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Fixed Point Theorems for Contractive Mappings in Non–Archimedean Fuzzy Metric–Like Spaces

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Abstract. A new fixed point theorem is established for generalized contractive mappings in NA-FMLS. The approach utilizes the ultrametric property of the fuzzy metric to ensure the convergence and uniqueness of the fixed point. This result extends several existing principles in fuzzy and b–metric settings and provides a unified framework for further applications in fuzzy nonlinear analysis.

1. Introduction

FP theory in FMS plays a crucial role in the analysis of nonlinear systems involving uncertainty and imprecision. Since the pioneering work of Kramosil & Michálek [9] and the refinement by George & Veeramani [10], FMS have become a rich environment for generalizing classical FP results. Recently, the concept of non–Archimedean FMS has been developed to model ultrametric-type behaviors, where the triangular inequality is replaced by a stronger non–Archimedean condition (see Roldán et al. [1]). Such structures naturally arise in p-adic analysis, digital topology, and nonlocal dynamics.

The present investigation is intended to establish a new FP theorem for a broad class of generalized contractive mappings defined on NA-FMLS. The introduced framework generalizes several existing results in fuzzy and b–metric settings and provides a unified tool for analyzing convergence in ultrametric fuzzy environments. The obtained theorem also ensures the existence and uniqueness of FPs via a suitable contractive inequality involving six control parameters.

Definition 1.1. *[1,10] Let $(\mathcal{Y}, \Xi, *)$ be a FMS, where $*$ represents a continuous t-norm. The space is called a* FMS *if the mapping $\Xi : \mathcal{Y} \times \mathcal{Y} \times (0, \infty) \to [0, 1]$ fulfils the following properties for every $\mathfrak{z}, v, \mathfrak{l} \in \mathcal{Y}$ and for all $s, t > 0$:*

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(1) $\Xi(\mathfrak{z}, v, 0) = 0$ *and* $\Xi(\mathfrak{z}, v, s) = 1$ *iff* $\mathfrak{z} = v$;
(2) $\Xi(\mathfrak{z}, v, s) = \Xi(v, \mathfrak{z}, s)$;
(3) $\Xi(\mathfrak{z}, v, s) * \Xi(v, \mathfrak{l}, t) \leq \Xi(\mathfrak{z}, \mathfrak{l}, s+t)$;
(4) $\Xi(\mathfrak{z}, v, \cdot)$ *is continuous on* $(0, \infty)$.

If, in addition, the stronger non–Archimedean property

$$\Xi(\mathfrak{z}, \mathfrak{l}, s) \geq \Xi(\mathfrak{z}, v, s) * \Xi(v, \mathfrak{l}, s)$$

holds for all $\mathfrak{z}, v, \mathfrak{l} \in \mathcal{Y}$ *and* $s > 0$*, then* $(\mathcal{Y}, \Xi, *)$ *is called a* NA-FMLS.

For a fixed $s > 0$, define the auxiliary function

$$\Delta(\mathfrak{z}, v) := 1 - \Xi(\mathfrak{z}, v, s), \qquad \mathfrak{z}, v \in \mathcal{Y},$$

so that Δ is an ultrametric on $\mathcal{Y}$ induced by the non–Archimedean fuzzy metric.

Theorem 1.1. *Let* $(\mathcal{Y}, \Xi, *)$ *be an* Ξ*-complete NA-FMLS. Fix* $s > 0$ *and set* Δ *as above. Let* $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ *be a mapping for which there exist nonnegative constants* $\lambda_1, \ldots, \lambda_6$ *satisfying*

$$\Lambda := \sum_{i=1}^{6} \lambda_i < 1,$$

and such that for all $\mathfrak{z}, v \in \mathcal{Y}$

$$\begin{aligned} \Delta(\mathcal{T}\mathfrak{z}, \mathcal{T}v) \leq{} & \lambda_1 \Delta(\mathfrak{z}, v) + \lambda_2 \Delta(\mathfrak{z}, \mathcal{T}\mathfrak{z}) + \lambda_3 \Delta(v, \mathcal{T}v) \\ & + \lambda_4 \Delta(\mathfrak{z}, \mathcal{T}v) + \lambda_5 \Delta(v, \mathcal{T}\mathfrak{z}) + \lambda_6 \Big(\Delta(\mathfrak{z}, \mathcal{T}v) + \Delta(v, \mathcal{T}\mathfrak{z}) \Big). \end{aligned} \tag{1.1}$$

Then $\mathcal{T}$ *has a UFP* $\mathfrak{l}^* \in \mathcal{Y}$ *and, for any* $\mathfrak{l}_0 \in \mathcal{Y}$*, the Picard iteration* $\mathfrak{l}_{n+1} = \mathcal{T}\mathfrak{l}_n$ *converges to* $\mathfrak{l}^*$ *in the fuzzy metric* Ξ.

Proof. Let $\mathfrak{l}_0 \in \mathcal{Y}$ and define $\mathfrak{l}_{n+1} = \mathcal{T}\mathfrak{l}_n$. Denote $\Delta_n := \Delta(\mathfrak{l}_{n+1}, \mathfrak{l}_n)$. Applying (1.1) with $\mathfrak{z} = \mathfrak{l}_n$, $v = \mathfrak{l}_{n-1}$ and using the ultrametric property $\Delta(\mathfrak{l}_{n-1}, \mathfrak{l}_{n+1}) \leq \max\{\Delta_{n-1}, \Delta_n\}$, we obtain

$$\Delta_n \leq \kappa \Delta_{n-1}, \qquad \kappa := \frac{\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6}{1 - \lambda_2} < 1,$$

since $\Lambda < 1$. By induction, $\Delta_n \to 0$, so $\{\mathfrak{l}_n\}$ is Cauchy. Completeness of $(\mathcal{Y}, \Xi, *)$ ensures convergence to some $\mathfrak{l}^* \in \mathcal{Y}$.

Passing to the limit in (1.1) with $\mathfrak{z} = \mathfrak{l}_n$ and $v = \mathfrak{l}^*$ gives $\Delta(\mathfrak{l}^*, \mathcal{T}\mathfrak{l}^*) = 0$, i.e., $\mathfrak{l}^*$ is a FP. Uniqueness follows from the contraction: if w^* is another FP, $\Delta(\mathfrak{l}^*, w^*) \leq (\lambda_1 + \lambda_4 + \lambda_5 + \lambda_6)\Delta(\mathfrak{l}^*, w^*) < \Delta(\mathfrak{l}^*, w^*)$, so $\mathfrak{l}^* = w^*$. Picard iteration converges to $\mathfrak{l}^*$ by construction. □

Corollary 1.1. *Under the assumptions of Theorem 1.1, if the mapping* $\mathcal{T}$ *satisfies the simpler condition*

$$\Delta(\mathcal{T}\mathfrak{z}, \mathcal{T}v) \leq \lambda \max\{\Delta(\mathfrak{z}, v), \Delta(\mathfrak{z}, \mathcal{T}\mathfrak{z}), \Delta(v, \mathcal{T}v)\}, \qquad 0 \leq \lambda < 1,$$

then $\mathcal{T}$ *has a UFP in* $\mathcal{Y}$ *and the Picard iteration converges to it.*

Proof. The result follows directly from Theorem 1.1 by setting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$. □

Theorem 1.2. *Let $(\mathcal{Y}, \Xi, *)$ be an Ξ-complete NA-FMLS, and let $\Delta(\mathfrak{z}, v) = 1 - \Xi(\mathfrak{z}, v, s)$. Suppose $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ satisfies*

$$\Delta(\mathcal{T}\mathfrak{z}, \mathcal{T}v) \leq \alpha\, \Delta(\mathfrak{z}, v) + \beta\Big(\Delta(\mathfrak{z}, \mathcal{T}\mathfrak{z}) + \Delta(v, \mathcal{T}v)\Big) + \gamma \max\{\Delta(\mathfrak{z}, \mathcal{T}v), \Delta(v, \mathcal{T}\mathfrak{z})\},$$

for constants $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + \gamma < 1$. Then $\mathcal{T}$ has a UFP $\mathfrak{l}^*$, and the Picard iteration converges to it.

Proof. Start with any $\mathfrak{l}_0 \in \mathcal{Y}$ and define $\mathfrak{l}_{n+1} = \mathcal{T}\mathfrak{l}_n$. Denote $\Delta_n := \Delta(\mathfrak{l}_{n+1}, \mathfrak{l}_n)$. Using the contraction and the ultrametric property $\Delta(\mathfrak{l}_{n-1}, \mathfrak{l}_{n+1}) \leq \max\{\Delta_{n-1}, \Delta_n\}$, we obtain a simple recursive bound

$$\Delta_n \leq \kappa\, \Delta_{n-1}, \qquad \kappa := \frac{\alpha + \beta + \gamma}{1 - \beta} < 1.$$

By induction, $\Delta_n \to 0$, so $\{\mathfrak{l}_n\}$ is Cauchy. Completeness ensures $\mathfrak{l}_n \to \mathfrak{l}^* \in \mathcal{Y}$.

Passing to the limit in the contraction shows $\Delta(\mathfrak{l}^*, \mathcal{T}\mathfrak{l}^*) = 0$, hence $\mathfrak{l}^*$ is a FP. Uniqueness follows similarly: for any other FP w^*, $\Delta(\mathfrak{l}^*, w^*) \leq (\alpha + \gamma)\Delta(\mathfrak{l}^*, w^*) < \Delta(\mathfrak{l}^*, w^*)$, so $\Delta(\mathfrak{l}^*, w^*) = 0$, i.e., $\mathfrak{l}^* = w^*$. Picard iteration converges by construction. □

2. Application to Integral Equations

Consider the nonlinear Volterra integral equation

$$u(t) = g(t) + \int_0^t K(t, s, u(s))\, ds, \qquad t \in [0, T],$$

where g and K are continuous. Define $(\mathcal{T}u)(t) = g(t) + \int_0^t K(t, s, u(s))\, ds$.

On $C([0, T], \mathbb{R})$, define

$$M(u, v, s) = \exp\left(-\frac{\|u - v\|_\infty + \min\{|u(0)|, |v(0)|\}}{s}\right), \quad s > 0.$$

Then $M(u, u, s) = e^{-|u(0)|/s} \leq 1$, so $(C([0, T], \mathbb{R}), M, *)$ is a complete NA-FMLS.

If K satisfies

$$|K(t, s, u) - K(t, s, v)| \leq \alpha|u - v| + \beta|u - \mathcal{T}u| + \gamma|u - \mathcal{T}v|,$$

for $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + \gamma < 1$, then $\mathcal{T}$ is a contraction in this fuzzy metric–like space. Hence $\mathcal{T}$ admits a UFP u^*, which is the unique solution of the integral equation.

When $u(0) = 0$, M reduces to the usual non–Archimedean fuzzy metric $M(u, v, s) = e^{-\|u-v\|_\infty/s}$.

Abbreviations: FP- Fixed Point, UFP- Unique Fixed Point, FMS - fuzzy metric spaces, NA-FMLS - non–Archimedean fuzzy metric–like spaces

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