

Applications of Neutrosophic in Third-Order Differential Subordinations of Multivalent Functions

Suha Hammad¹, Ala Amourah^{2,*}, Mohammad El-Itan³, Abdullah Alsoboh⁴,
Mohammed M. Al-Hatmi^{4,*}, Fadi M.A. Al-Zu'bi⁵, Tariq Al-Hawary⁶

¹Department of Mathematics, College of Education for Pure Sciences, University of Tikrit, Iraq

²Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman

³Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Jordan

⁴Department of Basic and Applied Sciences, College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra 413, Oman

⁵Zayed University, Department of Mathematics and Statistics, Abu Dhabi, United Arab Emirates

⁶Department of Applied Science, Ajloun College, Al-Balqa Applied University, Jordan

*Corresponding authors: AAmourah@su.edu.om, mohammed.alhatmi@asu.edu.om

Abstract. In this work, we investigate third-order differential subordination for multivalent functions using the Cho-Kwon-Srivastava operator with the neutrosophic parameter $M_{\mathfrak{N}}$. By incorporating uncertainty, our approach extends classical models and provides a flexible framework to analyze the behavior of multivalent functions under both precise and uncertain settings. These results generalize previous studies, offering deeper insights into the structure and properties of such functions.

1. INTRODUCTION

Recently, a new approach has been developed to handle situations where data might be imprecise, while still relying on classical probabilistic models. This method combines neutrosophic set theory with standard probability distributions, such as uniform, exponential, and Poisson. A notable example of this idea is the neutrosophic Poisson distribution [1].

Let $T(\mathcal{D})$ denote the class of holomorphic functions in the unit disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $j \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $\eta \in \mathbb{C}$, define

Received: Nov. 11, 2025.

2020 Mathematics Subject Classification. 30C45; 30C50; 39B12.

Key words and phrases. multivalent functions; holomorphic functions; quasi-subordination; Poisson distribution.

$$T(\eta, j) = \left\{ f \in T(\mathcal{D}) : f(z) = \eta + \sum_{\varsigma=j}^{\infty} \eta_{\varsigma} z^{\varsigma} \right\},$$

and set $T = T(1, 1)$.

If there exists a Schwarz function $\pi(z)$ analytic in $\mathcal{D}$ with $|\pi(z)| < 1$ and $\pi(0) = 0$, then an analytic function f_1 is said to be subordinate to f_2 if

$$f_1(z) = f_2(\pi(z)), \quad z \in \mathcal{D},$$

and we write

$$f_1(z) < f_2(z), \quad z \in \mathcal{D}.$$

The set of all μ -valent holomorphic functions in $\mathcal{D}$ is denoted by $\mathcal{A}_{\mu}$ and is given by

$$f(z) = z^{\mu} + \sum_{\varsigma=1}^{\infty} c_{\mu+\varsigma} z^{\mu+\varsigma}. \quad (1.1)$$

For functions f and ξ of the form

$$\xi(z) = z^{\mu} + \sum_{\varsigma=1}^{\infty} d_{\mu+\varsigma} z^{\mu+\varsigma}, \quad (1.2)$$

their convolution is defined by

$$(f * \xi)(z) = z^{\mu} + \sum_{\varsigma=1}^{\infty} c_{\mu+\varsigma} d_{\mu+\varsigma} z^{\mu+\varsigma}, \quad z \in \mathcal{D}.$$

Let $h(z)$ be a univalent function in $\mathcal{D}$, and define the mapping

$$Y(\tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}; z) : \mathbb{C}^4 \times \mathcal{D} \rightarrow \mathbb{C}.$$

Suppose $k(z)$ is analytic in $\mathcal{D}$ and satisfies

$$Y(k(z), zk'(z), z^2k''(z), z^3k'''(z); z) < h(z), \quad (1.3)$$

then the set of all solutions $k(z)$ satisfying (1.3) is considered. If $k(z)$ is subordinate to a univalent function $\varphi(z)$, then φ is called the dominant of (1.3). Moreover, if the univalent dominant $\check{k}$ satisfies $\check{k} < \varphi$ for all dominants, it is referred to as the best dominant [2].

Neutrosophic Approach and Cho-Kwon-Srivastava Operator. Neutrosophic science offers a modern way to handle uncertainty and indeterminacy, common challenges in many research areas. Introduced by Florentin Smarandache [3–6], it studies the concept of neutrality, its properties, and its relation to various spectra. Researchers like Salama have applied neutrosophic principles in information systems, neutrosophic crisp sets, and computer science [7–14]. At its core, a neutrosophic set consists of three components: $\mathcal{T}$ (membership), $\mathcal{I}$ (indeterminacy), and $\mathcal{F}$ (non-membership) [15].

The neutrosophic transformation replaces a real parameter λ with a neutrosophic number $\aleph = d + \mathcal{I}$, where d is deterministic and $\mathcal{I}$ captures indeterminacy. This allows for modeling uncertain or ambiguous parameters effectively.

Previously, Cho-Kwon-Srivastava introduced the operator $\mathcal{J}_\mu^\lambda(s, r)$ [2]:

$$\mathcal{J}_\mu^\lambda(s, r)f(z) = \Theta(s, r, z) * f(z) = z^\mu + \sum_{\varsigma=1}^{\infty} \frac{(\lambda + \mu)_\varsigma (r)_\varsigma}{\varsigma! (s)_\varsigma} c_{\mu+\varsigma} z^{\mu+\varsigma}, \quad (1.4)$$

where $s, r \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -\mu, \mu \in \mathbb{C}$, and $(y)_\varsigma$ is the Pochhammer symbol:

$$(y)_\varsigma = \frac{\Gamma(y + \varsigma)}{\Gamma(y)} = \begin{cases} 1, & \varsigma = 0, \\ y(y+1) \cdots (y + \varsigma - 1), & \varsigma \in \mathbb{N}. \end{cases} \quad (1.5)$$

The generating function $\Theta(s, r, z)$ is given by

$$\Theta(s, r, z) = \sum_{\varsigma=0}^{\infty} \frac{(s)_\varsigma}{(r)_\varsigma} z^{\mu+\varsigma}, \quad z \in \mathcal{D}. \quad (1.6)$$

Additionally, the conjugate operator satisfies

$$\Theta_\mu^+(s, r, z) * \Theta_\mu(s, r, z) = \frac{z^\mu}{(1-z)^{\lambda+\mu}}, \quad z \in \mathcal{D}. \quad (1.7)$$

By replacing λ with the neutrosophic parameter $\aleph$, we define the neutrosophic version:

$$\tilde{N}\left(\Theta_\mu^+(s, r, z) * \Theta_\mu(s, r, z)\right) = \frac{z^\mu}{(1-z)^{\aleph+\mu}}, \quad \mu \in \mathbb{N}, \quad (1.8)$$

and thus

$$\mathcal{J}_\mu^{\aleph}(s, r)f(z) = z^\mu + \sum_{\varsigma=1}^{\infty} \frac{(\aleph + \mu)_\varsigma (r)_\varsigma}{\varsigma! (s)_\varsigma} c_{\mu+\varsigma} z^{\mu+\varsigma}. \quad (1.9)$$

From (1.9), we also have

$$z\left(\mathcal{J}_\mu^{\aleph}(s, r)f(z)\right)' = (\mu + s + \varsigma)\mathcal{J}_\mu^{\aleph}(s, r)f(z) - (s + \varsigma)\mathcal{J}_\mu^{\aleph}(s+1, r)f(z).$$

Definition 1.1. [16] Let $\mathcal{Q}$ denote the set of univalent and analytic functions φ defined on $\overline{\mathcal{D}} \setminus B(\varphi)$, where

$$B(\varphi) = \left\{ \ell \in \partial\mathcal{D} : \lim_{z \rightarrow \ell} \varphi(z) = \infty \right\},$$

and

$$\min |\varphi'(\ell)| = \partial > 0, \quad \text{for } \ell \in \partial\mathcal{D} \setminus B(\varphi).$$

The subclass $\mathcal{Q}(a) \equiv \mathcal{Q}_0$ consists of functions $\varphi \in \mathcal{Q}$ with $\varphi(0) = a$, and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.2. [16] Let $z \in \mathbb{C}$, $\varphi \in \mathcal{Q}$, and $j \geq 2$. Define $\psi_j[z, \varphi]$ as the class of admissible functions $Y : \mathbb{C}^4 \times \mathcal{D} \rightarrow \mathbb{C}$ satisfying:

$$Y(\tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}; z) \notin z,$$

whenever

$$\tilde{t} = \varphi(\ell), \quad \tilde{u} = n\ell\varphi'(\ell), \quad \operatorname{Re}\left\{\frac{\tilde{v}}{\tilde{u}} + 1\right\} > n\operatorname{Re}\left\{1 + \frac{\ell\varphi''(\ell)}{\varphi'(\ell)}\right\},$$

and

$$\operatorname{Re}\left\{\frac{\tilde{w}}{\tilde{u}}\right\} \geq n^2\operatorname{Re}\left\{\frac{\ell^2\varphi'''(\ell)}{\varphi'(\ell)}\right\},$$

where $\ell \in \partial\mathcal{D} \setminus B(\varphi)$ and $n \geq j$.

Lemma 1.1. [16] Let $k \in \mathcal{T}(\eta, j)$ with $j \geq 2$, and let $\varphi \in \mathcal{Q}(a)$. Assume

$$\operatorname{Re}\left\{\frac{\ell\varphi''(\ell)}{\varphi'(\ell)}\right\} \geq 0, \quad \left|\frac{\ell k'(\ell)}{\varphi'(\ell)}\right| \leq n,$$

for $\ell \in \partial\mathcal{D} \setminus B(\varphi)$ and $n \geq j$. If $Y \in \psi_j[z, \varphi]$ and

$$Y(k(z), zk'(z), z^2k''(z), z^3k'''(z); z) \in z,$$

then $k(z) < \varphi(z)$.

Extensive studies have advanced our understanding of operators linked to differential subordination and superordination (see [17–25]).

Definition 1.3. Let $z \in \mathbb{C}$ and $\varphi \in \mathcal{Q}_1 \cap \mathcal{T}_j$. Define $\partial_1[z, \varphi]$ as the class of admissible functions $\psi : \mathbb{C}^4 \times \mathcal{D} \rightarrow \mathbb{C}$ that satisfy:

$$\psi(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; z) \notin z,$$

whenever

$$\tilde{a} = \varphi(\ell), \quad \tilde{b} = \varphi(\ell) - \frac{n\ell\varphi'(\ell)}{s + \varsigma},$$

$$\operatorname{Re}\left\{\frac{(\tilde{c} - \tilde{a})(s + \varsigma + 1) + (\tilde{a} - \tilde{b})(2s + 2\varsigma + 1)}{\tilde{a} - \tilde{b}}\right\} > n\operatorname{Re}\left\{1 + \frac{\ell\varphi''(\ell)}{\varphi'(\ell)}\right\},$$

and

$$\operatorname{Re}\left\{-\frac{2\tilde{d} + (s + \varsigma)(3\tilde{b}(-1 + s + \varsigma) - 3\tilde{c}(1 + s + \varsigma) + \tilde{d}(3 + s + \varsigma))}{\tilde{a} - \tilde{b}} + \frac{\tilde{a}(-2 + 3\varsigma + 3(s + s^2 + 2s\varsigma + \varsigma^2))}{\tilde{a} - \tilde{b}}\right\} \geq n^2\operatorname{Re}\left\{1 + \frac{\ell^2\varphi'''(\ell)}{\varphi'(\ell)}\right\},$$

for $\ell \in \partial\mathcal{D} \setminus B(\varphi)$ and $n \geq j \geq 2$.

Key references on differential subordination and superordination in p -valent functions include the works of Tayyah and Atshan [26] and Tayyah, Atshan & Oros [27].

2. MATERIALS AND METHODS

Theorem 2.1. Let $z \in \mathbb{C}$ and $\psi \in \partial_1[z, \varphi]$. Suppose $f \in \mathcal{A}_\mu$ and $\varphi \in \mathcal{Q}_1$ satisfy

$$\operatorname{Re} \left\{ \frac{\ell \varphi''(\ell)}{\varphi'(\ell)} \right\} \geq 0, \quad \text{and} \quad \left| z \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}} f(z) \right)' \right| \leq n |\varphi'(\ell)|, \quad (2.1)$$

and

$$\left\{ \psi \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s, r) f(z), z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+1, r) f(z), \right. \right. \\ \left. \left. z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+2, r) f(z), z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+3, r) f(z); z \right) : z \in \mathcal{D} \right\} \subset z, \quad (2.2)$$

then

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s, r) f(z) < \varphi(\ell), \quad z \in \mathcal{D}. \quad (2.3)$$

Proof. Let us define the function

$$k(z) = z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s, r) f(z), \quad (z \in \mathcal{D}). \quad (2.4)$$

Using the recurrence relation of (1.9) and differentiating, we obtain

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+1, r) f(z) = k(z) - \frac{1}{s+\varsigma} z k'(z), \quad (2.5)$$

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+2, r) f(z) = k(z) - \frac{2}{s+\varsigma+1} z k'(z) + \frac{1}{(s+\varsigma)(s+\varsigma+1)} z^2 k''(z), \quad (2.6)$$

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+3, r) f(z) = k(z) - \frac{3}{s+\varsigma+2} z k'(z) + \frac{3}{(s+\varsigma+1)(s+\varsigma+2)} z^2 k''(z) \\ - \frac{1}{(s+\varsigma)(s+\varsigma+1)(s+\varsigma+2)} z^3 k'''(z). \quad (2.7)$$

Now, define

$$\begin{aligned} \tilde{a} &= \tilde{t}, \\ \tilde{b} &= \tilde{t} - \frac{1}{s+\varsigma} \tilde{u}, \\ \tilde{c} &= \tilde{t} - \frac{2}{s+\varsigma+1} \tilde{u} + \frac{1}{(s+\varsigma)(s+\varsigma+1)} \tilde{v}, \\ \tilde{d} &= \tilde{t} - \frac{3}{s+\varsigma+2} \tilde{u} + \frac{3}{(s+\varsigma+1)(s+\varsigma+2)} \tilde{v} - \frac{1}{(s+\varsigma)(s+\varsigma+1)(s+\varsigma+2)} \tilde{w}. \end{aligned} \quad (2.8)$$

Consider the transformation

$$Y(\tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}; z) = \psi(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; z), \quad (2.9)$$

which, via (2.4)–(2.7), gives

$$Y(k(z), z k'(z), z^2 k''(z), z^3 k'''(z); z) = \psi \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s, r) f, \dots, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s+3, r) f; z \right). \quad (2.10)$$

From Definitions 1.2 and 1.3, the admissibility of $\psi \in \partial_1[z, \varphi]$ implies that $Y \in \psi_j[z, \varphi]$, so Lemma 1.1 yields

$$k(z) = z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\mathbb{N}}(s, r) f(z) < \varphi(z). \quad (2.11)$$

□

Corollary 2.1. Let $z \in \mathcal{C}$ and $\varphi \in \mathcal{Q}_1$ be univalent in $\mathcal{D}$. For $\rho \in (0, 1)$, define $\varphi_\rho(z) = \varphi(\rho z)$ and let $\psi \in \partial_1[z, \varphi_\rho]$. If $f \in \mathcal{A}_\mu$ satisfies

$$\operatorname{Re} \left\{ \frac{\ell \varphi_\rho''(\ell)}{\varphi_\rho'(\ell)} \right\} \geq 0, \quad \left| z \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f \right)' \right| \leq n |\varphi_\rho'(\ell)|, \quad (2.12)$$

then

$$\left\{ \psi \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+1, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+2, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+3, r) f; z \right) : z \in \mathcal{D} \right\} \subset z, \quad (2.13)$$

which implies

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f(z) < \varphi(z). \quad (2.14)$$

Proof. By Theorem 2.1, we have

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f(z) < \varphi_\rho(z). \quad (2.15)$$

Since $\varphi_\rho(z) < \varphi(z)$, it follows that

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f(z) < \varphi(z). \quad (2.16)$$

Assume $z \neq C$ is a simply connected domain. A conformal mapping $h : \mathcal{D} \rightarrow \mathcal{C}$ can be constructed with $h(\mathcal{D}) = \mathcal{C}$.

Denote the class $\partial_1[h(\mathcal{D}), \varphi]$ by $\partial_1[h, \varphi]$. Then, from Theorem 2.1 and Corollary 2.1, we directly obtain the following corollaries. □

Corollary 2.2. Let h be univalent in $\mathcal{D}$ and $\psi \in \partial_1[h, \varphi]$. If $\varphi \in \mathcal{Q}_1$ satisfies condition (2.1), then

$$\left\{ \psi \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+1, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+2, r) f, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+3, r) f; z \right) : z \in \mathcal{D} \right\} < h(z), \quad (2.17)$$

which implies

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f(z) < \varphi(z). \quad (2.18)$$

Corollary 2.3. Let $\varphi \in \mathcal{Q}_1$ be univalent in $\mathcal{D}$ and $\psi \in \partial_1[h, \varphi_\rho]$ for some $\rho \in (0, 1)$, where $\varphi_\rho(z) = \varphi(\rho z)$. If φ_ρ satisfies condition (2.12), then the subordination

$$\left\{ \psi \left(z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f, \dots, z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s+3, r) f; z \right) : z \in \mathcal{D} \right\} < h(z) \quad (2.19)$$

implies

$$z^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_\infty}(s, r) f(z) < \varphi(z). \quad (2.20)$$

Corollary 2.4. Let h be as defined in (2.10) and assume Y is univalent in $\mathcal{D}$ so that $\psi \in \partial_1[h, \varphi]$. If there exists a solution $\varphi \in \mathcal{Q}_1$ to the differential equation

$$Y(k, zk', z^2k'', z^3k'''; z) = h(z) \quad (2.21)$$

satisfying condition (2.1), then

$$z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s, r) f(z) < \varphi(z), \quad (2.22)$$

and φ is the best dominant for this subordination.

Proof. If

$$\psi(z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s, r) f, z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+1, r) f, \\ z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+2, r) f, z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+3, r) f; z)$$

satisfies

$$Y(k, zk', z^2 k'', z^3 k'''; z) < h(z), \quad (2.23)$$

then $k(z)$ is a solution and, by Corollary 2.2, we get $k(z) < \varphi(z)$. That is, φ is a dominant of this subordination. Similarly,

$$Y(k, zk', z^2 k'', z^3 k'''; z) < h(z) = Y(\varphi, z\varphi', z^2 \varphi'', z^3 \varphi'''; z),$$

so φ is the best dominant. $\square$

When $\varphi(z) = 1 + Ve^{i\theta}$ with $V > 0$, Theorem 2.1 applies. The admissible class $\partial_1[z, \varphi]$ can be written as $\partial_1[z, V]$:

Definition 2.1. Let $z \subseteq C$ and $V > 0$. The set $\partial_1[z, V]$ consists of all $\psi : C^4 \times \mathcal{D} \rightarrow C$ satisfying

$$\psi \left(\begin{array}{c} 1 + Ve^{i\theta}, 1 + \frac{(s+\zeta-n)Ve^{i\theta}}{s+\zeta}, \\ 1 + \frac{L+(s+\zeta)[(s+\zeta+1)-2n]Ve^{i\theta}}{(s+\zeta+1)(s+\zeta)}, \\ 1 + \frac{3L(s+\zeta)-N+(s+\zeta+1)(s+\zeta)[(s+\zeta+2)-3n]Ve^{i\theta}}{(s+\zeta+2)(s+\zeta+1)(s+\zeta)} \end{array} ; z \right) \notin z,$$

for all $z \in \mathcal{D}$, whenever $\operatorname{Re}\{Le^{-i\theta}\} \geq n(n-1)V$ and $\operatorname{Re}\{Ne^{-i\theta}\} \geq 0$, with $0 \leq \theta \leq \pi$ and $2 \leq j \leq n$.

Corollary 2.5. Let $z \subseteq C$ and $\psi \in \partial_1[z, V]$. If

$$\left| z \left(z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s, r) f(z) \right)' \right| \leq nV,$$

then

$$\psi \left(\begin{array}{c} z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s, r) f, z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+1, r) f, \\ z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+2, r) f, z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s+3, r) f; z \end{array} \right) \in z, \quad (z \in \mathcal{D}),$$

which implies

$$z^{-\mu} \mathcal{J}_{\mu}^{\mathcal{M}_s}(s, r) f(z) < 1 + Ve^{i\theta}.$$

3. APPLICATIONS OF THE OPERATOR $\mathcal{J}_\mu^{\mathcal{M}_s}(s, r)$

Let

$$\mathbf{z} = \{\varphi \in \mathcal{C} : |\varphi - 1| < V\},$$

and denote $\partial_1[\mathbf{z}, V]$ by $\partial_1[V]$. Using Corollary 2.5, we have the following results.

Corollary 3.1. *Let $\psi \in \partial_1[V]$ and suppose*

$$\left| \mathbf{z} \left(\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) \right)' \right| \leq nV.$$

Then

$$\left| \psi \left(\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f, \dots, \mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s+3, r) f; \mathbf{z} \right) - 1 \right| < V, \quad \mathbf{z} \in \mathcal{D},$$

which implies

$$\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) < 1 + V\mathbf{z}.$$

Corollary 3.2. *If*

$$\left| \mathbf{z} \left(\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) \right)' \right| \leq nV,$$

then

$$\left| \mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) - 1 \right| < V, \quad \mathbf{z} \in \mathcal{D},$$

and therefore

$$\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) < 1 + V\mathbf{z}.$$

Proof. Take $\psi(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; \mathbf{z}) = \tilde{b}$. Then Corollary 3.1 gives the desired result.

If we set $\mathcal{M}_s = 1$, $s = \mu + 1$, and $r = 1$, then

$$\begin{aligned} \mathcal{J}_\mu^1(\mu + 1, 1) f(\mathbf{z}) &= f(\mathbf{z}), \\ \mathcal{J}_\mu^1(\mu + 2, 1) f(\mathbf{z}) &= \frac{s + \varsigma + \mu}{s + \varsigma} f(\mathbf{z}) - \frac{1}{s + \varsigma} \mathbf{z} f'(\mathbf{z}). \end{aligned}$$

□

Example 3.1. *Let $f \in \mathcal{A}_\mu$ satisfy*

$$\left| \mathbf{z} \left(\mathbf{z}^{-\mu} \mathcal{J}_\mu^{\mathcal{M}_s}(s, r) f(\mathbf{z}) \right)' \right| \leq nV,$$

and

$$\left| \frac{s + \varsigma + \mu}{s + \varsigma} \mathbf{z}^{-\mu} f(\mathbf{z}) - \frac{1}{s + \varsigma} \mathbf{z}^{-\mu+1} f'(\mathbf{z}) - 1 \right| < V.$$

Then

$$\left| \mathbf{z}^{-\mu} f(\mathbf{z}) - 1 \right| < V.$$

Example 3.2. Taking $M_{\mathfrak{N}} = 1$, $s = \mu + 1$, $r = 1$, and $\mu = j = 1$, we get: If $f \in \mathcal{A}_{\mu}$ satisfies

$$|f'(z) - z^{-1}f(z)| \leq nV,$$

and

$$\left| \frac{s + \varsigma + 1}{s + \varsigma} z^{-1} f(z) - \frac{1}{s + \varsigma} f'(z) - 1 \right| < V,$$

then

$$|z^{-1}f(z) - 1| < V.$$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] A.T. Oladipo, Bounds for Poisson and Neutrosophic Poisson Distributions Associated with Chebyshev Polynomials, *Palestine J. Math.* 10 (2021), 169–174. <https://zbmath.org/7335998>.
- [2] J.A. Antonino, S.S. Miller, Third-Order Differential Inequalities and Subordinations in the Complex Plane, *Complex Var. Elliptic Equ.* 56 (2011), 439–454. <https://doi.org/10.1080/17476931003728404>.
- [3] F. Smarandache, J. Dezert, S. Bhattacharya, A. Buller, M. Khoshnevisan, et al., Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, arXiv:math/0306384, 2003. <https://doi.org/10.48550/arXiv.math/0306384>.
- [4] F. Smarandache, Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, Sitech, (2013). <https://doi.org/10.48550/arXiv.1311.7139>.
- [5] K.F. Alhasan, A.A. Salama, Introduction to Neutrosophic Reliability Theory, *Int. J. Neutrosophic Sci.* 15 (2021), 52–61. <https://doi.org/10.5281/zenodo.5033829>.
- [6] F. Smarandache, Neutropsychic Personality: A Mathematical Approach to Psychology, Pons, (2018). https://digitalrepository.unm.edu/math_fsp/29/.
- [7] I.M. Hanafy, A.A. Salama, K.M. Mahfouz, Correlation Coefficients of Neutrosophic Sets by Centroid Method, *Int. J. Probab. Stat.* 2 (2013), 9–12.
- [8] I.M. Hanafy, A.A. Salama, K. Mahfouz, Correlation of Neutrosophic Data, *Int. Ref. J. Eng. Sci.* 1 (2012), 39–43.
- [9] H. ElWahsh, M. Gamal, A.A. Salama, I.M. El-Henawy, Intrusion Detection System and Neutrosophic Theory for MANETs: a Comparative Study, *Neutrosophic Sets Syst.* 23 (2018), 16–22. <https://doi.org/10.5281/zenodo.2155075>.
- [10] H. Elwahsh, M. Gamal, A.A. Salama, I.M. El-Henawy, Modeling Neutrosophic Data by Self-Organizing Feature Map: MANETs Data Case Study, *Procedia Comput. Sci.* 121 (2017), 152–159. <https://doi.org/10.1016/j.procs.2017.11.021>.
- [11] A. Salama, H.A. El-Ghareeb, A.M. Manie, F. Smarandache, Introduction to Develop Some Software Programs for Dealing with Neutrosophic Sets, *Neutrosophic Sets Syst.* 3 (2014), 51–52.
- [12] A.B. AL-Nafee, F. Smarandache, A.A. Salama, New Types of Neutrosophic Crisp Closed Sets, *Neutrosoph. Sets Syst.* 36 (2020), 175–183. <https://doi.org/10.5281/zenodo.4065428>.
- [13] A.A. Salama, M. Eisa, H. ElGhawalby, A.E. Fawzy, A New Approach in Content-Based Image Retrieval Neutrosophic Domain, in: C. Kahraman, İ. Otay (eds.), *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, Studies in Fuzziness and Soft Computing, Springer, Cham, pp. 361–369, (2019). https://doi.org/10.1007/978-3-030-00045-5_14.
- [14] F.M. Mohammed, S.F. Matar, Fuzzy Neutrosophic Alpha-m-Closed Sets in Fuzzy Neutrosophic Topological Spaces, *Neutrosophic Sets Syst.* 21 (2018), 56–65. <https://fs.unm.edu/NSS/FuzzyNeutrosophicAlphaM.pdf>.

- [15] N.E. Cho, O.S. Kwon, H.M. Srivastava, Inclusion Relationships and Argument Properties for Certain Subclasses of Multivalent Functions Associated with a Family of Linear Operators, *J. Math. Anal. Appl.* 292 (2004), 470–483. <https://doi.org/10.1016/j.jmaa.2003.12.026>.
- [16] A. Çetinkaya, L.-I. Cotîrlă, Briot–Bouquet Differential Subordinations for Analytic Functions Involving the Struve Function, *Fractal Fract.* 6 (2022), 540. <https://doi.org/10.3390/fractalfract6100540>.
- [17] H.A. Farzana, M.P. Jeyaraman, T. Bulboacă, On Certain Subordination Properties of a Linear Operator, *J. Fract. Calc. Appl.* 12 (2021), 148–162.
- [18] S. Porwal, G. Murugusundaramoorthy, Geometric Properties of a New Integral Operator Associated with Dini Functions, *Electron. J. Math. Anal. Appl.* 10 (2022), 267–274. <https://doi.org/10.21608/ejmaa.2022.292057>.
- [19] T.M. Seoudy, Second Order Differential Subordination and Superordination of Liu-Srivastava Operator on Meromorphic Functions, *Afr. Mat.* 32 (2021), 1399–1408. <https://doi.org/10.1007/s13370-021-00907-4>.
- [20] H.M. Srivastava, A. Prajapati, P. Gochhayat, Third-Order Differential Subordination and Differential Superordination Results for Analytic Functions Involving the Srivastava-Attiya Operator, *Appl. Math. Inf. Sci.* 12 (2018), 469–481. <https://doi.org/10.18576/amis/120301>.
- [21] A. Almalkawi, A. Alsoboh, A. Amourah, T. Sasa, Estimates for the Coefficients of Subclasses Defined by the q -Babalola Convolution Operator of Bi-Univalent Functions Subordinate to the q -Fibonacci Analogue, *Eur. J. Pure Appl. Math.* 18 (2025), 6499. <https://doi.org/10.29020/nybg.ejpam.v18i3.6499>.
- [22] M. Ahmed, A. Alsoboh, A. Amourah, J. Salah, On the Fractional q -Differintegral Operator for Subclasses of Bi-Univalent Functions Subordinate to q -Ultraspherical Polynomials, *Eur. J. Pure Appl. Math.* 18 (2025), 6586. <https://doi.org/10.29020/nybg.ejpam.v18i3.6586>.
- [23] A. Alsoboh, M. Çağlar, M. Buyankara, Fekete–Szegő Inequality for a Subclass of Bi-Univalent Functions Linked to q -Ultraspherical Polynomials, *Contemp. Math.* 5 (2024), 2531–2545. <https://doi.org/10.37256/cm.5220243737>.
- [24] A. Alsoboh, A. Amourah, W.G. Atshan, J. Salah, T. Sasa, A Study of Bi-Univalent Class in Leaf-Like Domains Using Quantum Calculus Through Subordination, *Int. J. Anal. Appl.* 23 (2025), 293. <https://doi.org/10.28924/2291-8639-23-2025-293>.
- [25] A. Alsoboh, A. Amourah, A. Al Kasbi, J. Salah, A.A.-R. Malkawi, T. Sasa, A Novel Subclass of Bi-Univalent Functions Defined by the q -Wright Operator and the q -Analogue of Fibonacci Numbers, *Int. J. Anal. Appl.* 24 (2026), 78. <https://doi.org/10.28924/2291-8639-24-2026-78>.
- [26] A.S. Tayyah, W.G. Atshan, Differential Subordination and Superordination Results for p -Valent Analytic Functions Associated with (r, k) -Srivastava Fractional Integral Calculus, *MethodsX* 13 (2024), 103079. <https://doi.org/10.1016/j.mex.2024.103079>.
- [27] A.S. Tayyah, W.G. Atshan, G.I. Oros, Third-Order Differential Subordination Results for Meromorphic Functions Associated with the Inverse of the Legendre Chi Function via the Mittag-Leffler Identity, *Mathematics* 13 (2025), 2089. <https://doi.org/10.3390/math13132089>.