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Integral Contraction Techniques for Twisted Coupling in C^*-Algebra Valued G-Metric Spaces and Their Applications

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Abstract. This paper presents novel twisted (α, β)-φ-ψ-integral type contractive conditions based on $\mathbb{T}$-coupling. A main theorem is established to guarantee the existence and uniqueness of strong coupled coincidence and common fixed points within C^*-algebra valued G-metric spaces ($\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$). The results extend previous studies, are illustrated with examples, and demonstrate the framework's applicability to functional equations and homotopy theory.

1. Introduction

Fixed point theory is a core area of modern analysis with wide applications in differential equations, optimization, image processing, computer science, and dynamical systems. Rooted in the Banach contraction principle, it has been extensively generalized to address problems beyond the classical metric framework. Integral-type contractions, first introduced by Branciari [1] and further studied in ([2]- [6]), expanded its scope. In recent decades, new distance structures such as G_b-metric, S_b-metric, fuzzy metric, and C^*-algebra-valued metric spaces have been developed to model complex and nonlinear systems. Notably, Shen *et al.* [9] unified the C^*-algebra-valued metric spaces of Zhenhua, Jiang, and Sun [8] with the G-metric framework of Mustafa and Sims [7], yielding significant theoretical and practical applications, particularly in differential equations.

The theory of coupled fixed points has evolved as an important extension within metric fixed point analysis. Originating from the work of Guo *et al.* [11] in 1987, the concept gained momentum when Kirk *et al.* [12] introduced cyclic contractions in 2003, establishing the existence of fixed points for such mappings. Subsequently, Bhaskar *et al.* [13] formulated the coupled contraction mapping theorem, laying the foundation for further research. In 2017, S. Binayak Choudhury *et*

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al. [14] defined couplings between two non-empty subsets of a metric space and demonstrated the existence of unique strong fixed points under Banach- and Chatterjea-type conditions. This idea was extended to complete S-metric spaces by G. V. R. Babu *et al.* [15] and S. Mary Anushia *et al.* [16]. Later developments focused on relaxing classical contraction requirements through altering and ultra-altering distance functions introduced by Khan *et al.* [17] and Ansari *et al.* [18,19]. Choudhury *et al.* [20] presented open problems involving inequality-based couplings, which were subsequently addressed by Aydi *et al.* [21] through strong coupled fixed point results for (ϕ, ψ)-contraction-type couplings in partial metric spaces. Further contributions were made by Rashid *et al.* [22] and D. Eshi *et al.* [23], who introduced SCC-Map and ϕ-contraction-type T-coupling to establish coincidence point theorems, while Fuad Abdulkerim *et al.* [24] expanded this framework by proving fixed point results for (ϕ, ψ)-contraction-type T-coupling mappings.

The aim of this study is to derive unique strong common coupled fixed point (USCCFP) theorems within the framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$, focusing on twisted (α, β)-φ-ψ-integral contractive type $\mathbb{T}$-coupling SCC-maps. In addition, the paper explores applications of these results to system of nonlinear Fredholm integral equations and homotopy theory, emphasizing their significance and potential impact on further research developments.

2. Preliminaries

This section provides a brief introduction to some fundamental aspects of C^*-algebra theory [25]. Let $\mathscr{A}$ be a unital C^*-algebra with the unit element $1_{\mathscr{A}}$. Define $\mathscr{A}_h = \{\mathfrak{l} \in \mathscr{A} : \mathfrak{l} = \mathfrak{l}^*\}$. An element $\mathfrak{l} \in \mathscr{A}$ is considered positive, denoted as $\mathfrak{l} \geq 0_{\mathscr{A}}$, if $\mathfrak{l} = \mathfrak{l}^*$ and its spectrum $\eta(\mathfrak{l}) \subseteq [0, \infty)$. Here, $0_{\mathscr{A}}$ in $\mathscr{A}$ represents the zero element in $\mathscr{A}$, and $\eta(\mathfrak{l})$ denotes the spectrum of $\mathfrak{l}$. On $\mathscr{A}_h$, a natural partial ordering is defined by $\varkappa \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \varkappa \succeq 0_{\mathscr{A}}$. We denote $\mathscr{A}_+ = \{\mathfrak{l} \in \mathscr{A} : \mathfrak{l} \succeq 0_{\mathscr{A}}\}$ and $\mathscr{A}' = \{\mathfrak{l} \in \mathscr{A} : \mathfrak{l}\mathfrak{d} = \mathfrak{d}\mathfrak{l}\ \forall \mathfrak{d} \in \mathscr{A}\}$.

Definition 2.1. *([9,10]) Let $\mathbb{V}$ be a non-empty set and denote the associated C^*-algebra by $\mathscr{A}$. A mapping $\varrho_{c^*} : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathscr{A}$ that satisfies the required conditions is referred to as a C^*-algebra-valued G-metric.*

(i) $\varrho_{c^*}(\varkappa_1, \varkappa_2, \varkappa_3) = 0_{\mathscr{A}}$ *if* $\varkappa_1 = \varkappa_2 = \varkappa_2$,
(ii) $0_{\mathscr{A}} \prec \varrho_{c^*}(\varkappa_1, \varkappa_1, \varkappa_2)$ *for all* $\varkappa_1, \varkappa_2 \in \mathbb{V}$ *with* $\varkappa_1 \neq \varkappa_2$,
(iii) $\varrho_{c^*}(\varkappa_1, \varkappa_1, \varkappa_2) \preceq \varrho_{c^*}(\varkappa_1, \varkappa_2, \varkappa_3)$ *for all* $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{V}$ *with* $\varkappa_1 \neq \varkappa_3$,
(iv) $\varrho_{c^*}(\varkappa_1, \varkappa_2, \varkappa_3) = \varrho_{c^*}(P[\varkappa_1, \varkappa_2, \varkappa_3])$ *where P is a permutation of* $\varkappa_1, \varkappa_2, \varkappa_3$ *(symmetry),*
(v) $\varrho_{c^*}(\varkappa_1, \varkappa_2, \varkappa_3) \preceq \varrho_{c^*}(\varkappa_1, \varkappa_4, \varkappa_4) + \varrho_{c^*}(\varkappa_4, \varkappa_2, \varkappa_3)$ *for all* $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in \mathbb{V}$ *(rectangle inequality)*

Then the structure $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ *is called a* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$.

Example 2.1. *([9,10]) Let $\mathbb{V} = \mathbb{R}$ and define $\varrho_{c^*} : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathscr{A}$ as $\varrho_{c^*}(\varkappa_1, \varkappa_2, \varkappa_3) = \|\varkappa_1 - \varkappa_2\| I_{\mathscr{A}} + \|\varkappa_2 - \varkappa_3\| I_{\mathscr{A}} + \|\varkappa_3 - \varkappa_1\| I_{\mathscr{A}}$ for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{V}$ then $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}\mathbb{G}\mathbb{MS}$. ϱ_{c^*} is a C^*-algebra valued G-metric.*

Definition 2.2. *([9,10])A $\mathscr{C}^*$-$\mathscr{A}$$\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is said to be symmetric if*

$$\varrho_{c^*}(\varkappa_1, \varkappa_1, \varkappa_2) = \varrho_{c^*}(\varkappa_2, \varkappa_2, \varkappa_1) \ \forall \ \varkappa_1, \varkappa_2 \in \mathbb{V}$$

Definition 2.3. *([9,10]) Assume that $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}$$\mathbb{V}$-G-$\mathbb{MS}$. According to $\mathscr{A}$ a sequence $\{\varkappa_k\}$ in $\mathbb{V}$ is defined as:*

(1) *C^*-algebra valued G-convergent to a point $\varkappa \in \mathbb{V}$ if, for each $0_{\mathscr{A}} \prec \epsilon$, there exist $x, y \in \mathbb{N}$ such that $\varrho_{c^*}(\varkappa, \varkappa_x, \varkappa_y) \prec \epsilon$. We can also use different presentations for that as follows:*

$$\varkappa_x \to \varkappa \text{ or } \lim_{x\to\infty} \varrho_{c^*}(\varkappa, \varkappa_x, \varkappa_y) = 0_{\mathscr{A}} \text{ or } \lim_{x\to\infty} \varkappa_x = \varkappa.$$

(2) *C^*-algebra valued G-Cauchy sequence, if for $0_{\mathscr{A}} \prec \epsilon$, there exists positive integer $x^* \in \mathbb{N}$ such that $\varrho_{c^*}(\varkappa_x, \varkappa_y, \varkappa_z) \prec \epsilon \ \forall \ x, y, z \geq x^*$ or $\varrho_{c^*}(\varkappa_x, \varkappa_y, \varkappa_z) \to 0_{\mathscr{A}}$ as $x, y, z \to \infty$ or $\|\varrho_{c^*}(\varkappa_x, \varkappa_y, \varkappa_z)\| \to 0$.*

(3) *It is referred to as being complete when a $\mathscr{C}^*$-$\mathscr{A}$$\mathbb{VGMS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is present. If each Cauchy sequence in $\mathbb{V}$ converges to a point in $\mathbb{V}$.*

Lemma 2.1. *([9,10]) Let $\mathscr{A}$ be a C^*-algebra with the identity element $I_{\mathscr{A}}$ and $\mathfrak{v}$ be a positive element of $\mathscr{A}$.Then*

(i) *There is a unique element $\mathfrak{u} \in \mathscr{A}_+$ such that $\mathfrak{u}^2 = \mathfrak{v}$.*

(ii) *The set $\mathscr{A}_+ = \{\mathfrak{v}^*\mathfrak{v} / \mathfrak{v} \in \mathscr{A}\}$ with a conjugate-linear involution $* : \mathscr{A} \to \mathscr{A}$.*

(iii) *$\mathfrak{v}, \mathfrak{u} \in \mathscr{A}$, and $0_{\mathscr{A}} \leq \mathfrak{v} \leq \mathfrak{u}$ then $\|\mathfrak{v}\| \leq \|\mathfrak{u}\|$.*

(iv) *If $\mathfrak{v} \in \mathscr{A}_+$ with $\|\mathfrak{v}\| < \frac{1}{2}$ then $(I - \mathfrak{v})$ is invertible and $\|\mathfrak{v}(I - \mathfrak{v})^{-1}\| < 1$*

3. Main Results

Definition 3.1. *Let $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}$$\mathbb{V}$-G-$\mathbb{MS}$ and a pair $(\varkappa, \mathfrak{v}) \in \mathbb{V} \times \mathbb{V}$ is called*

(a) *a CFP of mapping $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ if $\mathbb{Q}(\varkappa, \mathfrak{v}) = \varkappa$, $\mathbb{Q}(\mathfrak{v}, \varkappa) = \mathfrak{v}$;*

(a_i) *a SCFP of mapping $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ if $(\varkappa, \mathfrak{v})$ is CFP and $\varkappa = \mathfrak{v}$ i.e $\mathbb{Q}(\varkappa, \varkappa) = \varkappa$;*

(b) *a CCIP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathbb{Q}(\varkappa, \mathfrak{v}) = \mathbb{T}\varkappa$, $\mathbb{Q}(\mathfrak{v}, \varkappa) = \mathbb{T}\mathfrak{v}$;*

(b_i) *a SCCIP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\varkappa = \mathfrak{v}$. i.e $\mathbb{Q}(\varkappa, \varkappa) = \mathbb{T}\varkappa$;*

(c) *a CCFP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathbb{Q}(\varkappa, \mathfrak{v}) = \mathbb{T}\varkappa = \varkappa$, $\mathbb{Q}(\mathfrak{v}, \varkappa) = \mathbb{T}\mathfrak{v} = \mathfrak{v}$;*

(c_i) *a SCCFP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\varkappa = \mathfrak{v}$. i.e $\mathbb{Q}(\varkappa, \varkappa) = \mathbb{T}\varkappa = \varkappa$;*

(d) *the pair $(\mathbb{Q}, \mathbb{T})$ is weakly compatible (ω-compt) if $\mathbb{T}(\mathbb{Q}(\varkappa, \mathfrak{v})) = \mathbb{Q}(\mathbb{T}\varkappa, \mathbb{T}\mathfrak{v})$ and $\mathbb{T}(\mathbb{Q}(\mathfrak{v}, \varkappa)) = \mathbb{Q}(\mathbb{T}\mathfrak{v}, \mathbb{T}\varkappa)$ whenever $\mathbb{Q}(\varkappa, \mathfrak{v}) = \mathbb{T}\varkappa$, $\mathbb{Q}(\mathfrak{v}, \varkappa) = \mathbb{T}\mathfrak{v}$.*

Definition 3.2. *Let $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}$$\mathbb{V}$-G-$\mathbb{MS}$, $\mathcal{F}$ and $\mathcal{G}$ be two nonempty subsets of $\mathbb{V}$. Then a function $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be a coupling with respect to $\mathcal{F}$ and $\mathcal{G}$ if $\mathbb{Q}(\varkappa, \mathfrak{v}) \in \mathcal{G}$ and $\mathbb{Q}(\mathfrak{v}, \varkappa) \in \mathcal{F}$ where $\varkappa \in \mathcal{F}$ and $\mathfrak{v} \in \mathcal{G}$.*

Definition 3.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonempty subsets of $\mathbb{V}$. Any function $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is said to be*

(i) *a cyclic (with respect to $\mathcal{F}$ and $\mathcal{G}$) if $\mathbb{T}(\mathcal{F}) \subset \mathcal{G}$ and $\mathbb{T}(\mathcal{G}) \subset \mathcal{F}$.*

(ii) *a self-cyclic (with respect to $\mathcal{F}$ and $\mathcal{G}$) if $\mathbb{T}(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathbb{T}(\mathcal{G}) \subseteq \mathcal{G}$.*

Definition 3.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonempty subsets of a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ and a self map $\mathbb{T}: \mathbb{V} \to \mathbb{V}$ is said to be (self-cyclic compatible map) SCC-Map with respect to $\mathcal{F}$ and $\mathcal{G}$, if*

(*i*) *$\mathbb{T}$ is self-cyclic with respect to $\mathcal{F}$ and $\mathcal{G}$ i.e $\mathbb{T}(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathbb{T}(\mathcal{G}) \subseteq \mathcal{G}$*

(*ii*) *$\mathbb{T}(\mathcal{F})$ and $\mathbb{T}(\mathcal{G})$ are closed in $\mathbb{V}$.*

Definition 3.5. *Let $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$, $\mathbb{T}: \mathbb{V} \to \mathbb{V}$ and $\alpha, \beta : \mathbb{V}^3 \to \mathscr{A}_+$ be a functions then $\mathbb{T}$ is called a twisted (α, β)-adm, if for all $\varkappa, \mathfrak{l} \in \mathbb{V}$*

$$\begin{cases} \alpha(\varkappa, \varkappa, \mathfrak{l}) \geq 1_{\mathscr{A}} \\ \beta(\varkappa, \varkappa, \mathfrak{l}) \geq 1_{\mathscr{A}} \end{cases} \Rightarrow \begin{cases} \alpha(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}, \mathbb{T}\varkappa) \geq 1_{\mathscr{A}} \\ \beta(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}, \mathbb{T}\varkappa) \geq 1_{\mathscr{A}} \end{cases}$$

Example 3.1. *Let $\mathbb{V} = \mathbb{R}$ and $\mathscr{A} = \mathbb{M}_2(\mathbb{C})$, the 2×2 complex matrices, which form a C^*-algebra. Define the $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-metric $\varrho_{c^*} : \mathbb{R}^3 \to \mathbb{M}_2(\mathbb{C})$ by*

$$\varrho_{c^*}(x, y, z) = \begin{pmatrix} |x - y| & 0 \\ 0 & |y - z| + |z - x| \end{pmatrix}.$$

Define $\mathbb{T}(x) = -x$ for $x \in \mathbb{R}$, and functions $\alpha, \beta : \mathbb{R}^3 \to \mathbb{M}_2(\mathbb{C})_+$ by

$$\alpha(x, y, z) = \begin{pmatrix} 1 + |x| & 0 \\ 0 & 1 + |z| \end{pmatrix}, \quad \beta(x, y, z) = \begin{pmatrix} 2 + |y| & 0 \\ 0 & 2 + |z| \end{pmatrix}.$$

Choose $\varkappa, \mathfrak{l} \in \mathbb{R}$ such that

$$\alpha(\varkappa, \varkappa, \mathfrak{l}) \geq I_{\mathscr{A}} \quad \text{and} \quad \beta(\varkappa, \varkappa, \mathfrak{l}) \geq I_{\mathscr{A}}.$$

This is true since

$$\alpha(\varkappa, \varkappa, \mathfrak{l}) = \begin{pmatrix} 1 + |\varkappa| & 0 \\ 0 & 1 + |\mathfrak{l}| \end{pmatrix} \geq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathscr{A}},$$

and

$$\beta(\varkappa, \varkappa, \mathfrak{l}) = \begin{pmatrix} 2 + |\varkappa| & 0 \\ 0 & 2 + |\mathfrak{l}| \end{pmatrix} \geq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathscr{A}}.$$

Now verify the twisted (α, β)-admissibility:

$$\alpha(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}, \mathbb{T}\varkappa) = \begin{pmatrix} 1 + |-\mathfrak{l}| & 0 \\ 0 & 1 + |-\varkappa| \end{pmatrix} = \begin{pmatrix} 1 + |\mathfrak{l}| & 0 \\ 0 & 1 + |\varkappa| \end{pmatrix} \geq I_{\mathscr{A}},$$

$$\beta(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}, \mathbb{T}\varkappa) = \begin{pmatrix} 2 + |-\mathfrak{l}| & 0 \\ 0 & 2 + |-\varkappa| \end{pmatrix} = \begin{pmatrix} 2 + |\mathfrak{l}| & 0 \\ 0 & 2 + |\varkappa| \end{pmatrix} \geq I_{\mathscr{A}}.$$

Hence, $\mathbb{T}$ is a twisted (α, β)-admissible mapping on $(\mathbb{R}, \mathbb{M}_2(\mathbb{C}), \varrho_{c^*})$ with respect to α and β.

Definition 3.6. *A function $\varphi : \mathscr{A}_+ \to \mathscr{A}_+$ is called an altering distance function if the following properties are satisfied:*

(*a*) *φ is monotonically non-deceasing and continuous;*

(*b*) *$\varphi(\varkappa) = 0_{\mathscr{A}}$ if and only if $\varkappa = 0_{\mathscr{A}}$.*

The family of all altering distance functions is denoted by Ω

Lemma 3.1. *Let $\{\varkappa_n\}_{n=1}^{\infty} \subset \mathscr{A}_+$ be a sequence of non-negative elements in a unital C*-algebra $\mathscr{A}$, and let $\psi : \mathscr{A}_+ \to \mathscr{A}_+$ denote a Lebesgue-integrable function that is summable on every compact subset of $\mathscr{A}_+$. Suppose further that for each $\epsilon > 0_{\mathscr{A}}$, we have $\int_0^{\epsilon} \psi(\mathfrak{k})\, d\mathfrak{k} > 0_{\mathscr{A}}$. If $\varkappa_n \to \varkappa$ in norm as $n \to \infty$, then*

$$\lim_{n\to\infty} \int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} = \int_0^{\varkappa} \psi(\mathfrak{k})\, d\mathfrak{k}.$$

Proof. Consider that the norm convergence $\varkappa_n \to \varkappa$ ensures that for any $\delta > 0$ there exists N such that $\|\varkappa_n - \varkappa\|_{\mathscr{A}} < \delta$ for all $n \geq N$.

Assume for large n that $\varkappa_n \preceq \varkappa$. Using the additivity of the integral and the properties of C*-algebra-valued functions, one has

$$\int_0^{\varkappa} \psi(\mathfrak{k})\, d\mathfrak{k} - \int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} = \int_{\varkappa_n}^{\varkappa} \psi(\mathfrak{k})\, d\mathfrak{k}.$$

Thanks to the integrability and compact summability, the integral over $[\varkappa_n, \varkappa]$ vanishes as $\varkappa_n$ approaches $\varkappa$. Thus,

$$\left\| \int_0^{\varkappa} \psi(\mathfrak{k})\, d\mathfrak{k} - \int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} \right\|_{\mathscr{A}} \to 0$$

as $n \to \infty$, establishing that

$$\int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} \to \int_0^{\varkappa} \psi(\mathfrak{k})\, d\mathfrak{k}$$

in the norm topology. An analogous argument applies if $\varkappa \preceq \varkappa_n$ by exchanging the bounds of the difference integral. This completes the proof. □

Lemma 3.2. *Let $\{\varkappa_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{A}_+$ such that each term is non-negative. If $\psi : \mathscr{A}_+ \to \mathscr{A}_+$ is a Lebesgue-integrable map, summable over compact subsets, and for every $\epsilon > 0_{\mathscr{A}}$, the integral $\int_0^{\epsilon} \psi(\mathfrak{k})\, d\mathfrak{k}$ is strictly positive, then*

$$\lim_{n\to\infty} \int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} = 0_{\mathscr{A}} \iff \lim_{n\to\infty} \varkappa_n = 0_{\mathscr{A}}.$$

Proof. First, suppose $\varkappa_n \to 0_{\mathscr{A}}$ in norm. By norm continuity of integration, for any $\eta > 0$ there exists a stage N such that for all $n \geq N$, the quantity $\|\varkappa_n\| < \eta$. As ψ is integrable and positive over compact intervals, it follows that $\int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k}$ attains values with arbitrarily small norm as $\|\varkappa_n\| \to 0$, and thus the sequence of integrals approaches $0_{\mathscr{A}}$.

Conversely, assume $\lim_{n\to\infty} \int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k} = 0_{\mathscr{A}}$ but suppose that $\varkappa_n$ does not approach $0_{\mathscr{A}}$. This means, for some $\delta > 0_{\mathscr{A}}$, infinitely many $\varkappa_n$ have norm at least δ. Considering such an $\epsilon := \delta 1_{\mathscr{A}}$, and utilizing the positivity of the integral over every neighborhood of $0_{\mathscr{A}}$, we see that each corresponding $\int_0^{\varkappa_n} \psi(\mathfrak{k})\, d\mathfrak{k}$ is bounded below by a positive element, thus the sequence cannot converge to $0_{\mathscr{A}}$, yielding a contradiction.

Therefore, the two forms of convergence are equivalent under the given conditions. □

Definition 3.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonempty subsets of a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is a SCC-map on $\mathbb{V}$ (with respect to $\mathcal{F}$ and $\mathcal{G}$). Then a coupling $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be twisted* (α, β)-φ-ψ-*integral type contractive mapping of $\mathbb{T}$-coupling (with respect to $\mathcal{F}$ and $\mathcal{G}$) if there exist* *altering distance functions* $\varphi \in \Omega$, $\alpha, \beta : \mathbb{V}^3 \to \mathscr{A}_+$ *and* $\psi : \mathscr{A}_+ \to \mathscr{A}_+$ *be a Lebesgue-integrable function that is summable on every compact subset of* $\mathscr{A}_+$ *and for each* $\epsilon > 0_{\mathscr{A}}$, *we have* $\int_0^{\epsilon} \psi(\mathfrak{k})\, d\mathfrak{k} > 0_{\mathscr{A}}$ *such that for all* $\varkappa, \mathfrak{z} \in \mathcal{F}$, $\mathfrak{l}, \mathfrak{v} \in \mathcal{G}$ *and* $a \in \mathscr{A}$ *with* $\|a\| < 1$,

$$\int_0^{\alpha(\mathbb{T}\varkappa,\mathbb{T}\varkappa,\mathbb{T}\mathfrak{l})\beta(\mathbb{T}\mathfrak{v},\mathbb{T}\mathfrak{v},\mathbb{T}\mathfrak{z})\varphi(\varrho_{C^*}(\mathbb{Q}(\varkappa,\mathfrak{v}),\mathbb{Q}(\varkappa,\mathfrak{v}),\mathbb{Q}(\mathfrak{l},\mathfrak{z})))} \psi(\mathfrak{k})\, d\mathfrak{k} \leq a^{\star} \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{C^*}(\mathbb{T}\varkappa, \mathbb{T}\varkappa, \mathbb{T}\mathfrak{l}), \\ \varrho_{C^*}(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z}) \end{array} \right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right) a$$

Theorem 3.1. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be nonempty closed subsets of a complete* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ *space* $(\mathbb{V}, \mathscr{A}, \varrho_{C^*})$. *Assume that* $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ *is an SCC-map with respect to* $\mathcal{F}$ *and* $\mathcal{G}$. *Let* $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ *be a twisted* (α, β)-φ-ψ-*integral type contractive* $\mathbb{T}$-*coupling with respect to* $\mathcal{F}$ *and* $\mathcal{G}$. *Suppose the following conditions hold:*

1. $\mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G}) \neq \emptyset$, *and* $\mathbb{Q}(\mathcal{F} \times \mathcal{G}) \subseteq \mathbb{T}(\mathcal{G})$, $\quad \mathbb{Q}(\mathcal{G} \times \mathcal{F}) \subseteq \mathbb{T}(\mathcal{F})$.
2. $\mathbb{T}$ *is a twisted* (α, β)-*admissible mapping.*
3. *The pair* $(\mathbb{Q}, \mathbb{T})$ *has a common coincidence point(CCIP) in* $\mathcal{F} \times \mathcal{G}$.
4. *The pair* $\{\mathbb{Q}, \mathbb{T}\}$ *is* ω-*compatible.*
5. *For sequences* $\{\varkappa_n\} \subseteq \mathcal{F}$ *and* $\{\mathfrak{v}_n\} \subseteq \mathcal{G}$, *if*
   $\alpha(\mathbb{T}\varkappa_n, \mathbb{T}\varkappa_{n+1}, \mathbb{T}\varkappa_{n+1}) \geq I_{\mathscr{A}}$, $\quad \beta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}_{n+1}, \mathbb{T}\mathfrak{v}_{n+1}) \geq I_{\mathscr{A}}$, *for all n, and*
   $\lim_{n\to\infty} \mathbb{T}\varkappa_n = \mathbb{T}\varkappa \in \mathbb{T}(\mathcal{F})$, $\quad \lim_{n\to\infty} \mathbb{T}\mathfrak{v}_n = \mathbb{T}\mathfrak{v} \in \mathbb{T}(\mathcal{G})$, *then*
   $\alpha(\mathbb{T}\varkappa_n, \mathbb{T}\varkappa, \mathbb{T}\varkappa) \geq I_{\mathscr{A}}$, $\quad \beta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}) \geq I_{\mathscr{A}}$.
6. *For any distinct points* $\mathbb{T}\varkappa \neq \mathbb{T}\varkappa^{\star}$ *and* $\mathbb{T}\mathfrak{l} \neq \mathbb{T}\mathfrak{l}^{\star}$, $\alpha(\mathbb{T}\varkappa, \mathbb{T}\varkappa^{\star}, \mathbb{T}\varkappa^{\star}) \geq I_{\mathscr{A}}$, $\quad \beta(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}^{\star}, \mathbb{T}\mathfrak{l}^{\star}) \geq I_{\mathscr{A}}$.

Then, the pair $(\mathbb{Q}, \mathbb{T})$ *admits a unique strong coupled common fixed point (SCCFP) in* $\mathcal{F} \times \mathcal{G}$.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are non-empty subsets of $\mathbb{V}$ and $\mathbb{Q}$ is twisted (α, β)-φ-ψ-integral type contractive mapping of $\mathbb{T}$-coupling with respect $\mathcal{F}$ and $\mathcal{G}$, then for $\varkappa_0 \in \mathcal{F}$ and $\mathfrak{l}_0 \in \mathcal{G}$ such that $\alpha(\mathbb{Q}(\varkappa_0, \mathfrak{l}_0), \mathbb{Q}(\varkappa_0, \mathfrak{l}_0), \mathbb{T}\mathfrak{l}_0) \geq I_{\mathscr{A}}$ and $\beta(\mathbb{Q}(\mathfrak{l}_0, \varkappa_0), \mathbb{Q}(\mathfrak{l}_0, \varkappa_0), \mathbb{T}\varkappa_0) \geq I_{\mathscr{A}}$, we define the sequence $\{\varkappa_{\mathfrak{v}}\}$ and $\{\mathfrak{l}_{\mathfrak{v}}\}$ in $\mathcal{F}$ and $\mathcal{G}$ respectively such that

$$\mathbb{T}\varkappa_{\mathfrak{v}+1} = \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}}) \quad \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1} = \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}}) \ \forall\ \mathfrak{v} \in \mathbb{N} \cup \{0\}.$$

If for some $\mathfrak{v}$, $\mathbb{T}\varkappa_{\mathfrak{v}+1} = \mathbb{T}\mathfrak{l}_{\mathfrak{v}}$ and $\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1} = \mathbb{T}\varkappa_{\mathfrak{v}}$ then, we have $\mathbb{T}\varkappa_{\mathfrak{v}} = \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1} = \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}})$ and $\mathbb{T}\mathfrak{l}_{\mathfrak{v}} = \mathbb{T}\varkappa_{\mathfrak{v}+1} = \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}})$. This show that $(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}})$ is a coupled coincidence point of $\mathbb{Q}$ and $\mathbb{T}$. So, we are done in this case. Thus we assume that $\mathbb{T}\varkappa_{\mathfrak{v}+1} \neq \mathbb{T}\mathfrak{l}_{\mathfrak{v}}$ and $\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1} \neq \mathbb{T}\varkappa_{\mathfrak{v}}$ for all $\mathfrak{v} \geq 0$.

Since $\mathbb{T}$ is a twisted (α, β)-adm, then $\alpha(\mathfrak{l}_0, \varkappa_1, \varkappa_1) = \alpha(\mathfrak{l}_0, \mathbb{T}\varkappa_0, \mathbb{T}\varkappa_0) \geq 1_{\mathscr{A}}$ then
$\alpha(\varkappa_2, \varkappa_2, \mathfrak{l}_1) = \alpha(\mathbb{T}\varkappa_1, \mathbb{T}\varkappa_1, \mathbb{T}\mathfrak{l}_0) \geq 1_{\mathscr{A}} \implies \alpha(\mathfrak{l}_2, \varkappa_3, \varkappa_3) = \alpha(\mathbb{T}\mathfrak{l}_1, \mathbb{T}\varkappa_2, \mathbb{T}\varkappa_2) \geq 1_{\mathscr{A}}$.
By repeating similar process, we obtain $\alpha(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}) \geq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}+1}, \mathbb{T}\varkappa_{\mathfrak{v}+1}) \geq 1_{\mathscr{A}}$ $\forall$ $\mathfrak{v} \in \mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}-1}) \geq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}) \geq 1_{\mathscr{A}}$ $\forall$ $\mathfrak{v} \in \mathbb{N}$. and again $\alpha(\mathfrak{l}_1, \mathfrak{l}_1, \varkappa_0) = \alpha(\mathbb{T}\mathfrak{l}_0, \mathbb{T}\mathfrak{l}_0, \varkappa_0) \geq 1_{\mathscr{A}}$ then $\alpha(\varkappa_1, \mathfrak{l}_2, \mathfrak{l}_2) = \alpha(\mathbb{T}\varkappa_0, \mathbb{T}\mathfrak{l}_1, \mathbb{T}\mathfrak{l}_1) \geq 1_{\mathscr{A}} \implies \alpha(\mathfrak{l}_3, \mathfrak{l}_3, \varkappa_2) =$

$\alpha(\mathbb{T}\mathfrak{l}_2, \mathbb{T}\mathfrak{l}_2, \mathbb{T}\varkappa_1) \geq 1_{\mathscr{A}}$. By repeating similar process, we obtain $\alpha(\mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}) \geq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\varkappa_{\mathfrak{v}}) \geq 1_{\mathscr{A}}\ \forall\ \mathfrak{v} \in \mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}) \geq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\varkappa_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}) \geq 1_{\mathscr{A}}\ \forall\ \mathfrak{v} \in \mathbb{N}$.
Also, $\alpha(\mathfrak{l}_0, \varkappa_0, \varkappa_0) \geq 1_{\mathscr{A}}$ then $\alpha(\varkappa_1, \varkappa_1, \mathfrak{l}_1) = \alpha(\mathbb{T}\varkappa_0, \mathbb{T}\varkappa_0, \mathbb{T}\mathfrak{l}_0) \geq 1_{\mathscr{A}} \implies \alpha(\mathfrak{l}_2, \varkappa_2, \varkappa_2) = \alpha(\mathbb{T}\mathfrak{l}_1, \mathbb{T}\varkappa_1, \mathbb{T}\varkappa_1) \geq 1_{\mathscr{A}}$. By repeating similar process, we obtain $\alpha(\mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}) \geq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}) \geq 1_{\mathscr{A}}\ \forall\ \mathfrak{v} \in \mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}-1}) \geq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}) \geq 1_{\mathscr{A}}\ \forall\ \mathfrak{v} \in \mathbb{N}$.

Now, fact that $\varkappa_{\mathfrak{v}} \in \mathcal{F}$ and $\mathfrak{l}_{\mathfrak{v}} \in \mathcal{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}
\int_0^{\varphi(\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}))} \psi(\mathfrak{t})\, d\mathfrak{t} &= \int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1}, \varkappa_{\mathfrak{v}-1}), \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}}), \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}})))} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&= \int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}})), \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1}, \varkappa_{\mathfrak{v}-1}))} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\preceq \int_0^{\left(\begin{array}{c} \alpha(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}-1}) \\ \varphi\left(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}})), \mathbb{Q}(\varkappa_{\mathfrak{v}}, \mathfrak{l}_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1}, \varkappa_{\mathfrak{v}-1})\right) \end{array}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\preceq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}), \\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}-1}), \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a. \qquad (3.1)
\end{aligned}$$

Now, fact that $\varkappa_{\mathfrak{v}} \in \mathcal{F}$ and $\mathfrak{l}_{\mathfrak{v}} \in \mathcal{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}
\int_0^{\varphi(\varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}+1}, \mathbb{T}\varkappa_{\mathfrak{v}+1}))} \psi(\mathfrak{t})\, d\mathfrak{t} &= \int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}-1}, \mathfrak{l}_{\mathfrak{v}-1}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}})))} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\preceq \int_0^{\left(\begin{array}{c} \alpha(\mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}})\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}) \\ \varphi\left(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}-1}, \mathfrak{l}_{\mathfrak{v}-1}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{l}_{\mathfrak{v}}, \varkappa_{\mathfrak{v}}))\right) \end{array}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\preceq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}), \\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}), \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a. \qquad (3.2)
\end{aligned}$$

By using (3.1) and (3.2), we get

$$\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}) \\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}+1}, \mathbb{T}\varkappa_{\mathfrak{v}+1}) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \preceq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}-1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}), \\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}, \mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}}), \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a$$

$$\preceq (a^*)^2\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}-2},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}),\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-2},\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\varkappa_{\mathfrak{v}-1}),\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^2$$

$$\vdots$$

$$\preceq (a^*)^{\mathfrak{v}}\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_1,\mathbb{T}\mathfrak{l}_1),\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_0,\mathbb{T}\varkappa_1,\mathbb{T}\varkappa_1),\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^{\mathfrak{v}}.$$

Using Lemma 2.1, we have

$$\begin{aligned}\left\|\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1})\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\| &\preceq \left\|(a^*)^{\mathfrak{v}}\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_1,\mathbb{T}\mathfrak{l}_1),\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_0,\mathbb{T}\varkappa_1,\mathbb{T}\varkappa_1),\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^{\mathfrak{v}}\right\|\\ &\preceq \|a\|^{2\mathfrak{v}}\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_1,\mathbb{T}\mathfrak{l}_1),\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_0,\mathbb{T}\varkappa_1,\mathbb{T}\varkappa_1),\end{array}\right\}\right)}\|\psi(\mathfrak{k})\|\,d\mathfrak{k}.\end{aligned}$$

Since $\|a\|<1$ and passing to the limit as $\mathfrak{v}\to\infty$ in above equation, we obtain
$\lim_{\mathfrak{v}\to\infty}\left\|\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1})\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\|=0.$

Hence, by the property of integral ψ and by Lemma 3.2, we obtain
$\lim_{\mathfrak{v}\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1})\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)=0_{\mathscr{A}}$. Again by the properties of altering distance function φ, we obtain $\lim_{\mathfrak{v}\to\infty}\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1})\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1})\end{array}\right\}=0_{\mathscr{A}}$.

Thus, $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1})=0_{\mathscr{A}}$ and $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1})=0_{\mathscr{A}}$.

Now, we define a sequence $\{\Gamma_{\mathfrak{v}}\}$ by $\Gamma_{\mathfrak{v}}=\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}})$ and show that $\Gamma_{\mathfrak{v}}\to 0_{\mathscr{A}}$ as $\mathfrak{v}\to\infty$. For each $\varkappa_{\mathfrak{v}}\in\mathcal{F}$ and $\mathfrak{l}_{\mathfrak{v}}\in\mathcal{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}\int_0^{\varphi(\Gamma_{\mathfrak{v}})}\psi(\mathfrak{k})\,d\mathfrak{k} &= \int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}}))}\psi(\mathfrak{k})\,d\mathfrak{k}=\int_0^{\varphi(\varrho_{C^*}(\mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1},\varkappa_{\mathfrak{v}-1}),\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1}),\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1})))}\psi(\mathfrak{k})\,d\mathfrak{k}\\ &= \int_0^{\varphi(\varrho_{C^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1})),\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1},\varkappa_{\mathfrak{v}-1}))}\psi(\mathfrak{k})\,d\mathfrak{k}\\ &\preceq \int_0^{\left(\begin{array}{c}\alpha(\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\varkappa_{\mathfrak{v}-1})\\ \varphi(\varrho_{C^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1})),\mathbb{Q}(\varkappa_{\mathfrak{v}-1},\mathfrak{l}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{l}_{\mathfrak{v}-1},\varkappa_{\mathfrak{v}-1}))\end{array}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\end{aligned}$$

$$
\begin{aligned}
&\preceq a^{\star}\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1})\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\varkappa_{\mathfrak{v}-1})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)a\\
&\preceq a^{\star}\left(\int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-1}))}\psi(\mathfrak{k})\,d\mathfrak{k}\right)a\\
&\preceq (a^{\star})^2\left(\int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}-2},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-2},\mathbb{T}\mathfrak{l}_{\mathfrak{v}-2}))}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^2\\
&\vdots\\
&\preceq (a^{\star})^{\mathfrak{v}}\left(\int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_0,\mathbb{T}\mathfrak{l}_0))}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^{\mathfrak{v}}
\end{aligned}
$$

Using Lemma 2.1, we have

$$
\begin{aligned}
\left\|\int_0^{\varphi(\Gamma_{\mathfrak{v}})}\psi(\mathfrak{k})\,d\mathfrak{k}\right\| &\le \left\|(a^{\star})^{\mathfrak{v}}\left(\int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_0,\mathbb{T}\mathfrak{l}_0))}\psi(\mathfrak{k})\,d\mathfrak{k}\right)(a)^{\mathfrak{v}}\right\|\\
&\le \|a\|^{2\mathfrak{v}}\int_0^{\varphi(\varrho_{C^*}(\mathbb{T}\varkappa_0,\mathbb{T}\mathfrak{l}_0,\mathbb{T}\mathfrak{l}_0))}\|\psi(\mathfrak{k})\|\,d\mathfrak{k}.
\end{aligned}
$$

Since $\|a\| < 1$ and passing to the limit as $\mathfrak{v}\to\infty$ in above equation, we obtain

$\lim_{\mathfrak{v}\to\infty}\|\int_0^{\varphi(\Gamma_{\mathfrak{v}})}\psi(\mathfrak{k})\,d\mathfrak{k}\| = 0$. Hence, by the property of integral ψ and by Lemma 3.2, we obtain $\lim_{\mathfrak{v}\to\infty}\varphi(\Gamma_{\mathfrak{v}}) = 0_{\mathscr{A}}$. Again by the properties of altering distance function φ, we obtain $\lim_{\mathfrak{v}\to\infty}\Gamma_{\mathfrak{v}} = 0_{\mathscr{A}}$. That is, $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}}) = 0_{\mathscr{A}}$. Now, we have

$$\lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1}) \preceq \lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}}) + \lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}+1},\mathbb{T}\varkappa_{\mathfrak{v}+1}) = 0_{\mathscr{A}}$$

and

$$\lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}) \preceq \lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\varkappa_{\mathfrak{v}}) + \lim_{\mathfrak{v}\to\infty}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}) = 0_{\mathscr{A}}$$

Now, we will prove that the sequences $\{\mathbb{T}\varkappa_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathfrak{l}_{\mathfrak{v}}\}$ are Cauchy sequences in $\mathbb{T}(\mathcal{F})$ and $\mathbb{T}(\mathcal{G})$ with regard to $\tilde{\mathscr{A}}$ respectively. If possible, let $\{\mathbb{T}\varkappa_{\mathfrak{v}}\}$ or $\{\mathbb{T}\mathfrak{l}_{\mathfrak{v}}\}$ is not a Cauchy sequence. Then there exist $\epsilon \succ 0_{\mathscr{A}}$, and a sequence of positive integer there exists two subsequences $\{\mathfrak{u}(j)\}$ and $\{\mathfrak{v}(j)\}$ such that for all positive integers j with $\mathfrak{v}(j) > \mathfrak{u}(j) > j$, we have

$$\max\left\{\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\right\} \succeq \epsilon. \tag{3.3}$$

Furthermore, corresponding to $\mathfrak{u}(j)$, we can choose $\mathfrak{v}(j)$ such that j is the smallest positive integer with $\mathfrak{v}(j) \ge \mathfrak{u}(j) > j$ and satisfying (3.3), then

$$\max\left\{\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\right\} \prec \epsilon \tag{3.4}$$

Now, from (3.3) and (3.4), we have

$$\epsilon \preceq \max\left\{\begin{array}{c}\varrho_{C^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{C^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}$$

$$
\begin{aligned}
&\preceq \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1})+\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1})+\varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}\\
&\preceq \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1})\end{array}\right\}+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}\\
&\prec \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1})\end{array}\right\}+\epsilon.
\end{aligned}
$$

Taking the limit as $j\to\infty$ in above expression, we obtain

$$
\lim_{\mathfrak{v}\to\infty}\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}=\epsilon
$$

By the rectangle inequality, we have

$$
\begin{aligned}
\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\} &\preceq \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}\\
&\prec \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\} \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1})\end{array}\right\} &\preceq \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1})\end{array}\right\}\\
&\preceq \max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{v}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)})\end{array}\right\}\\
&\quad+\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{u}(j)},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1})\end{array}\right\}. \qquad (3.6)
\end{aligned}
$$

Taking the limit as $j\to\infty$ in equations (3.5) and (3.6), we obtain

$$
\lim_{\mathfrak{v}\to\infty}\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1},\mathbb{T}\varkappa_{\mathfrak{u}(j)-1}),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1},\mathbb{T}\mathfrak{l}_{\mathfrak{u}(j)-1})\end{array}\right\}=\epsilon.
$$

Since, $\alpha(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}) \geq 1_{\mathscr{A}}$ and $\beta(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}) \geq 1_{\mathscr{A}}$. Then, we have

$$\begin{aligned}
\int_0^{\varphi\left(\varrho_{\mathcal{C}^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)}, \mathbb{T}\varkappa_{\mathfrak{u}(j)}, \mathbb{T}\varkappa_{\mathfrak{u}(j)})\right)} \psi(\mathfrak{t})\, d\mathfrak{t} &= \int_0^{\varphi\left(\varrho_{\mathcal{C}^*}(\mathbb{Q}(\mathrm{I}_{\mathfrak{v}(j)-1}, \varkappa_{\mathfrak{v}(j)-1}), \mathbb{Q}(\mathrm{I}_{\mathfrak{u}(j)-1}, \varkappa_{\mathfrak{u}(j)-1}), \mathbb{Q}(\mathrm{I}_{\mathfrak{u}(j)-1}, \varkappa_{\mathfrak{u}(j)-1}))\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\leq \int_0^{\left(\begin{array}{c} \alpha(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1})\beta(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}) \\ \varphi\left(\varrho_{\mathcal{C}^*}(\mathbb{Q}(\mathrm{I}_{\mathfrak{v}(j)-1}, \varkappa_{\mathfrak{v}(j)-1}), \mathbb{Q}(\mathrm{I}_{\mathfrak{u}(j)-1}, \varkappa_{\mathfrak{u}(j)-1}), \mathbb{Q}(\mathrm{I}_{\mathfrak{u}(j)-1}, \varkappa_{\mathfrak{u}(j)-1}))\right) \end{array}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\leq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{\mathcal{C}^*}(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}), \\ \varrho_{\mathcal{C}^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a.
\end{aligned}$$

Similarly, we can show by the same steps that

$$\int_0^{\varphi\left(\varrho_{\mathcal{C}^*}(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)})\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \leq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{\mathcal{C}^*}(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}), \\ \varrho_{\mathcal{C}^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a.$$

Therefore, we conclude that

$$\begin{aligned}
&\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{\mathcal{C}^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)}, \mathbb{T}\varkappa_{\mathfrak{u}(j)}, \mathbb{T}\varkappa_{\mathfrak{u}(j)}), \\ \varrho_{\mathcal{C}^*}(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)}) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\leq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{\mathcal{C}^*}(\mathbb{T}\mathrm{I}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}, \mathbb{T}\mathrm{I}_{\mathfrak{u}(j)-1}), \\ \varrho_{\mathcal{C}^*}(\mathbb{T}\varkappa_{\mathfrak{v}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}, \mathbb{T}\varkappa_{\mathfrak{u}(j)-1}) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a.
\end{aligned}$$

Now, passing to the limit as $j \to \infty$ in above equation and using property of φ and Lemma 3.1, we obtain

$$\int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \leq a^* \left(\int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a.$$

which implies that

$$\| \int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \| \leq \| a^* \left(\int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a \| \leq \|a\|^2 \| \int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \|$$

which is a contradiction, since $\|a\| < 1$. So, we have $\| \int_0^{\varphi(\epsilon)} \psi(\mathfrak{t})\, d\mathfrak{t} \| = 0$. Again by the property of integral ψ, we obtain $\varphi(\epsilon) = 0_{\mathscr{A}}$ and so by the property of φ, we have $\epsilon = 0_{\mathscr{A}}$, which is a contradiction, since $\epsilon > 0_{\mathscr{A}}$. Hence, $\{\mathbb{T}\varkappa_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathrm{I}_{\mathfrak{v}}\}$ are Cauchy sequences in $\mathbb{T}(\mathcal{F})$ and $\mathbb{T}(\mathcal{G})$ with regard to $\tilde{\mathscr{A}}$ respectively. Since $\mathbb{T}(\mathcal{F})$ and $\mathbb{T}(\mathcal{G})$ are closed subset of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{\mathcal{C}^*})$, $\{\mathbb{T}\varkappa_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathrm{I}_{\mathfrak{v}}\}$ are convergent in $\mathbb{T}(\mathcal{F})$ and $\mathbb{T}(\mathcal{G})$ respectively. Thus, there exist

$\mathfrak{p} \in \mathbb{T}(\mathcal{F})$ and $\mathfrak{q} \in \mathbb{T}(\mathcal{G})$ such that

$$\lim_{\mathfrak{v}\to\infty} \mathbb{T}\varkappa_{\mathfrak{v}} = \mathfrak{p} \text{ and } \lim_{\mathfrak{v}\to\infty} \mathbb{T}\mathfrak{l}_{\mathfrak{v}} = \mathfrak{q}. \tag{3.7}$$

Since $\lim_{\mathfrak{v}\to\infty} \varrho_{c^*}(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}}) = 0_{\mathscr{A}} \implies \varrho_{c^*}(\mathfrak{p}, \mathfrak{q}, \mathfrak{q}) = 0_{\mathscr{A}}$ then, we have $\mathfrak{p} = \mathfrak{q}$.
As $\mathfrak{p} \in \mathbb{T}(\mathcal{F})$ and $\mathfrak{q} \in \mathbb{T}(\mathcal{G})$ it follows that $\mathfrak{p} = \mathfrak{q} \in \mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G})$ and hence,
$\mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G}) \neq \emptyset$. Now, since $\mathfrak{p} \in \mathbb{T}(\mathcal{F})$ and $\mathfrak{q} \in \mathbb{T}(\mathcal{G})$, there exist $\varkappa \in \mathcal{F}$ and $\mathfrak{l} \in \mathcal{G}$ such that $\mathfrak{p} = \mathbb{T}(\varkappa)$ and $\mathfrak{q} = \mathbb{T}(\mathfrak{l})$. From (3.7), we ahve $\lim_{\mathfrak{v}\to\infty} \mathbb{T}\varkappa_{\mathfrak{v}} = \mathfrak{p} = \mathbb{T}(\varkappa)$ and $\lim_{\mathfrak{v}\to\infty} \mathbb{T}\mathfrak{l}_{\mathfrak{v}} = \mathfrak{q} = \mathbb{T}(\mathfrak{l})$ and hence, $\mathbb{T}(\varkappa) = \mathbb{T}(\mathfrak{l})$. From condition (v), we have $\{\varkappa_{\mathfrak{v}}\}_{\mathfrak{v}=1}^{\infty} \subseteq \mathcal{F}$ and $\{\mathfrak{l}_{\mathfrak{v}}\}_{\mathfrak{v}=1}^{\infty} \subseteq \mathcal{G}$ with $\alpha(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa_{\mathfrak{v}+1}, \mathbb{T}\varkappa_{\mathfrak{v}+1}) \geq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}) \geq I_{\mathscr{A}}$ for all $\mathfrak{v}$ and $\lim_{\mathfrak{v}\to\infty} \mathbb{T}\varkappa_{\mathfrak{v}} = \mathfrak{p} = \mathbb{T}\varkappa \in \mathbb{T}(\mathcal{F})$ and $\lim_{\mathfrak{v}\to\infty} \mathbb{T}\mathfrak{l}_{\mathfrak{v}} = \mathfrak{q} = \mathbb{T}\mathfrak{l} \in \mathbb{T}(\mathcal{G})$ then $\alpha(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa, \mathbb{T}\varkappa) \geq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}) \geq I_{\mathscr{A}}$. Then, we have
$\varrho_{c^*}\left(\mathfrak{p}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right) \leq \varrho_{c^*}\left(\mathfrak{p}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}\right) + \varrho_{c^*}\left(\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right).$
Letting $\mathfrak{v} \to \infty$, we get
$\varrho_{c^*}\left(\mathfrak{p}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right) \leq \lim_{\mathfrak{v}\to\infty} \varrho_{c^*}\left(\mathbb{T}\mathfrak{l}_{\mathfrak{v}+1}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right).$
It follows that $\mathbb{Q}$ is a twisted (α,β)-φ-ψ-integral type contractive mapping of $\mathbb{T}$-coupling, then, we have

$$\begin{aligned}
\int_0^{\varphi(\varrho_{c^*}(\mathfrak{p},\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa,\mathfrak{l})))} \psi(\mathfrak{k})\, d\mathfrak{k} &\leq \lim_{\mathfrak{v}\to\infty} \int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}}),\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa,\mathfrak{l})))} \psi(\mathfrak{k})\, d\mathfrak{k} \\
&\leq \lim_{\mathfrak{v}\to\infty} \int_0^{\alpha(\mathbb{T}\varkappa_{\mathfrak{v}},\mathbb{T}\varkappa,\mathbb{T}\varkappa)\beta(\mathbb{T}\mathfrak{l}_{\mathfrak{v}},\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l})\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}}),\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa,\mathfrak{l})))} \psi(\mathfrak{k})\, d\mathfrak{k} \\
&\leq \lim_{\mathfrak{v}\to\infty} a^{\star} \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}\left(\mathbb{T}\varkappa_{\mathfrak{v}}, \mathbb{T}\varkappa, \mathbb{T}\varkappa\right), \\ \varrho_{c^*}\left(\mathbb{T}\mathfrak{l}_{\mathfrak{v}}, \mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}\right) \end{array} \right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right) a \\
&\leq a^{\star} \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}\left(\mathbb{T}\varkappa, \mathbb{T}\varkappa, \mathbb{T}\varkappa\right), \\ \varrho_{c^*}\left(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}\right) \end{array} \right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right) a = 0_{\mathscr{A}}.
\end{aligned}$$

Hence, $\int_0^{\varphi(\varrho_{c^*}(\mathfrak{p},\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa,\mathfrak{l})))} \psi(\mathfrak{k})\, d\mathfrak{k} = 0_{\mathscr{A}}$. Again by the property of integral ψ, we obtain $\varphi\left(\varrho_{c^*}\left(\mathfrak{p}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right)\right) = 0_{\mathscr{A}}$ and so by the property of φ, we have $\varrho_{c^*}\left(\mathfrak{p}, \mathbb{Q}(\varkappa,\mathfrak{l}), \mathbb{Q}(\varkappa,\mathfrak{l})\right) = 0_{\mathscr{A}}$ implies $\mathbb{Q}(\varkappa,\mathfrak{l}) = \mathfrak{p}$. Similarly, we can prove $\mathbb{Q}(\mathfrak{l},\varkappa) = \mathfrak{q}$. Thus, $\mathbb{Q}(\varkappa,\mathfrak{l}) = \mathfrak{p} = \mathbb{T}\varkappa$ and $\mathbb{Q}(\mathfrak{l},\varkappa) = \mathfrak{q} = \mathbb{T}\mathfrak{l}$.Therefore, $(\varkappa,\mathfrak{l}) \in \mathcal{F} \times \mathcal{G}$ is the coupled coincidence point, and $(\mathbb{T}(\varkappa), \mathbb{T}(\mathfrak{l}))$ is the coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$. Now, we will show that the coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$ is unique. Let $(\varkappa^{\star}, \mathfrak{l}^{\star})$ be another coupled coincidence point of $\mathbb{Q}$ and $\mathbb{T}$. So, we will prove that $\mathbb{T}(\varkappa) = \mathbb{T}(\varkappa^{\star})$ and $\mathbb{T}(\mathfrak{l}) = \mathbb{T}(\mathfrak{l}^{\star})$. Suppose $\mathbb{T}(\varkappa) \neq \mathbb{T}(\varkappa^{\star})$ or $\mathbb{T}(\mathfrak{l}) \neq \mathbb{T}(\mathfrak{l}^{\star})$, from condition (vi), we have $\alpha(\mathbb{T}\varkappa, \mathbb{T}\varkappa^{\star}, \mathbb{T}\varkappa^{\star}) \geq I_{\mathscr{A}}$ and $\beta(\mathbb{T}\mathfrak{l}, \mathbb{T}\mathfrak{l}^{\star}, \mathbb{T}\mathfrak{l}^{\star}) \geq I_{\mathscr{A}}$. Then,

$$\int_0^{\varphi(\varrho_{c^*}(\mathbb{T}(\varkappa),\mathbb{T}(\varkappa^{\star}),\mathbb{T}(\varkappa^{\star})))} \psi(\mathfrak{k})\, d\mathfrak{k} = \int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa^{\star},\mathfrak{l}^{\star}),\mathbb{Q}(\varkappa^{\star},\mathfrak{l}^{\star})))} \psi(\mathfrak{k})\, d\mathfrak{k}$$

$$\leq \int_0^{\left(\begin{array}{c}\alpha(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star)\beta(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\\ \varphi\left(\varrho_{c^*}\left(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\varkappa^\star,\mathfrak{l}^\star),\mathbb{Q}(\varkappa^\star,\mathfrak{l}^\star)\right)\right)\end{array}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}$$

$$\leq a^\star\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)a.$$

Hence, we conclude that

$$\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k} \leq a^\star\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)a$$

implies that

$$\begin{aligned}\left\|\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\| &\leq \left\|a^\star\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)a\right\|\\ &\leq \|a\|^2\left\|\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\|\end{aligned}$$

which is a contradiction, since $\|a\| < 1$. So, we have $\left\|\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\| = 0$. Again by the property of integral ψ, we obtain $\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\}\right) = 0_{\mathscr{A}}$ and so by the property of φ, we have $\max\left\{\begin{array}{c}\varrho_{c^*}(\mathbb{T}\varkappa,\mathbb{T}\varkappa^\star,\mathbb{T}\varkappa^\star),\\ \varrho_{c^*}(\mathbb{T}\mathfrak{l},\mathbb{T}\mathfrak{l}^\star,\mathbb{T}\mathfrak{l}^\star)\end{array}\right\} = 0_{\mathscr{A}}$. Hence, $\mathbb{T}(\varkappa) = \mathbb{T}(\varkappa^\star)$ and $\mathbb{T}(\mathfrak{l}) = \mathbb{T}(\mathfrak{l}^\star)$. Applying integral type contraction and following above steps, we get $\mathbb{T}(\varkappa) = \mathbb{T}(\mathfrak{l})$. Thus, $(\mathbb{T}(\varkappa), \mathbb{T}(\varkappa))$ is the unique coupled point of coincidence of the mapping $\mathbb{Q}$ and $\mathbb{T}$ with respect to $\mathcal{F}$ and $\mathcal{G}$. Now, we show that $\mathbb{Q}$ and $\mathbb{T}$ have unique coupled common fixed point. For this let $\mathbb{T}(\varkappa) = \mathfrak{z}$, then, we have $\mathfrak{z} = \mathbb{T}(\varkappa) = \mathbb{Q}(\varkappa, \varkappa)$, by the weakly compatibility of $\mathbb{Q}$ and $\mathbb{T}$, we have $\mathbb{T}\mathfrak{z} = \mathbb{T}(\mathbb{T}(\varkappa)) = \mathbb{T}\mathbb{Q}(\varkappa,\varkappa) = \mathbb{Q}(\mathbb{T}\varkappa, \mathbb{T}\varkappa) = \mathbb{Q}(\mathfrak{z},\mathfrak{z})$. Thus, $(\mathbb{T}(\mathfrak{z}), \mathbb{T}(\mathfrak{z}))$ is coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$. By the uniqueness of coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$, we have $\mathbb{T}(\mathfrak{z}) = \mathbb{T}(\varkappa)$. Thus, we obtain $\mathfrak{z} = \mathbb{T}(\mathfrak{z}) = \mathbb{Q}(\mathfrak{z},\mathfrak{z})$. Therefore, $(\mathfrak{z},\mathfrak{z})$ is the unique strong coupled common fixed point of $\mathbb{Q}$ and $\mathbb{T}$. □

Corollary 3.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be a nonempty closed subsets of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$, $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is a SCC-map on $\mathbb{V}$ (with respect to $\mathcal{F}$ and $\mathcal{G}$), and a coupling $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be a φ-ψ-integral contractive type mapping of $\mathbb{T}$-coupling (with respect to $\mathcal{F}$ and $\mathcal{G}$) and assume that* *(a)* $\mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G}) \neq \emptyset$ *and* $\mathbb{Q}(\mathcal{F} \times \mathcal{G}) \subseteq \mathbb{T}(\mathcal{G})$, $\mathbb{Q}(\mathcal{G} \times \mathcal{F}) \subseteq \mathbb{T}(\mathcal{F})$;

(b) $\mathbb{Q}$ *and* $\mathbb{T}$ *have a CCIP in* $\mathcal{F} \times \mathcal{G}$;

(c) $\{\mathbb{Q}, \mathbb{T}\}$ *is* ω*-compatible pair.*

Then $\mathbb{Q}$ *and* $\mathbb{T}$ *have a USCCFP in* $\mathcal{F} \times \mathcal{G}$.

Proof. The proof follows from Theorems(3.1) by taking $\alpha(\mathbb{T}\varkappa, \mathbb{T}\varkappa, \mathbb{T}\mathfrak{l}) = 1_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z}) = 1_{\mathscr{A}}$ in integral type contraction of definition (3.7). □

Corollary 3.2. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be a nonempty closed subsets of a complete* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ *with* $\mathcal{F} \cap \mathcal{G} \neq \emptyset$, *let a coupling* $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ *satisfying* φ- ψ*-integral type contractive mapping (with respect to* $\mathcal{F}$ *and* $\mathcal{G}$*). Then* $\mathbb{Q}$ *has a USCFP in* $\mathcal{F} \times \mathcal{G}$.

Proof. Using the identity map on $\mathbb{V}$ w.r.t $\mathscr{A}$ and $\mathbb{T} = I_{\mathscr{A}}$, we can determine from Corollary (3.1) that $\mathbb{Q}$ has a USCFP. □

Example 3.2. *Let* $\mathbb{V} = M_2(\mathbb{C})$ *be the space of all* 2×2 *complex matrices, and let* $\mathscr{A} = M_2(\mathbb{C})$ *equipped with the operator norm* $\|\cdot\|$. *Define the* C^**-algebra valued G-metric* $\varrho_{c^*} : \mathbb{V}^3 \to \mathscr{A}_+$ *by* $\varrho_{c^*}(A, B, C) = |A - C| + |B - C|$, *where* $|\cdot|$ *denotes the positive square root of* A^*A *(i.e., the modulus of a matrix). Then* $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ *is a complete* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. *Define the subsets:*

$$\mathcal{F} = \left\{ A \in M_2(\mathbb{C}) : A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \|A\| \leq 1 \right\}, \quad \mathcal{G} = \left\{ B \in M_2(\mathbb{C}) : B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \|B\| \leq 1 \right\}.$$

Define the SCC-map $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ *by* $\mathbb{T}(X) = \frac{1}{2}X$. *Then clearly,* $\mathbb{T}(\mathcal{F}) \subseteq \mathcal{F}$, $\mathbb{T}(\mathcal{G}) \subseteq \mathcal{G}$, *and* $\mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G}) = \{0\} \neq \emptyset$. *Define the coupling* $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ *by* $\mathbb{Q}(A, B) = \frac{1}{4}(A + B)$. *Then obviously,* $\mathbb{Q}(\mathcal{F} \times \mathcal{G}) \subseteq \mathbb{T}(\mathcal{G})$ *and* $\mathbb{Q}(\mathcal{G} \times \mathcal{F}) \subseteq \mathbb{T}(\mathcal{F})$. *Define the functions:* $\alpha(X, Y, Z) = I + |X - Z|$, $\beta(X, Y, Z) = I + |Y - Z|$, *where* I *is the identity matrix in* $M_2(\mathbb{C})$. *Let* $\varphi(T) = T$, *which is continuous, non-decreasing, and satisfies* $\varphi(T) = 0$ *iff* $T = 0$. *Let* $\psi(T) = T^2$, *which is Lebesgue integrable and satisfies* $\int_0^{\epsilon} \psi(k)\, dk = \frac{\epsilon^3}{3} I > 0$ *for all* $\epsilon > 0$. *Thus,* $\varphi \in \Omega$ *and* ψ *satisfies the integral condition. Now we verify that* $\mathbb{T}$ *is twisted* (α, β)*-admissible: If* $\alpha(A, A, B) \geq I$ *and* $\beta(A, A, B) \geq I$, *then* $\alpha(\mathbb{T}B, \mathbb{T}B, \mathbb{T}A) = I + \left|\frac{B-A}{2}\right| \geq I$, *and similarly for* β, *so the admissibility condition holds. The contractive condition from Definition 3.7 is satisfied due to the sub-multiplicative property of the norm and the fact that* $\|\mathbb{T}\| < 1$. *Hence, All conditions of Theorem 3.1 are satisfied and by Theorem 3.1,* $\mathbb{Q}$ *and* $\mathbb{T}$ *have a unique strong common coupled fixed point (USCCFP) in* $\mathcal{F} \times \mathcal{G}$, *which is* $(0_{\mathscr{A}},\ 0_{\mathscr{A}})$.

Example 3.3. *Let* $\mathbb{V} = [0, 1]$ *be a compact metric space with the usual metric* $\varrho_{c^*}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = |\mathfrak{x} - \mathfrak{z}| + |\mathfrak{y} - \mathfrak{z}|$, *and let* $\mathscr{A} = \mathbb{R}$ *so that* $\mathscr{A}_+ = [0, \infty)$. *Then* $(\mathbb{V}, \mathscr{A}, \varrho_{c^*})$ *is a complete* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. *Define the subsets* $\mathcal{F} = [0, 1/2]$, $\quad \mathcal{G} = [1/2, 1]$ *which are nonempty and closed in* $\mathbb{V}$. *Define the SCC-map* $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ *by* $\mathbb{T}(x) = \frac{x}{2} + \frac{1}{4}$. *Then* $\mathbb{T}(\mathcal{F}) = [1/4, 1/2]$, $\mathbb{T}(\mathcal{G}) = [1/2, 3/4]$ *So* $\mathbb{T}(\mathcal{F}) \cap \mathbb{T}(\mathcal{G}) = \{1/2\} \neq \emptyset$. *Define the coupling* $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ *by* $\mathbb{Q}(x, y) = \frac{x+y}{4} + \frac{1}{4}$. *Then,* $\mathbb{Q}(\mathcal{F} \times \mathcal{G}) \subseteq [1/4, 3/8] \subseteq \mathbb{T}(\mathcal{G})$, $\mathbb{Q}(\mathcal{G} \times \mathcal{F}) \subseteq [1/4, 3/8] \subseteq \mathbb{T}(\mathcal{F})$. *Let* $\varphi(t) = t$, $\quad \psi(t) = t^2$ *Then,* $\varphi \in \Omega$, $\quad \int_0^{\epsilon} \psi(k)\, dk = \frac{\epsilon^3}{3} > 0$ *for all* $\epsilon > 0$. *Now check for a CCIP. Let* $x^* = \frac{1}{2} \in \mathcal{F} \cap \mathcal{G}$. *Then,*

$\mathbb{T}(x^*) = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4} = \frac{1}{2}$, $\mathbb{Q}(x^*,x^*) = \frac{1}{2}+\frac{1}{4} = \frac{1}{2}$. *So* $\mathbb{Q}(x^*,x^*) = \mathbb{T}(x^*)$ *and since* $x^* = \frac{1}{2} \in \mathcal{F}\cap\mathcal{G}$, *we have* $(x^*,x^*) \in \mathcal{F}\times\mathcal{G}$. *Also, for any* $x,y \in \mathbb{V}$, $\mathbb{T}(\mathbb{Q}(x,y)) = \frac{1}{2}\left(\frac{x+y}{4}+\frac{1}{4}\right)+\frac{1}{4} = \frac{x+y}{8}+\frac{3}{8}$ *and* $\mathbb{Q}(\mathbb{T}(x),\mathbb{T}(y)) = \frac{1}{4}\left(\frac{x}{2}+\frac{1}{4}+\frac{y}{2}+\frac{1}{4}\right)+\frac{1}{4} = \frac{x+y}{8}+\frac{3}{8}$. *So* $\mathbb{T}$ *and* $\mathbb{Q}$ *are* ω*-compatible.*

Hence, all conditions of Corollary 3.1 are satisfied, and the unique strong common coupled fixed point (USCCFP) of $\mathbb{Q}$ *and* $\mathbb{T}$ *in* $\mathcal{F}\times\mathcal{G}$ *is* $(\frac{1}{2},\frac{1}{2})$.

4. Applications

4.1. **Existence and uniqueness of solutions for a system of nonlinear Fredholm integral equations.**

Let $\mathbb{V} = C([0,1],\mathbb{R}^+) \subseteq L^\infty([0,1],\mathbb{R}^+)$ and let $\mathscr{A} = \mathbb{B}(L^2([0,1]))$ denote the C^*-algebra of bounded linear operators on $L^2([0,1])$ with operator norm

$$\|\mathbb{A}\| = \sup_{\|a\|=1} \|\mathbb{A}a\|.$$

Define the map $\varrho_{C^*} : \mathbb{V}^3 \to \mathscr{A}$ by

$$\varrho_{C^*}(\mathfrak{p},\mathfrak{r},\mathfrak{q}) = \mathbb{M}_{|\mathfrak{p}-\mathfrak{q}|+|\mathfrak{r}-\mathfrak{q}|},$$

where, for $\phi \in \mathscr{A}$, $\mathbb{M}_\phi$ is the operator acting on $L^2([0,1])$ via composition: $\mathbb{M}_\phi(\alpha) = \phi\diamond\alpha$. It follows that $(\mathbb{V},\mathscr{A},\varrho_{C^*})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$.

We consider the nonlinear system of Fredholm integral equations

$$\begin{cases} \mathfrak{x}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_0^1 \mathfrak{T}(\mathfrak{v},\mathfrak{u})\mathscr{K}(\mathfrak{v},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))\,d\mathfrak{u}, \\ \mathfrak{y}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_0^1 \mathfrak{T}(\mathfrak{v},\mathfrak{u})\mathscr{K}(\mathfrak{v},\mathfrak{y}(\mathfrak{u}),\mathfrak{x}(\mathfrak{u}))\,d\mathfrak{u}, \end{cases} \tag{4.1}$$

where $\mathscr{K} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$, $\mathfrak{f} : [0,1] \to \mathbb{R}$, and $\mathfrak{T} : [0,1]\times[0,1] \to \mathbb{R}$ are continuous functions.

Define operators

$$\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}, \quad \mathbb{Q}(\mathfrak{x},\mathfrak{y})(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_0^1 \mathfrak{T}(\mathfrak{v},\mathfrak{u})\mathscr{K}(\mathfrak{v},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))\,d\mathfrak{u}$$

and

$$\mathbb{T} : \mathbb{V} \to \mathbb{V}, \quad \mathbb{T}\mathfrak{x}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_0^1 \mathfrak{T}(\mathfrak{v},\mathfrak{u})\mathscr{K}(\mathfrak{v},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))\,d\mathfrak{u}.$$

Note that the coupled system (4.1) has a solution if and only if $\mathbb{Q}$ and $\mathbb{T}$ share a common coupled fixed point.

Theorem 4.1. *Let* $\mathcal{F},\mathcal{G}$ *be nonempty closed subsets of* $\mathbb{V}$ *and assume* $\iota \in (0,1)$ *such that for all* $\varkappa,\mathfrak{z} \in \mathcal{F}$, $\mathfrak{l},\mathfrak{x} \in \mathcal{G}$, *and* $\mathfrak{v},\mathfrak{u} \in [0,1]$,

$$\left|\mathscr{K}(\mathfrak{v},\varkappa(\mathfrak{u}),\mathfrak{l}(\mathfrak{u})) - \mathscr{K}(\mathfrak{v},\mathfrak{x}(\mathfrak{u}),\mathfrak{z}(\mathfrak{u}))\right| \le \iota\max\{\|\mathbb{T}(\varkappa)(\mathfrak{u})-\mathbb{T}(\mathfrak{x})(\mathfrak{u})\|, \|\mathbb{T}(\mathfrak{l})(\mathfrak{u})-\mathbb{T}(\mathfrak{z})(\mathfrak{u})\|\}.$$

Moreover, let $\delta \ge 0$ *satisfy*

$$\sup_{\mathfrak{v},\mathfrak{u}\in[0,1]} \mathfrak{T}(\mathfrak{v},\mathfrak{u}) \le \delta,$$

and assume that

$$\iota\delta < 1.$$

Then, the system (4.1) *admits a unique solution.*

Proof. Since $\mathscr{K}$, $\mathfrak{T}$, and $\mathfrak{f}$ are continuous, the operators $\mathbb{Q}$ and $\mathbb{T}$ are continuous on $\mathbb{V}$.

From the definition of the $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-metric, we have

$$\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z})) = \mathbb{M}_{2|\mathbb{Q}(\varkappa,\mathfrak{l})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})|}.$$

Hence, using the operator norm,

$$\|\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))\| = \sup_{\|h\|=1}\left\langle \mathbb{M}_{2|\mathbb{Q}(\varkappa,\mathfrak{l})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})|}h, h\right\rangle.$$

By integrating and applying the given Lipschitz condition on $\mathscr{K}$, we get

$$\|\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))\| \leq 2\iota \sup_{\|h\|=1}\int_0^1 |h(\varsigma)|^2 d\varsigma \int_0^1 \mathfrak{T}(\mathfrak{v},\mathfrak{u})d\mathfrak{u}\max\left\{\begin{array}{c}\|\mathbb{T}(\varkappa)-\mathbb{T}(\mathfrak{x})\|\\ \|\mathbb{T}(\mathfrak{l})-\mathbb{T}(\mathfrak{z})\|\end{array}\right\}.$$

Setting $\kappa = \iota\delta < 1$ and denoting $a = \kappa 1_{\mathbb{B}(L^2([0,1]))} \in \mathscr{A}$, we have $\|a\| = \kappa < 1$, and thus

$$\|\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{l}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))\| \leq \|a\|^2 \max\{\|\varrho_{c^*}(\mathbb{T}(\varkappa),\mathbb{T}(\mathfrak{x}),\mathbb{T}(\mathfrak{x}))\|, \|\varrho_{c^*}(\mathbb{T}(\mathfrak{l}),\mathbb{T}(\mathfrak{z}),\mathbb{T}(\mathfrak{z}))\|\}.$$

Choosing altering distance functions $\varphi(t) = t$ is altering distance function and homomorphism or preserving multiplication compatibly and $\psi(t) = t^2$, which satisfy the integral positivity condition

$$\int_0^{\epsilon} \psi(k)dk = \frac{\epsilon^3}{3} > 0, \quad \forall \epsilon > 0,$$

one obtains the contractive-type integral inequality

$$\int_0^{\varphi(\varrho_{c^*}(\mathbb{Q}(\varkappa,\mathfrak{v}),\mathbb{Q}(\varkappa,\mathfrak{v}),\mathbb{Q}(\mathfrak{l},\mathfrak{z})))} \psi(\mathfrak{k})d\mathfrak{k} \leq a^*\left(\int_0^{\varphi\left(\max\{\varrho_{c^*}(\mathbb{T}(\varkappa),\mathbb{T}(\mathfrak{x}),\mathbb{T}(\mathfrak{x})),\varrho_{c^*}(\mathbb{T}(\mathfrak{l}),\mathbb{T}(\mathfrak{z}),\mathbb{T}(\mathfrak{z}))\}\right)} \psi(\mathfrak{k})d\mathfrak{k}\right)a,$$

which confirms that $\mathbb{Q}$ is a φ-ψ-integral type contractive mapping with respect to $\mathbb{T}$-coupling.

By Corollary 3.1, this guarantees $\mathbb{Q}$ and $\mathbb{T}$ have a unique common coupled fixed point, concluding that the system (4.1) has a unique solution. □

4.2. **Application to Homotopy.**

In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem 4.2. *Let* $(\mathbb{V},\mathscr{A},\varrho_{c^*})$ *be a complete* $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$ *space. Let* (Δ_1,Δ_2) *and* $(\overline{\Delta}_1,\overline{\Delta}_2)$ *be open and closed subsets of* $\mathbb{V}$ *such that* $(\Delta_1,\Delta_2) \subseteq (\overline{\Delta}_1,\overline{\Delta}_2)$ *with* $\Delta_1 \cap \Delta_2 \neq \emptyset$. *Suppose the operator*

$$\mathcal{H} : \left((\overline{\Delta}_1,\overline{\Delta}_2) \cup (\overline{\Delta}_2,\overline{\Delta}_1)\right) \times [0,1] \to \mathbb{V}$$

satisfies:

(i) *For each* $\varkappa \in \partial\Delta_1$, $\mathfrak{l} \in \partial\Delta_2$ *and* $\iota \in [0,1]$, $\varkappa \neq \mathcal{H}(\varkappa,\mathfrak{l},\iota)$, $\quad \mathfrak{l} \neq \mathcal{H}(\mathfrak{l},\varkappa,\iota)$.

(ii) *For all* $\varkappa, \mathfrak{z} \in \overline{\Delta}_1$, $\mathfrak{l}, \mathfrak{x} \in \overline{\Delta}_2$, $\iota \in [0,1]$, *with functions* $\varphi \in \Omega$, *and Lebesgue-integrable* $\psi : \mathscr{A}_+ \to \mathscr{A}_+$ *summable on each compact subset and such that for each* $\epsilon > 0_{\mathscr{A}}$, $\int_0^\epsilon \psi(\mathfrak{k})\, d\mathfrak{k} > 0_{\mathscr{A}}$, *and* $a \in \mathscr{A}$ *with* $\|a\| < 1$, *the following integral contractive condition holds:*

$$\int_0^{\varphi(\varrho_{c^*}(\mathcal{H}(\varkappa,\mathfrak{l},\iota),\mathcal{H}(\mathfrak{x},\mathfrak{z},\iota),\mathcal{H}(\mathfrak{x},\mathfrak{z},\iota)))} \psi(\mathfrak{k})\, d\mathfrak{k} \leq a^* \left(\int_0^{\varphi(\max\{\varrho_{c^*}(\varkappa,\mathfrak{x},\mathfrak{x}),\varrho_{c^*}(\mathfrak{l},\mathfrak{z},\mathfrak{z})\})} \psi(\mathfrak{k})\, d\mathfrak{k} \right) a.$$

(iii) *There exists* $\mathbb{M} \in \mathscr{A}_+$ *such that for every* $\varkappa \in \overline{\Delta}_1$, $\mathfrak{l} \in \overline{\Delta}_2$, $\iota, \ell \in [0,1]$,

$$\varrho_{c^*}\left(\mathcal{H}(\varkappa,\mathfrak{l},\iota),\mathcal{H}(\varkappa,\mathfrak{l},\ell),\mathcal{H}(\varkappa,\mathfrak{l},\ell)\right) \leq \|\mathbb{M}\||\iota - \ell|.$$

Then, $\mathcal{H}(\cdot,0)$ *has a coupled fixed point if and only if* $\mathcal{H}(\cdot,1)$ *has a coupled fixed point.*

Proof. Let us consider the set

$$\mathbb{B} = \{\iota \in [0,1] : \text{there exist } \varkappa \in \Delta_1, \mathfrak{l} \in \Delta_2 \text{ such that } \mathcal{H}(\varkappa,\mathfrak{l},\iota) = \varkappa,\ \mathcal{H}(\mathfrak{l},\varkappa,\iota) = \mathfrak{l}\}.$$

Assuming $\mathcal{H}(\cdot,0)$ admits a coupled fixed point in $\Delta_1 \times \Delta_2$, it follows that $0 \in \mathbb{B}$, ensuring $\mathbb{B}$ is not empty.

To prove that $\mathbb{B}$ is closed in $[0,1]$, let $\{\iota_v\} \subseteq \mathbb{B}$ be a sequence converging to some $\iota \in [0,1]$. By definition of $\mathbb{B}$, there exist corresponding sequences $\{\varkappa_v\} \subseteq \Delta_1$ and $\{\mathfrak{l}_v\} \subseteq \Delta_2$ satisfying

$$\varkappa_v = \mathcal{H}(\varkappa_v,\mathfrak{l}_v,\iota_v), \quad \mathfrak{l}_v = \mathcal{H}(\mathfrak{l}_v,\varkappa_v,\iota_v).$$

Using the contractive conditions on $\mathcal{H}$ and the completeness of the space, one can show that these sequences are Cauchy and thus converge to some $\mathfrak{p} \in \overline{\Delta}_1$, $\mathfrak{q} \in \overline{\Delta}_2$. By continuity of $\mathcal{H}$ in ι, these limits satisfy

$$\mathfrak{p} = \mathcal{H}(\mathfrak{p},\mathfrak{q},\iota), \quad \mathfrak{q} = \mathcal{H}(\mathfrak{q},\mathfrak{p},\iota),$$

which implies $\iota \in \mathbb{B}$. Therefore, $\mathbb{B}$ is closed. Consider

$$\begin{aligned}
\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}) &= \varrho_{c^*}\left(\mathcal{H}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1})\right)\\
&\leq \begin{array}{c} \varrho_{c^*}\left(\mathcal{H}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}})\right)\\ +\varrho_{c^*}\left(\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1})\right)\end{array}\\
&\leq \varrho_{c^*}\left(\mathcal{H}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}})\right) + \|\mathbb{M}\||\mathfrak{i}_{\mathfrak{v}} - \mathfrak{i}_{\mathfrak{v}+1}|.
\end{aligned}$$

Letting $\mathfrak{v} \to \infty$, and applying φ properties, we get

$$\begin{aligned}
\lim_{\mathfrak{v}\to\infty} \int_0^{\varphi(\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}))} \psi(\mathfrak{k})\, d\mathfrak{k} &\leq \lim_{\mathfrak{v}\to\infty} \int_0^{\varphi(\varrho_{c^*}(\mathcal{H}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}}),\mathcal{H}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}})))} \psi(\mathfrak{k})\, d\mathfrak{k}\\
&\leq \lim_{\mathfrak{v}\to\infty} \left(a^* \left(\int_0^{\varphi\left(\max\left\{\begin{array}{l} \varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1}) \end{array}\right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right) a \right).
\end{aligned}$$

Hence, we conclude that

$$\lim_{\mathfrak{v}\to\infty} \left\| \int_0^{\varphi\left(\max\left\{\begin{array}{l} \varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1}) \end{array}\right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right\| \leq \lim_{\mathfrak{v}\to\infty} \|a\|^2 \left\| \int_0^{\varphi\left(\max\left\{\begin{array}{l} \varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1}) \end{array}\right\}\right)} \psi(\mathfrak{k})\, d\mathfrak{k} \right\|$$

which implies $(1-\|a\|^2)\lim\limits_{\mathfrak{v}\to\infty}\left\| \int_0^{\varphi\left(\max\left\{\begin{array}{l}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\|\leq 0$

since, $\|a\|<1$ so that $\lim\limits_{\mathfrak{v}\to\infty}\left\| \int_0^{\varphi\left(\max\left\{\begin{array}{l}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right\|=0.$

Hence, by the property of integral ψ and by Lemma 3.2, we obtain $\lim_{\mathfrak{v}\to\infty}\varphi\left(\max\left\{\begin{array}{l}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})\end{array}\right\}\right)=0_{\mathscr{A}}$. Again by the properties of altering distance function φ, we obtain $\lim_{\mathfrak{v}\to\infty}\max\left\{\begin{array}{l}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})\end{array}\right\}=0_{\mathscr{A}}$.

Thus, $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1})=0_{\mathscr{A}}$ and $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})=0_{\mathscr{A}}$.

By following similar steps, we can establish the result $\lim\limits_{p\to\infty}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}})=\tilde{0}_{\tilde{\mathscr{A}}}$ and $\lim\limits_{p\to\infty}\varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}},\varkappa_{\mathfrak{v}})=\tilde{0}_{\tilde{\mathscr{A}}}$. Now for $\mathfrak{u}>\mathfrak{v}$, by use of rectangle inequality , we have

$$\begin{aligned}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\varkappa_{\mathfrak{u}},\varkappa_{\mathfrak{u}}) &\leq \varrho_{c^*}(\varkappa_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})+\varrho_{c^*}(\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+2},\varkappa_{\mathfrak{v}+2})+\ldots+\varrho_{c^*}(\varkappa_{\mathfrak{u}-2},\varkappa_{\mathfrak{u}-1},\varkappa_{\mathfrak{u}-1})\\ &\quad+\varrho_{c^*}(\varkappa_{\mathfrak{u}-1},\varkappa_{\mathfrak{u}},\varkappa_{\mathfrak{u}})\\ &\leq \varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}})+\varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\varkappa_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+1})+\varrho_{c^*}(\varkappa_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1},\mathfrak{l}_{\mathfrak{v}+1})\\ &\quad+\varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}+1},\varkappa_{\mathfrak{v}+2},\varkappa_{\mathfrak{v}+2})\ldots+\varrho_{c^*}(\varkappa_{\mathfrak{u}-2},\mathfrak{l}_{\mathfrak{u}-2},\mathfrak{l}_{\mathfrak{u}-2})\\ &\quad+\varrho_{c^*}(\mathfrak{l}_{\mathfrak{u}-2},\varkappa_{\mathfrak{u}-1},\varkappa_{\mathfrak{u}-1})+\varrho_{c^*}(\varkappa_{\mathfrak{u}-1},\mathfrak{l}_{\mathfrak{u}-1},\mathfrak{l}_{\mathfrak{u}-1})+\varrho_{c^*}(\mathfrak{l}_{\mathfrak{u}-1},\varkappa_{\mathfrak{u}},\varkappa_{\mathfrak{u}})\\ &\quad\to 0_{\mathscr{A}}\text{ as }\mathfrak{v},\mathfrak{u}\to\infty.\end{aligned}$$

Hence $\{\varkappa_{\mathfrak{v}}\}$ is a Cauchy sequence in $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$ $(\mathbb{V},\mathscr{A},\varrho_{c^*})$. Similarly we can show that $\left\{\mathfrak{l}_{\mathfrak{v}_p}\right\}$ is CS in $(\mathbb{V},\mathscr{A},\varrho_{c^*})$ and by the completeness of $(\mathbb{V},\mathscr{A},\varrho_{c^*})$, there exist $\mathfrak{p}\in\Delta_1$ and $\mathfrak{q}\in\Delta_2$ such that $\lim_{\mathfrak{v}\to\infty}\varkappa_{\mathfrak{v}}=\mathfrak{p}$ and $\lim_{\mathfrak{v}\to\infty}\mathfrak{l}_{\mathfrak{v}}=\mathfrak{q}$. Since $\lim\limits_{\mathfrak{v}\to\infty}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}})=0_{\mathscr{A}}\implies\varrho_{c^*}(\mathfrak{p},\mathfrak{q},\mathfrak{q})=0_{\mathscr{A}}$ then, we have $\mathfrak{p}=\mathfrak{q}$.

As $\mathfrak{p}\in\Delta_1$ and $\mathfrak{q}\in\Delta_2$ it follows that $\mathfrak{p}=\mathfrak{q}\in\Delta_1\cap\Delta_2$ and hence, $\Delta_1\cap\Delta_2\neq\emptyset$. Now, we have

$$\begin{aligned}\int_0^{\varphi(\varrho_{c^*}(\mathfrak{p},\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})))}\psi(\mathfrak{k})\,d\mathfrak{k} &= \lim_{\mathfrak{v}\to\infty}\int_0^{\varphi(\varrho_{c^*}(\mathcal{H}(\varkappa_{\mathfrak{v}},\mathfrak{l}_{\mathfrak{v}},\mathfrak{i}),\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})))}\psi(\mathfrak{k})\,d\mathfrak{k}\\ &\leq \lim_{\mathfrak{v}\to\infty}\left(a^*\left(\int_0^{\varphi\left(\max\left\{\begin{array}{c}\varrho_{c^*}(\varkappa_{\mathfrak{v}},\mathfrak{p},\mathfrak{p}),\\ \varrho_{c^*}(\mathfrak{l}_{\mathfrak{v}},\mathfrak{q},\mathfrak{q})\end{array}\right\}\right)}\psi(\mathfrak{k})\,d\mathfrak{k}\right)\right)a=0_{\mathscr{A}}.\end{aligned}$$

By the property of integral ψ, we get $\varphi(\varrho_{c^*}(\mathfrak{p},\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})))=0_{\mathscr{A}}$ and so by the property of φ, we have $\varrho_{c^*}(\mathfrak{p},\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}))=0_{\mathscr{A}}$ implies that $\mathcal{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})=\mathfrak{p}$. Similarly, we can prove $\mathcal{H}(\mathfrak{q},\mathfrak{p},\mathfrak{i})=\mathfrak{q}$. Thus $\mathfrak{i}\in\mathbb{B}$. Hence $\mathbb{B}$ is closed in $[0,1]$.

Let $\mathfrak{i}_0\in\mathbb{B}$, then there exist $\varkappa_0\in\Delta_1,\mathfrak{l}_0\in\Delta_2$ with $\varkappa_0=\mathcal{H}(\varkappa_0,\mathfrak{l}_0,\mathfrak{i}_0)$, $\mathfrak{l}_0=\mathcal{H}(\mathfrak{l}_0,\varkappa_0,\mathfrak{i}_0)$. Since (Δ_1,Δ_2) is open, then there exist $r>0$ such that $B_{\varrho_{c^*}}(\varkappa_0,r)\subseteq\Delta_1$ and $B_{\varrho_{c^*}}(\mathfrak{l}_0,r)\subseteq\Delta_2$. Choose $\mathfrak{i}\in(\mathfrak{i}_0-\epsilon,\mathfrak{i}_0+\epsilon)$ such that $|\mathfrak{i}-\mathfrak{i}_0|\leq\frac{1}{\|\mathbb{M}^{\mathfrak{v}}\|}<\frac{\epsilon}{2}$, then for

$\varkappa\in\overline{B_{\varrho_{c^*}}(\varkappa_0,r)}=\{\varkappa\in\Delta_1/\varrho_{c^*}(\varkappa,\varkappa_0,\varkappa_0)\leq r+\varrho_{c^*}(\varkappa_0,\varkappa_0,\varkappa_0)\}$

and $\text{l} \in \overline{B_{\varrho_{c^*}}(\text{l}_0, r)} = \{\text{l} \in \Delta_2 / \varrho_{c^*}(\text{l}, \text{l}_0, \text{l}_0) \leq r + \varrho_{c^*}(\text{l}_0, \text{l}_0, \text{l}_0)\}$. Now we have

$$\begin{aligned}
\varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \varkappa_0, \varkappa_0) &= \varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}_0), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}_0)) \\
&\leq \varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i})) \\
&\quad + \varrho_{c^*}(\mathcal{H}(\varkappa_0, \text{l}_0, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}_0), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}_0)) \\
&\leq \varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i})) + \frac{1}{\|\mathbb{M}^{\mathfrak{v}-1}\|}
\end{aligned}$$

Letting $\mathfrak{v} \to \infty$ and applying φ properties, we obtain

$$\begin{aligned}
\int_0^{\varphi(\varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \varkappa_0, \varkappa_0))} \psi(\mathfrak{t})\, d\mathfrak{t} &\leq \int_0^{\varphi(\varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i}), \mathcal{H}_b(\varkappa_0, \text{l}_0, \text{i})))} \psi(\mathfrak{t})\, d\mathfrak{t} \\
&\leq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\varkappa, \varkappa_0, \varkappa_0), \\ \varrho_{c^*}(\text{l}, \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a
\end{aligned}$$

we conclude that

$$\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \varkappa_0, \varkappa_0), \\ \varrho_{c^*}(\mathcal{H}(\text{l}, \varkappa, \text{i}), \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \leq a^* \left(\int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\varkappa, \varkappa_0, \varkappa_0), \\ \varrho_{c^*}(\text{l}, \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right) a$$

which implies that

$$\begin{aligned}
\left\| \int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\mathcal{H}(\varkappa, \text{l}, \text{i}), \varkappa_0, \varkappa_0), \\ \varrho_{c^*}(\mathcal{H}(\text{l}, \varkappa, \text{i}), \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right\| &\leq \|a\|^2 \left\| \int_0^{\varphi\left(\max\left\{ \begin{array}{c} \varrho_{c^*}(\varkappa, \varkappa_0, \varkappa_0), \\ \varrho_{c^*}(\text{l}, \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right\| \\
&< \left\| \int_0^{\varphi\left(\max\left\{ \begin{array}{c} r + \varrho_{c^*}(\varkappa_0, \varkappa_0, \varkappa_0), \\ r + \varrho_{c^*}(\text{l}_0, \text{l}_0, \text{l}_0) \end{array} \right\}\right)} \psi(\mathfrak{t})\, d\mathfrak{t} \right\|.
\end{aligned}$$

For any fixed i in the interval $(\text{i}_0 - \epsilon, \text{i}_0 + \epsilon)$, the operator

$$\mathcal{H}(\cdot, \text{i}) : \overline{B_{\varrho_{c^*}}(\varkappa_0, r)} \to \overline{B_{\varrho_{c^*}}(\varkappa_0, r)} \quad \text{and} \quad \mathcal{H}(\cdot, \text{i}) : \overline{B_{\varrho_{c^*}}(\text{l}_0, r)} \to \overline{B_{\varrho_{c^*}}(\text{l}_0, r)}$$

is well defined. Given that condition (ii) holds and φ is a continuous, non-decreasing function, it follows that all prerequisites of Theorem 4.2 are met. Consequently, $\mathcal{H}(\cdot, \text{i})$ possesses a coupled fixed point in $\overline{\Delta_1} \times \overline{\Delta_2}$. Since condition (i) prevents such fixed points on the boundary, the coupled fixed point must lie in $\Delta_1 \times \Delta_2$. This shows that every i in the neighborhood $(\text{i}_0 - \epsilon, \text{i}_0 + \epsilon)$ belongs to $\mathbb{B}$, proving the openness of $\mathbb{B}$ in $[0, 1]$.

To complete the proof, the converse direction follows analogously using comparable arguments.

□

Conclusion

This study successfully establishes SUCCFP theorems for twisted (α, β)-φ-ψ-integral type contractive $\mathbb{T}$-Coupling SCC-maps within the framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. The results substantially advance fixed point theory and illustrate their practical relevance through applications involving integral equations and homotopy theory, thereby creating promising directions for future mathematical research efforts.

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