

Generalized n -th Kind Extended q -Difference Operator: Unified Theory and Applications

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Abstract. This work defines the extended q -difference operator of n -th kind $\Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_n)}$ and presents the discrete version of the Leibniz theorem according to $\Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_n)}$, for q and ℓ_i^s are positive reals. Derived some interesting results on the relation between the extended q -polynomial factorial of the first and n -th kind. The reciprocal of the extended q -difference operator of n -th kind, $\Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_n)}^{-1}$, is also described. The derivation of the formula for the sum of higher powers of arithmetic progressions is extended through numerical methods, supported by illustrative examples that emphasize the principal results.

1. INTRODUCTION

The study of linear q -difference equations started with the initial pioneering works of Adams and Carmichael. In [1], a systematic treatment of n -th order q -difference equations, introducing methods for constructing formal and convergent power-series solutions. The author made a distinction between regular and irregular cases depending on the type of characteristic roots and used Newton-polygon methods to study convergence and asymptotic behavior.

The analytical framework for q -difference equations was further strengthened by [7], who laid the groundwork for subsequent developments in both the algebraic and analytic theory of q -difference equations. Building on this foundation, researchers such as Van der Put and Singer (1997) developed the Galois theory for difference equations [26, 27]. Later, Ramis, Sauloy, and Zhang contributed to advancing the local analytic classification and summation theory for q -difference equations. Together, these studies helped establish a theoretical framework that links classical formal methods with modern algebraic and analytic approaches [23].

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Bangerezako's series of works provides a seminal framework for q -variational calculus and its relation to the Askey–Wilson operator [3–5]. In [12], the author developed a q -analogue of classical variational calculus on q -linear lattices, deriving q -Euler Lagrange equations, integration by parts formulas, and applications to isoperimetric and optimal-control problems. This framework was later extended [6] to q -nonuniform lattices, where the divided-difference operator and integral were adapted to the Askey–Wilson lattice, establishing a closer connection between variational formulations and orthogonal polynomial structures. In an earlier study [4], Bangerezako provided a factorization approach for the Askey–Wilson second order q -difference operator by developing raising and lowering operators and generating generalized eigenfunctions. Together, these studies reveal strong links between q -variational principles and spectral theory, suggesting that factorization techniques can be used to analyze q -Euler Lagrange systems and develop integrable models on q -lattices.

In approximation theory, the role of q -calculus has gained significant importance over the past three decades. The first q -analogue of the classical Bernstein polynomials was introduced by Lupaş (1987), followed by Phillips (1997), who developed another q -version of these polynomials. Since then, many researchers have proposed various q -extensions of exponential-type operators, such as the Baskakov, Szász–Mirakyan, Meyer–Konig–Zeller, Bleimann–Butzer–Hahn, Picard, and Weierstrass operators. Additionally, q -analogues of Kantorovich and Durrmeyer-type integral operators have been formulated, and their approximation properties have been extensively studied [3].

The extended q -difference operators of the n -th kind serve as a generalized form of classical q -difference operators. Incorporating higher-order terms and parameter-dependent modifications enables a more flexible and powerful framework for analyzing discrete systems. These operators are particularly beneficial in managing complicated functional relationships, such as those found in recursive sequences and multi-level transformations. Their structure renders them valuable tools in the study of special functions, orthogonal polynomials, and quantum calculus, where conventional operators may fall short [15, 16].

Several studies have emphasized the increasing relevance of q -calculus in both mathematics and physics, particularly in areas such as quantum computing, non-commutative probability, combinatorics, functional analysis, and approximation theory. In recent years, q -Hermite polynomials have attracted considerable attention due to their wide-ranging applications in non-commutative probability, quantum mechanics, and combinatorial analysis. Likewise, the classical three-variable Hermite polynomials are widely used in charged-beam transport problems in mechanics and in quantum phase-space calculations. In this context, umbral techniques have proven powerful for deriving and analyzing the properties of these polynomials [14].

Furthermore, recent research introduced three q -analogs of the generalized Bernoulli polynomials, whose generating functions involve Jackson q -Bessel functions. Researchers have also

investigated their key properties, including the development of a q -Fourier expansion for generalized Bernoulli numbers associated with the first and second Jackson q -Bessel functions. Their asymptotic behavior for large degrees was analyzed, and connection coefficients with certain q -orthogonal polynomials established [13].

During the formulation of the q -derivative operator, Δ_q , defined as

$$\Delta_q u(k) = \frac{u(kq) - u(k)}{(q-1)k}, \quad q \in (0, \infty), \quad (1.1)$$

researchers did not arrive at any striking conclusion in numerical analysis. Later, Chandrasekar and Suresh [10] expanded Δ_q by defining

$$\Delta_{q(\ell)} u(k) = \frac{u(kq) - \ell u(k)}{(q-\ell)k}, \quad (1.2)$$

where $\ell \in (0, \infty)$ and $u(k)$ is a real-valued function, establishing several arithmetic–geometric progressions.

Subsequently, in [8, 9], the first-kind extended q -difference operator was introduced as

$$\Delta_{q(\ell)} u(k) = u((k+\ell)q) - u(k), \quad k \in [0, \infty), \ell \in (0, \infty), \quad (1.3)$$

which generalized the basic q -difference operator. Recently, Kathiravan and Khadar Babu SK [19] extended the operator further to the second kind, defined by

$$\Delta_{q(\ell_1, \ell_2)} u(k) = u(((k+\ell_1)q + \ell_2)q) - u(k), \quad k \in [0, \infty), \ell \in (0, \infty), \quad (1.4)$$

and derived formulas for the sum of real numbers in arithmetic progressions within numerical methods.

In this study, we introduce an extended q -difference operator and employ its reciprocal form to derive a formula for fractional series within the framework of numerical analysis. The primary objective of this work is to establish significant results concerning the summation of fractional series involving the n -th kind extended q -difference operator, denoted by $\Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_n)}$.

The paper is organized as follows: Section 2 presents the fundamental concepts related to the n -th kind extended q -difference operator. Section 3 discusses the polynomial factorial associated with this operator. In Section 4, we introduce the reciprocal operator and describe the main theoretical results. Illustrative examples are provided in Section 5, and finally, Section 6 offers the concluding remarks.

The analysis in this paper is carried out under the following assumptions:

- (i) For non-negative real number a , the shifted set of natural numbers is defined as $N(a) = \{a, a+1, a+2, a+3, \dots\}$.
- (ii) If m and r are positive integers, then the binomial coefficient is given by

$$\binom{m}{r} = \frac{m!}{(m-r)!r!}, \quad \text{where } 0! = 1 \quad \text{and} \quad m! = 1 \cdot 2 \cdot 3 \cdots m.$$

- (iii) The term $\left[\frac{k}{\ell} \right] = j^*$ denotes the integer part of $\frac{k}{\ell}$, and $j_{rq(\ell)}$ represents a constant, depends upon k and ℓ .
- (iv) If the extended q -difference operator involves multiple parameters $\ell_1, \ell_2, \ell_3, \dots, \ell_n$. It may be conveniently represented as $\Delta_{q(L)}$, where $L = (\ell_1, \ell_2, \ell_3, \dots, \ell_n)$.
- (v) If all the parameters are equal, that is, $\ell_1 = \ell_2 = \ell_3 = \dots = \ell_n = \ell$, then we can write $\underline{L} = (\ell, \ell, \ell, \dots, \ell)$. Hence, the operator can be expressed as $\Delta_{q(\ell, \ell, \ell, \dots, \ell)} = \Delta_{q(\underline{L})}$.
- (vi) In this context, the total sum of the terms is denoted by R_n , that is, $r_1 + r_2 + r_3 + \dots + r_n = R_n$.

2. FOUNDATIONS

We outline the discrete versions of the Leibniz and Binomial theorems for the n -th kind of extended q -difference operator, emphasizing several key conclusions.

Definition 2.1. [9] *The extended q -difference operator, denoted by $\Delta_{q(\ell)}$, and applicable to a function $u(k)$ for $k \in [0, \infty)$, is defined as follows*

$$\Delta_{q(\ell)}u(k) = u((k + \ell)q) - u(k). \quad (2.1)$$

Definition 2.2. *For a function $u(k)$ with $k \in [0, \infty)$, the extended q -difference operator of the n -th kind, denoted by $\Delta_{q(L)}$, $L = (\ell_1, \ell_2, \ell_3, \dots, \ell_n)$, is defined as follows*

$$\Delta_{q(L)}u(k) = \sum_{r=0}^n (-1)^{n-r} \left[\prod_{s=1}^r \left(\sum_{i_s=i_{s-1}+1}^{n-r+s} \right) u \left(kq^r + \sum_{t=1}^r \ell_{i_t} q^{r-t+1} \right) \right], \quad (2.2)$$

where $i_0 = 0$, ℓ_i 's and q 's are positive reals and n is a positive integer.

Lemma 2.1. *The usual extended q -difference operator, $E^{q(\ell)}$ is defined as $E^{q(\ell)}u(k) = u((k + \ell)q)$. With this definition, the following results are easily obtained. Furthermore, if $\ell_j, j = 1, 2, 3, \dots, n$ are positive reals, then*

$$(i) \quad \Delta_{q(L)} = \sum_{r=0}^n (-1)^{n-r} \left[\prod_{s=1}^r \left(\sum_{i_s=i_{s-1}+1}^{n-r+s} \right) E^{q(\ell_{i_s})} \right], \text{ with } i_0 = 0 \quad (2.3)$$

$$(ii) \quad \Delta_{q(L)} = \sum_{r=1}^n (-1)^{n-r} \left[\prod_{s=1}^r \left(\sum_{i_s=i_{s-1}+1}^{n-r+s} \right) \Delta_{q(\ell_{i_s})} \right], \text{ with } i_0 = 0 \quad (2.4)$$

$$(iii) \quad \Delta_{q(L)} = \prod_{r=1}^n \Delta_{q(\ell_r)} \quad (2.5)$$

$$(iv) \quad \Delta_{q(L)} = \prod_{m=1}^n \left[\sum_{i=1}^{q(\ell_m)} \binom{q(\ell_m)}{i} \Delta^i \right] \quad (2.6)$$

Definition 2.3. *The second order extended q -difference operator of the n -th kind can be represented as*

$$\Delta_{q(L)}^2 = \Delta_{q(L)}(\Delta_{q(L)})$$

and in general the higher order of the extended q -difference operator of n -th kind is defined as

$$\Delta_{q(L)}^n = \Delta_{q(L)} \left(\Delta_{q(L)}^{n-1} \right)$$

Remark 2.1. For the positive integers r and s ,

$$\Delta_{q(L)}^r \left(\Delta_{q(L)}^s \right) = \Delta_{q(L)}^s \left(\Delta_{q(L)}^r \right)$$

Lemma 2.2. It is easily seen that the following results hold:

(i) For the positive integer m ,

$$\Delta_{q(L)}^m = \prod_{r=1}^n \left[\sum_{s=0}^m (-1)^s \binom{m}{s} E^{q(\ell_r)(m-s)} \right] \tag{2.7}$$

which is equivalent to

$$\Delta_{q(L)}^m = \prod_{t=1}^n \left[\sum_{r_t=0}^m (-1)^{R_t} \binom{m}{r_t} \right] u \left(kq^{mm-R_t} + \sum_{i=1}^n \left(\sum_{m_i=1}^{m-r_i} \ell_i q^{m_i+(n-i)m-\sum_{s=i}^n r_{s+1}} \right) \right), \tag{2.8}$$

where $R_t = r_1 + r_2 + r_3 + \dots + r_t$.

(ii) If $q(\ell_r) = \sum_{i=1}^m q(\ell_{ri})$, $r = 1, 2, 3, \dots, n$ then

$$\Delta_{q(L)} = \prod_{r=1}^n \left[\prod_{i=1}^m (\Delta_{q(\ell_{ri})} + 1) - 1 \right]$$

(iii) Given any positive real-valued function $u(k)$ and a constant c , then

$$\Delta_{q(L)}^m (cu(k)) = c \left(\Delta_{q(L)}^m u(k) \right)$$

(iv) For the positive integer n ,

$$\Delta_{mq(L)} = \prod_{r=1}^n [E^{mq(\ell_r)} - 1] = \prod_{r=1}^n [(1 + \Delta_{q(\ell_r)})^m - 1]$$

(v) $\Delta_{mq(L)} = \prod_{r=1}^n \left[\sum_{i_r=1}^m \binom{m}{i_r} \Delta_{q(i_r)}^{i_r} \right]$

(vi) $\Delta_{q(L)}^m = \prod_{r=1}^n \left[\sum_{i_r=0}^{m-1} (-1)^{i_r} \binom{m}{i_r} \Delta_{(m-i_r)q(\ell_r)}^{i_r} \right]$

Corollary 2.1. For the shift operator $E^{q(L)}$, we have

$$\begin{aligned} & \left[\sum_{r=0}^n (-1)^{n-r} \left(\prod_{s=1}^r \left(\sum_{i_s=i_{s-1}+1}^{n-r+s} E^{q(\ell_{i_s})} \right) \right) \right]^m u(k) \\ &= \sum_{r=0}^m (-1)^r \binom{m}{r} \Delta_{q(L)}^{(m-r)} u(k), \text{ with } i_0 = 0. \end{aligned} \tag{2.9}$$

Proof. Equation (2.9) follows by (2.2). □

According to $\Delta_{q(L)}$, the discrete form of the Leibniz theorem is as follows.

Theorem 2.1. *If $u(k)$ and $v(k)$ are any two real functions, then*

$$\Delta_{q(L)}^n [u(k)v(k)] = \sum_{r=0}^n \binom{n}{r} \Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_{n-1})}^{n-r} \left[\Delta_{q(\ell_n)}^r u(k) \Delta_{q(\ell_n)}^{n-r} v \left(kq^r + \ell_n \sum_{t=1}^r q^t \right) \right] \quad (2.10)$$

Proof. The proof follows from the extended q -difference Leibniz Theorem (Theorem 3.2 [19]) and (2.3). \square

Lemma 2.3. *If $a(k)$ is real-valued function and x is an positive integer, then*

$$\sum_{k=0}^{\infty} \frac{x^{kq(L)}}{k! [q(L)]^k} (a_{kq(L)}) = \exp \left(\frac{x^{q(L)} \left(\prod_{m=1}^n \Delta_{q(\ell_m)} + 1 \right)}{q(L)} \right) a(0). \quad (2.11)$$

Proof. The proof follows from the relation $E^{q(\ell)} u(k) = u((k + \ell)q)$, $a(k) = E^k a(0)$ and q -shift operator. \square

The extended n -th kind of Montmorte's theorem for the extended q -difference operator $\Delta_{q(L)}$, the following theorem gives.

Theorem 2.2. *If the series $\sum_{k=0}^{\infty} x^{kq(L)} a(kq(L))$ converges, it can be expressed as*

$$\sum_{k=0}^{\infty} x^{kq(L)} a(kq(L)) = \left[\sum_{k=0}^{\infty} \frac{x^{kq(L)} \left(\prod_{s=1}^r \left(\sum_{i_s=i_{s-1}+1}^{n-r+s} \Delta_{q(\ell_{i_s})} \right)^k \right)}{(1 - x^{q(L)})^{k+1}} \right] a(0).$$

Proof. The proof follows from the relation $\Delta_{q(\ell_1, \ell_2, \ell_3, \dots, \ell_n)} = \Delta_{q(\ell_1)} \cdot \Delta_{q(\ell_2)} \cdot \Delta_{q(\ell_3)} \cdots \Delta_{q(\ell_n)}$. \square

Corollary 2.2. *If the series $\sum_{k=0}^{\infty} x^{kq(L)} a(kq(L))$ converges, then*

$$\sum_{k=0}^{\infty} x^{kq(L)} a(kq(L)) = \left[\sum_{k=0}^{\infty} \frac{x^{kq(L)} \left(\Delta_{q(\ell)}^n + \binom{n}{1} \Delta_{q(\ell)}^{n-1} + \binom{n}{2} \Delta_{q(\ell)}^{n-2} + \cdots + \Delta_{q(\ell)} \right)^k}{(1 - x^{q(L)})^{k+1}} \right] a(0).$$

3. EXTENDED q -POLYNOMIAL FACTORIAL FOR n -th KIND

In this section, we define the extended q -polynomial factorial in k for the n -th kind as follows:

Definition 3.1. *If $\ell_1, \ell_2, \ell_3, \dots, \ell_n$ and q 's are positive reals, we define the extended q -polynomial factorial for the n -th kind as*

$$k_{q(L)}^{(t)} = \sum_{r=0}^{n-1} (-1)^{n-r-1} \sum_{j=1}^n \left[\prod_{s=1}^r \left(\sum_{\substack{i_s=i_{s-1}+1 \\ i_s \neq j}}^{n-r+s} \right) \left(kq^r + \sum_{t=1}^r \ell_{i_t} q^{r-t+1} \right)_{q(\ell_j)}^{(t)} \right] \quad (3.1)$$

Lemma 3.1. [8] *If m and n are positive integers, then*

$$\Delta_{q(\ell)}^m k_{q(\ell)}^{(n)} = \left[\frac{q^n - 1}{q^{n-1}} \Delta_{q(\ell)}^{m-1} \left(k_{q(\ell)}^{(n-1)} k_{q(\ell)}^1 \right) + \frac{(q^n + (n-1)) \ell}{q^{n-1}} \Delta_{q(\ell)}^{m-1} \left(k_{q(\ell)}^{(n-1)} \right) \right] \quad (3.2)$$

Proof. The validity of the statement is shown through a double induction process on m and n . \square

Lemma 3.2. *Let $L = (\ell_1, \ell_2, \dots, \ell_n)$, where $(\ell_1, \ell_2, \ell_3, \dots, \ell_n)$, t and m , are the positive integers. Then*

$$\Delta_{q(L)} k_t^{(m)} = \left[\frac{q^m - 1}{q^{m-1}} \left(\sum_{r=0}^{n-1} (-1)^r \sum_{j=1}^n \prod_{s=1}^r \left(\sum_{\substack{i_s=i_{s-1}+1 \\ i_s \neq j}}^{n-r+s} \right) \left(kq^r + \sum_{t=1}^r \ell_i q^{r-t+1} \right)_{q(\ell_j)}^{(1)} \right) + \frac{(q^m + (m-1)) \ell_j}{q^{m-1}} \right] \quad (3.3)$$

$$\times \left(kq^r + \sum_{t=1}^r \ell_i q^{r-t+1} \right)_{q(\ell_j)}^{(n-1)}, \text{ if } t = q(\ell_j), j = 1, 2, \dots, n.$$

Proof. By combining equations (2.2) and (3.1), the desired result follows. \square

Lemma 3.3. *For the real numbers $\ell_1, \ell_2, \ell_3, \dots, \ell_n$, q and positive integer n , we have*

$$\Delta_{q(L)} k_{q(L)}^{(n)} = \sum_{i=1}^n \Delta_{q(L \setminus \ell_i)}^{n-1} \left[\frac{q^n - 1}{q^{n-1}} k_{q(\ell_i)}^{(1)} + \frac{q^n + (n-1) \ell_i}{q^{n-1}} \right] \left(k_{q(\ell_i)}^{(n-1)} \right), \quad (3.4)$$

where $L = \{\ell_1, \ell_2, \dots, \ell_n\}$, $L \setminus \{\ell_i\} = \{\ell_j : j \neq i, j = 1, 2, \dots, n\}$.

If $\ell_1 = \ell_2 = \ell_3 = \dots = \ell_n = \ell$, then $\underline{L} = \{\ell, \ell, \ell, \dots, \ell\}$. We have

$$\Delta_{q(\underline{L})} k_{q(\underline{L})}^{(n)} = \underbrace{n \Delta_{q(\ell, \ell, \dots, \ell)}^{n-1}}_{(n-1)\text{-terms}} \left[\frac{q^n - 1}{q^{n-1}} k_{q(\ell)}^{(1)} + \frac{(q^n + (n-1)) \ell}{q^{n-1}} \right] \left(k_{q(\ell)}^{(n-1)} \right).$$

Proof. The proof is deduced by using equation (3.1). \square

Lemma 3.4. *For $\lambda \neq 1, k \geq 2q\ell$ and $P(k)$ is any function of k , then*

$$\prod_{i=1}^n \sum_{r_i=1}^{\lfloor \frac{k}{\ell} \rfloor} \lambda^{\left\lfloor \frac{k - \ell \sum_{p=1}^{R_n} q^p}{q^{R_n}} \right\rfloor} P \left(\frac{k - \ell \sum_{p=1}^{R_n} q^p}{q^{R_n}} \right)$$

$$= \frac{\lambda^k}{(\lambda^{\Delta_{q(\ell)}(k)} - 1)^n} \left[1 - \frac{\lambda^{\Delta_{q(\ell)}(k)} \Delta_{q(\ell)}}{(\lambda^{\Delta_{q(\ell)}(k)} - 1)} + \frac{\lambda^{2\Delta_{q(\ell)}(k)} \Delta_{q(\ell)}^2}{(\lambda^{\Delta_{q(\ell)}(k)} - 1)^2} + \dots + \frac{\lambda^{n\Delta_{q(\ell)}(k)} \Delta_{q(\ell)}^n}{(\lambda^{\Delta_{q(\ell)}(k)} - 1)^n} \right] P(k) + \sum_{i=1}^n j_{iq(\ell)}.$$

Proof. Let $\Delta_{q(\underline{L})} \lambda^k F(k) = \lambda^k P(k)$, where $P(k) = (\lambda^{\Delta_{q(\ell)}(k)} E^{q(\ell)} - 1)^n F(k)$.

Operating $\Delta_{q(\underline{L})}^{-1}$ on both sides of $\Delta_{q(\underline{L})} \lambda^k F(k) = \lambda^k P(k)$, we obtain

$$\Delta_{q(\underline{L})}^{-1} \lambda^k P(k) = \lambda^k F(k) + \sum_{i=1}^n j_{iq(\ell)} = (\lambda^{\Delta_{q(\ell)}(k)} E^{q(\ell)} - 1)^{-n} P(k) + \sum_{i=1}^n j_{iq(\ell)}.$$

The proof can now be obtained using $\Delta_{q(\ell)} = E^{q(\ell)} + 1$ and the Binomial theorem. \square

4. RECIPROCAL OF EXTENDED q -DIFFERENCE OPERATOR OF n -th KIND AND ITS APPLICATIONS

In this section, we introduce the reciprocal operator and derive several significant results based on it.

Definition 4.1. The reciprocal extended q -difference operator of n -th kind, denoted by $\Delta_{q(L)}^{-1}$, is defined as

$$\text{if } \Delta_{q(L)} v(k) = u(k) \text{ then } v(k) = \Delta_{q(L)}^{-1} u(k) + c_j, \quad (4.1)$$

and the n -th order reciprocal operator denoted by $\Delta_{q(L)}^{-n}$, is defined as

$$\text{if } \Delta_{q(L)}^n v(k) = u(k) \text{ then } v(k) = \Delta_{q(L)}^{-n} u(k) + c_j,$$

where c_j is constant, depends upon $k \in N_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right] \ell$.

Remark 4.1. Suppose $u(k) \in \mathbb{R}$ for all k , then

$$\Delta_{q(L)} \left[\Delta_{q(L)}^{-1} u(k) \right] \neq \Delta_{q(L)}^{-1} \left[\Delta_{q(L)} u(k) \right].$$

Theorem 4.1. Assume k, ℓ , and $q \in \mathbb{R}^+$. Then

$$\begin{aligned} & \sum_{r=1}^{j^*} \left[\frac{(r+n-2)^{(n-1)}}{(n-1)!} \right] \left(\frac{k - \ell \sum_{p=1}^{r+(n-1)} q^p}{q^{r+(n-1)}} \right) \\ & + \sum_{i=1}^{n-1} \left[\sum_{r=1}^{j^*} \left(\frac{\sum_{t=0}^i (-1)^t \binom{n}{t} [(i-t)j^* + r]^{n-1}}{(n-1)!} \right) \left(\frac{k - \ell \sum_{p=1}^{ij^*+r+n-1} q^p}{q^{ij^*+r+n-1}} \right) \right] \\ & = \Delta_{q(\ell)}^{-n} \left[u(k) + \sum_{r=1}^n (-1)^r \binom{n}{r} u(j_{rq(\ell)}) \right], \end{aligned} \quad (4.2)$$

where $j_{rq(\ell)} = \frac{k - \ell \sum_{p=1}^{rj^*} q^p}{q^{rj^*}}$, $r = 1, 2, \dots, n$, and $j^* = \left[\frac{k}{\ell}\right]$, is the integral part of $\frac{k}{\ell}$.

Proof. By applying equation (4.1) and the relation, the proof is established.

$$\prod_{i=1}^n \sum_{r_i=1}^{j^*} u \left(\frac{k - \ell \sum_{p=1}^{R_n} q^p}{q^{R_n}} \right) = \Delta_{q(\ell)}^{-n} u(k).$$

\square

Theorem 4.2. Let k, ℓ , and q denote positive real-valued parameters. Then

$$\begin{aligned} & \sum_{r=1}^{j^*} \left[\frac{(r+n-2)^{(n-1)}}{(n-1)!} \right] \left(\frac{k-\ell \sum_{p=1}^{r+(n-1)} q^p}{q^{r+(n-1)}} + \frac{\ell q}{q-1} \right) \\ & + \sum_{i=1}^{n-1} \left[\sum_{r=1}^{j^*} \frac{\sum_{t=0}^i (-1)^t \binom{n}{t} [(i-t)j^* + r]^{(n-1)}}{(n-1)!} \right] \left(\frac{k-\ell \sum_{p=1}^{ij^*+r+n-1} q^p}{q^{ij^*+r+n-1}} + \frac{\ell q}{q-1} \right) \\ & = \left[\frac{k}{(q-1)^n} + \sum_{r=1}^n (-1)^r \binom{n}{r} \frac{k-\ell \sum_{p=1}^{rj^*} q^p}{(q-1)^n q^{rj^*}} \right] \end{aligned} \tag{4.3}$$

Proof. From (4.1), we have

$$\Delta_{q(\underline{L})}^{-n} \left(k + \frac{\ell q}{q-1} \right) = \frac{k}{(q-1)^n} + c_j. \tag{4.4}$$

The proof follows from (4.2) and (4.4). □

Theorem 4.3. Assume $k \geq 0$ and define $j^* = \lfloor \frac{k}{\ell} \rfloor$. Then

$$\begin{aligned} & \sum_{r=1}^{j^*} \left[\frac{(n-2+r)^{(n-1)}}{(n-1)!} \right] \left[\sum_{s=0}^n (-1)^s \binom{n}{s} \left(\underbrace{\left(\left(\left(\frac{k-\ell \sum_{p=1}^{r+(n-1)} q^p}{q^{r+(n-1)}} + \ell \right) q + \ell \right) q + \dots + \ell \right) q}_{n-s \text{ times of } (\cdot + \ell)q} \right)^{-1} \right] \\ & + \sum_{i=1}^{n-1} \left[\sum_{r=1}^{j^*} \frac{\sum_{t=0}^1 (-1)^t \binom{n}{t} [(1-t)j^* + r]^{(n-1)}}{(n-1)!} \right] \\ & \left(\sum_{s=0}^n (-1)^s \binom{n}{s} \left(\underbrace{\left(\left(\left(\frac{k-\ell \sum_{p=1}^{ij^*+r+n-1} q^p}{q^{ij^*+r+n-1}} + \ell \right) q + \ell \right) q + \dots + \ell \right) q}_{n-s \text{ times of } (\cdot + \ell)q} \right)^{-1} \right) \end{aligned} \tag{4.5}$$

$$= \frac{1}{k} + \sum_{r=1}^n (-1)^r \binom{n}{r} \left(\frac{q^{rj^*}}{k - \ell \sum_{p=1}^{rj^*} q^p} \right)$$

Proof. Using equations (4.1) and (4.2), we have

$$\Delta_{q(\underline{L})} \left(\frac{1}{k} \right) = \left[\sum_{s=0}^n (-1)^s \binom{n}{s} \left(\underbrace{\left(((k + \ell)q + \ell)q + \cdots + \ell \right)q}_{n-s \text{ times of } (\cdot + \ell)q} \right)^{-1} \right] \quad (4.6)$$

By equation (4.1), we get

$$\Delta_{q(\underline{L})}^{-1} \left[\sum_{s=0}^n (-1)^s \binom{n}{s} \left(\underbrace{\left(((k + \ell)q + \ell)q + \cdots + \ell \right)q}_{n-s \text{ times of } (\cdot + \ell)q} \right)^{-1} \right] = \frac{1}{k} + c_j \quad (4.7)$$

The conclusion is derived directly from (4.2) and (4.5). \square

Theorem 4.4. For all $k \geq 0$ and integer m , we have

$$\sum_{r=1}^{j^*} \left[\frac{(n-2+r)^{(n-1)}}{(n-1)!} \right] \sum_{s=0}^n (-1)^s \binom{n}{s} \left(\frac{\left(\underbrace{\left(\left(\left(\frac{k - \ell \sum_{p=1}^{r+(n-1)} q^p}{q^{r+(n-1)}} + \ell \right) q + \ell \right) q + \cdots + \ell \right) q + m\ell}_{n-s \text{ times of } (\cdot + \ell)q} \right)}{\left(\underbrace{\left(\left(\left(\frac{k - \ell \sum_{p=1}^{r+(n-1)} q^p}{q^{r+(n-1)}} + \ell \right) q + \ell \right) q + \cdots + \ell \right) q + (m \pm 1)\ell}_{n-s \text{ times of } (\cdot + \ell)q} \right)} \right) \\ + \sum_{i=1}^{n-1} \left[\sum_{r=1}^{j^*} \left(\frac{\sum_{t=0}^i (-1)^t \binom{n}{t} [(1-t)j^* + r]^{(n-1)}}{(n-1)!} \right) \right]$$

$$\begin{aligned}
 & \left(\sum_{s=0}^n (-1)^s \binom{n}{s} \frac{\left(\underbrace{\left(\left(\frac{k-\ell \sum_{p=1}^{ij^*+r+n-1} q^p}{q^{ij^*+r+n-1}} + \ell \right) q + \ell \right) q + \dots + \ell \right) q + m\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)}{\left(\underbrace{\left(\left(\frac{k-\ell \sum_{p=1}^{ij^*+r+n-1} q^p}{q^{ij^*+r+n-1}} + \ell \right) q + \ell \right) q + \dots + \ell \right) q + (m \pm 1)\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)} \right) \\
 &= \left[\frac{k+m\ell}{k+(m \pm 1)\ell} + \sum_{r=1}^n (-1)^r \binom{n}{r} \frac{\left(\frac{k-\ell \sum_{p=1}^{rj^*} q^p}{q^{rj^*}} + m\ell \right)}{\left(\frac{k-\ell \sum_{p=1}^{rj^*} q^p}{q^{rj^*}} + (m \pm 1)\ell \right)} \right]
 \end{aligned} \tag{4.8}$$

Proof. By Quotient law, let us take $u(k) = k + m\ell$ and $v(k) = k + (m \pm 1)\ell$.

We get the following,

$$\Delta_{q(\underline{L})} \left[\frac{k+m\ell}{k+(m \pm 1)\ell} \right] = \sum_{s=0}^n (-1)^s \binom{n}{s} \frac{\left(\underbrace{\left(\left((k+\ell)q + \ell \right) q + \dots + \ell \right) q + m\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)}{\left(\underbrace{\left(\left((k+\ell)q + \ell \right) q + \dots + \ell \right) q + (m \pm 1)\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)} \tag{4.9}$$

In (4.8), we use equation (4.1) to obtain

$$\Delta_{q(\underline{L})}^{-1} \left[\sum_{s=0}^n (-1)^s \binom{n}{s} \frac{\left(\underbrace{\left(\left((k+\ell)q + \ell \right) q + \dots + \ell \right) q + m\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)}{\left(\underbrace{\left(\left((k+\ell)q + \ell \right) q + \dots + \ell \right) q + (m \pm 1)\ell}_{n-s \text{ times of } (\cdot+\ell)q} \right)} \right] = \left[\frac{k+m\ell}{k+(m \pm 1)\ell} \right] + c_j$$

(4.2) and (4.8) lead to the proof. \square

5. APPLICATIONS

The main subject of this section is the applications of the Extended q -difference operator for n -th kind.

Theorem (4.2) is verified in the case described below.

Example 5.1. Determine the progression

$$\begin{aligned} S = & 1 \binom{93}{2^{19}} + 10 \binom{93}{2^{21}} + 55 \binom{93}{2^{23}} + \dots + 49280065120 \binom{93}{2^{137}} + 20286591270 \binom{93}{2^{139}} \\ & + 23667689815 \binom{93}{2^{141}} + 27540584512 \binom{93}{2^{143}} + \dots + 10308761243560 \binom{93}{2^{255}} \\ & + 11993738148370 \binom{93}{2^{257}} + 12922816957215 \binom{93}{2^{259}} + 13913961442404 \binom{93}{2^{261}} + \dots \\ & + 342197325535900 \binom{93}{2^{373}} + \dots + 37014131440 \binom{93}{2^{1083}} + 31966749880 \binom{93}{2^{1085}} \\ & + 27540584512 \binom{93}{2^{1087}} + \dots + 1 \binom{93}{2^{1197}} \end{aligned}$$

Taking $k = 182$, $\ell = 3$, $q = 4$, $\left[\frac{k}{\ell}\right] = j^* = 60$ and $n = 10$ in (4.3), we get

$$\begin{aligned} & \sum_{r=1}^{60} \left[\frac{(8+r)^{(9)}}{9!} \right] \left(\frac{182 - 3 \sum_{p=1}^{r+9} 4^p}{4^{r+9}} + 4 \right) \\ & + \sum_{i=1}^9 \left[\sum_{r=1}^{60} \frac{1}{9!} \left(\sum_{t=0}^i (-1)^t \binom{10}{t} [(i-t)60+r]^{(9)} \right) \left(\frac{182 - 3 \sum_{p=1}^{i(60)+r+9} 4^p}{4^{i(60)+r+9}} + 4 \right) \right] \quad (5.1) \\ & = \frac{182}{3^{10}} + \sum_{r=1}^{10} (-1)^r \binom{10}{r} \left(\frac{182 - 3 \sum_{p=1}^{r(60)} 4^p}{(4-1)^{10} 4^{r(60)}} \right) \\ & = 0.003149926332368033. \end{aligned}$$

The next example makes Theorem (4.3) clearer for a reciprocal function.

Example 5.2. Determine the series

$$\begin{aligned} S = & (-2.586835 \times 10^3) + 15 (6.037535 \times 10^2) + 120 (-8.761690 \times 10^1) + \dots \\ & + 2403979904200 (-3.316791 \times 10^{-14}) + 52860229080 (1.393076 \times 10^1) \\ & + 78378960360 (4.953938 \times 10^0) + 114955808528 (2.038991 \times 10^0) + \dots \end{aligned}$$

$$\begin{aligned}
 &+ 1507804127129800 \left(2.282218 \times 10^{-20}\right) + 2197878544943100 \left(1.393076 \times 10^1\right) \\
 &+ 2643603014369700 \left(4.953938 \times 10^0\right) + 3172010101402200 \left(2.038991 \times 10^0\right) + \dots \\
 &+ 646793807501933760 \left(2.282218 \times 10^{-20}\right) + \dots + 1292706174900 \left(1.393076 \times 10^1\right) \\
 &+ 937845700000 \left(4.953938 \times 10^0\right) + 675248900000 \left(2.038991 \times 10^0\right) + \dots + \left(2.282218 \times 10^{-20}\right)
 \end{aligned}$$

With $k = 161, \ell = 4, q = 3, n = 15,$ and $j^* = 40,$ in (4.5), we find

$$\begin{aligned}
 &\sum_{r=1}^{40} \left[\frac{(r+13)^{(14)}}{14!} \right] \sum_{s=0}^{15} (-1)^s \binom{15}{s} \left(\underbrace{\left(\left(\left(\frac{161 - 4 \sum_{p=1}^{r+14} 3^p}{3^{r+14}} + 4 \right) 3 + 4 \right) 3 + \dots + 4 \right) 3}_{15-s \text{ times of } (\cdot+4)3} \right)^{-1} \Bigg] \\
 &+ \sum_{i=1}^{14} \left[\sum_{r=1}^{40} \left(\frac{\sum_{t=0}^i (-1)^t \binom{15}{t} [(i-t)40+r]^{(14)}}{14!} \right) \right. \\
 &\left. \left(\sum_{s=0}^{15} (-1)^s \binom{15}{s} \left(\underbrace{\left(\left(\left(\frac{161 - 4 \sum_{p=1}^{i(40)+r+14} 3^p}{3^{i(40)+r+14}} + 4 \right) 3 + 4 \right) 3 + \dots + 4 \right) 3}_{15-s \text{ times of } (\cdot+4)3} \right)^{-1} \right) \right] \\
 &= \frac{1}{161} + \sum_{r=1}^{15} (-1)^r \binom{15}{r} \left(\frac{3^{r(40)}}{161 - 4 \sum_{p=1}^{r(40)} 3^p} \right) \\
 &= 0.172877846790890.
 \end{aligned}$$

Theorem (4.4) is supported by the following example for a positive integer.

Example 5.3. Determine the series

$$\begin{aligned}
 S &= \left(\frac{5504512727329325}{10^{18}} \right) + 5 \left(\frac{-987585929182044416}{10^{20}} \right) + 15 \left(\frac{-31100509958131852}{10^{19}} \right) + \dots \\
 &+ 677040 \left(\frac{-33306690738754696}{10^{17}} \right) + 635376 \left(\frac{-33306690738754696}{10^{17}} \right) \\
 &+ 677040 \left(\frac{-33306690738754696}{10^{17}} \right) + 720715 \left(\frac{-33306690738754696}{10^{17}} \right) + \dots \\
 &+ 6592026 \left(\frac{-33306690738754696}{10^{17}} \right) + 6832245 \left(\frac{-33306690738754696}{10^{17}} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ 6949425 \left(\frac{-33306690738754696}{10^{17}} \right) + 7064410 \left(\frac{-33306690738754696}{10^{17}} \right) + \dots \\
 &+ 6949425 \left(\frac{-33306690738754696}{10^{17}} \right) + \dots + 595665 \left(\frac{-33306690738754696}{10^{17}} \right) \\
 &+ 557845 \left(\frac{-33306690738754696}{10^{17}} \right) + 521855 \left(\frac{-33306690738754696}{10^{17}} \right) + \dots + \left(\frac{-33306690738754696}{10^{17}} \right)
 \end{aligned}$$

With $k = 125, \ell = 2, q = 3, n = 5, m = 10$, and $j^* = 62$ in (4.8), we obtain

$$\begin{aligned}
 &\sum_{r=1}^{62} \left[\frac{(3+r)^{(4)}}{4!} \right] \sum_{s=0}^5 (-1)^s \binom{5}{s} \left(\frac{\left(\left(\left(\frac{125 - 2 \sum_{p=1}^{4+r} 3^p}{3^{4+r}} + 2 \right) 3 + 2 \right) 3 + \dots + 2 \right) 3 + 20}{5 - s \text{ times of } (\cdot + 2)3} \right)}{\left(\left(\left(\frac{125 - 2 \sum_{p=1}^{4+r} 3^p}{3^{4+r}} + 2 \right) 3 + 2 \right) 3 + \dots + 2 \right) 3 + 22} \right)} \\
 &+ \sum_{i=1}^4 \left[\sum_{r=1}^{62} \frac{\sum_{t=0}^i (-1)^t \binom{5}{t} [(i-t)(62+r)]^{(4)}}{4!} \right] \\
 &\left(\sum_{s=0}^5 (-1)^s \binom{5}{s} \left(\frac{\left(\left(\left(\frac{125 - 2 \sum_{p=1}^{i(62)+r+4}}{3^{i(62)+r+4}} + 2 \right) 3 + 2 \right) 3 + \dots + 2 \right) 3 + 20}{5 - s \text{ times of } (\cdot + 2)3} \right)}{\left(\left(\left(\frac{125 - 2 \sum_{p=1}^{i(62)+r+4}}{3^{i(62)+r+4}} + 2 \right) 3 + 2 \right) 3 + \dots + 2 \right) 3 + 22} \right)} \right)
 \end{aligned}$$

$$= \left[\frac{145}{147} + \sum_{r=1}^5 (-1)^r \binom{5}{r} \frac{\left(\left(\frac{125 - 2 \sum_{p=1}^{r \cdot 62} 3^p}{3^{r \cdot 62}} \right) + 20 \right)}{\left(\left(\frac{125 - 2 \sum_{p=1}^{r \cdot 62} 3^p}{3^{r \cdot 62}} \right) + 22 \right)} \right] = 0.09165771571786607.$$

The example below shows that Theorem (4.4) is true for negative integers.

Example 5.4. Determine the series

$$\begin{aligned} S = & \left(7.199842 \times 10^{-1} \right) + 8 \left(-3.585801 \times 10^{-2} \right) + 36 \left(-2.143241 \times 10^{-3} \right) + \dots \\ & + 1184040 \left(-1.391361 \times 10^{-10} \right) + 346104 \left(-6.956458 \times 10^{-11} \right) + 480700 \left(-3.478107 \times 10^{-11} \right) \\ & + 657800 \left(-1.738887 \times 10^{-11} \right) + \dots + 36956216 \left(4.218847 \times 10^{-15} \right) + 50755848 \left(-2.220446 \times 10^{-16} \right) \\ & + 59045899 \left(1.043610 \times 10^{-14} \right) + 68366672 \left(-3.774758 \times 10^{-15} \right) + \dots + 477000128 \left(3.330669 \times 10^{-15} \right) \\ & + \dots + 657800 \left(3.330669 \times 10^{-15} \right) + 480700 \left(3.330669 \times 10^{-15} \right) + 346104 \left(3.330669 \times 10^{-15} \right) \\ & + \dots + \left(3.330669 \times 10^{-15} \right) \end{aligned}$$

With $k = 111, \ell = 5, q = 2, n = 8, m = -15$, and $j^* = 22$ in (4.8), we find

$$\begin{aligned} & \sum_{r=1}^{22} \left[\frac{(6+r)^{(7)}}{7!} \right] \sum_{s=0}^8 (-1)^s \binom{8}{s} \frac{\left(\underbrace{\left(\left(\left(\frac{111 - 5 \sum_{p=1}^{r+7} 2^p}{2^{r+7}} + 5 \right) 2 + 5 \right) 2 + \dots + 5 \right) 2 - 75}_{8-s \text{ times of } (\cdot+5)2} \right)}{\left(\underbrace{\left(\left(\left(\frac{111 - 5 \sum_{p=1}^{r+7} 2^p}{2^{r+7}} + 5 \right) 2 + 5 \right) 2 + \dots + 5 \right) 2 - 80}_{8-s \text{ times of } (\cdot+5)2} \right)} \right) \\ & + \sum_{i=1}^7 \left[\sum_{r=1}^{22} \left(\frac{\sum_{t=0}^i (-1)^t \binom{8}{t} [(i-t)22+r]^{(7)}}{7!} \right) \right] \end{aligned}$$

$$\begin{aligned}
& \left(\sum_{s=0}^8 (-1)^s \binom{8}{s} \left(\underbrace{\left(\left(\left(\frac{111-5 \sum_{p=1}^{i(22)+r+7} 2^p}{2^{i(22)+r+7}} + 5 \right) 2 + 5 \right) 2 + \dots + 5 \right) 2 - 75}_{8-s \text{ times of } (\cdot+5)2} \right) \right. \\
& \left. \left(\underbrace{\left(\left(\left(\frac{111-5 \sum_{p=1}^{i(22)+r+7} 2^p}{2^{i(22)+r+7}} + 5 \right) 2 + 5 \right) 2 + \dots + 5 \right) 2 - 80}_{8-s \text{ times of } (\cdot+5)2} \right) \right) \right) \\
& = \left[\frac{36}{31} + \sum_{r=1}^8 (-1)^r \binom{8}{r} \left(\frac{\left(\frac{111-5 \sum_{p=1}^{r(22)} 2^p}{2^{r(22)}} - 75 \right)}{\left(\frac{111-5 \sum_{p=1}^{r(22)} 2^p}{2^{r(22)}} - 80 \right)} \right) \right] \\
& = 0.21684602059859603.
\end{aligned}$$

6. CONCLUSION

The research develops an n -th kind operator acting on the discrete function $u(k)$ to solve extended q -difference equations. The proposed operator framework enables a structured analysis of the solution space related to equation (4.2) and offers a versatile tool for dealing with complex discrete dynamical systems. The main theorem derived from this framework significantly simplifies the computation of fractional series, particularly when large values of k and small positive values of q and ℓ are employed.

The findings contribute both theoretical depth and practical utility to the field of numerical analysis. By bridging operator theory and extended q -calculus, this work provides a pathway for efficiently approximating solutions to classes of problems that are otherwise computationally intensive. These results may also serve as a foundation for future studies involving discrete models in applied mathematics, special function theory, and computational science.

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