
A_1	0.517	*III*
A_2	0.555	*II*
A_3	0.604	*I*
A_4	0.363	*IV*

According to the Table 7, we can rank the alternatives in descending order as $A_3 > A_2 > A_1 > A_4$. Therefore, according to the ranking order, A_3 is the best option among these three, and A_4 is the worst option. The MCDM problem is solved by the simplified bipolar neutrosophic TOPSIS method. The result is graphically represented in Figure 1.

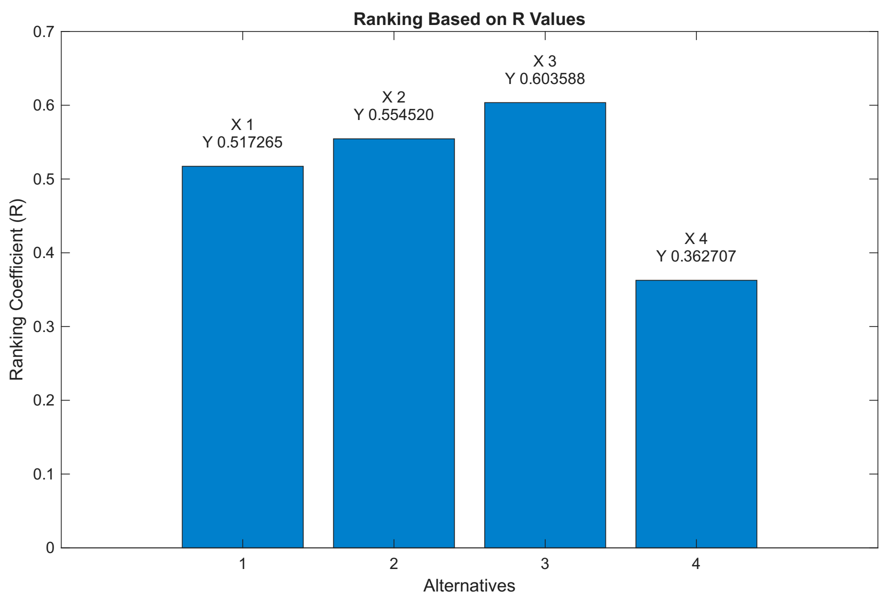

Figure 1. The bar graph representation of Table 7

.

7. Analysis and Discussion

For comparison, consider the previous study by Jeni Seles Martina and Deepa (2022); they proposed the neutrosophic simplified TOPSIS method for MVNNs and applied this to the best apartment problem. In this paper, we do the extension of this study and generalize the neutrosophic simplified TOPSIS method for MVNNs, which handles both the positive and negative membership degrees of alternatives along with multi-valued uncertainty, where here the opinions are not only uncertain but also have positive and negative tendencies.

The best apartment problem analyzed in section 6 is that we get the ranking coefficients of the corresponding alternatives shown in table 7, and based on that table, we rank the alternatives as $A_3 > A_2 > A_1 > A_4$. The same apartment problem was analyzed; based on the previous study, the ranking of the corresponding alternatives is given in Table 8 $A_3 > A_2 > A_1 > A_4$. The ranking order of the alternatives is identical in both approaches, confirming the stability and consistency of the decision outcomes across neutrosophic environments. However, compared to the MVNN model, the ranking coefficient is measured differently from the MVBN-TOPSIS model. This difference arises becauses the proposed study incorporates the range of uncertainty through multi-values, and score values exhibit the analytical sensitivity of the bipolar model.

Moreover, In both cases, the simplified TOPSIS method work faster and maintains robustness. The integration of the multi-valued bipolar framework enhances the clarity of results, allowing decision-makers to capture both the degree of support and opposition towards each alternative in a more realistic manner.

Table 8. Ranking coefficient measure.

Alternative	R_K	*Order*
A_1	0.533	*III*
A_2	0.763	*II*
A_3	0.867	*I*
A_4	0.000	*IV*

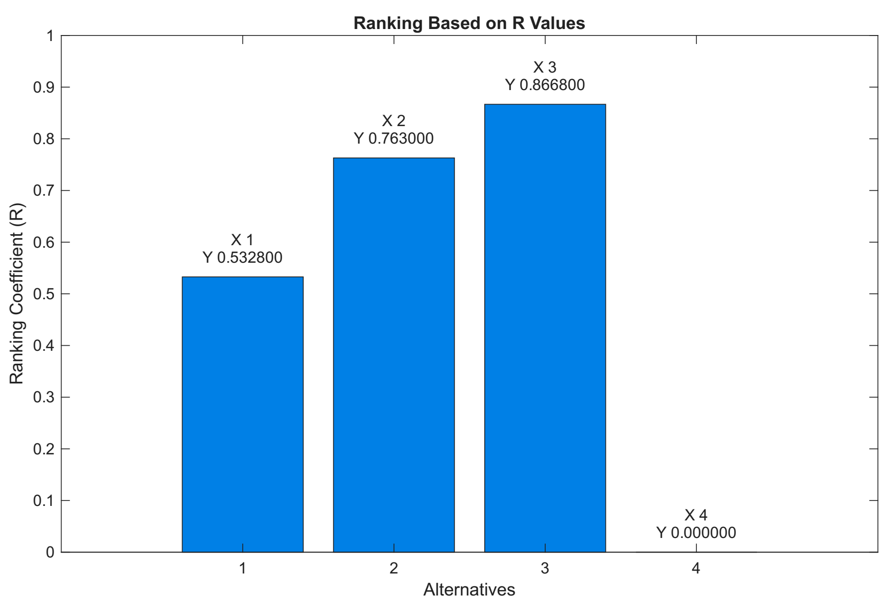

Figure 2. The bar graph representation of Table 8

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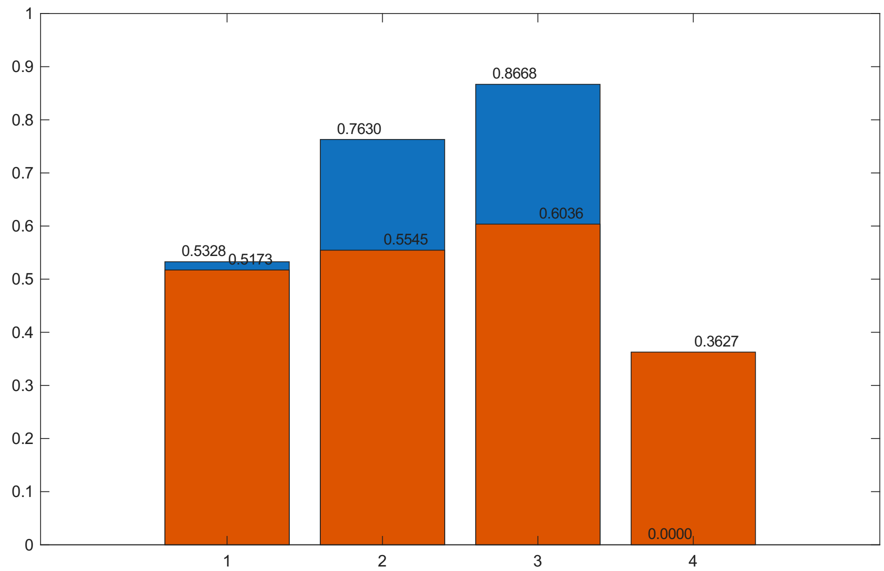

FIGURE 3. The comparison of figures 1 and 2

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8. Conclusion

This study extends the concept of multi-valued neutrosophic matrices into a multi-valued bipolar neutrosophic framework. It applies this to the simplified TOPSIS method to address multi-criteria decision-making problems. In this paper, multi-valued bipolar neutrosophic matrices and their basic definitions and fundamental concepts were introduced, and several properties associated with these operations were established. A numerical example was also presented to demonstrate the accuracy and reliability of the proposed method. The numerical calculation and tables are obtained using MATLAB. Moreover, the application of linguistic variables within the neutrosophic simplified TOPSIS framework successfully identified the best alternative among the given options, and the obtained result was compared with the existing MVN-simplified TOPSIS method, giving a consistent ranking, but the bipolar approach captures the variation in the evaluation values, making it a more powerful tool for multi-criteria decision-making under complex and uncertain environments. For future work, applying the concept of MVBNMs can be developed based on the decision-making to solve real applications, apply this in various fields, and further expand the concept.

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