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ON THE STABILIZATION OF THE LINEAR KAWAHARA EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. We study the stabilization of global solutions of the linear Kawahara equation (K) with periodic boundary conditions under the effect of a localized damping mechanism. The Kawahara equation is a model for small amplitude long waves. Using separation of variables, the Ingham inequality, multiplier techniques and compactness arguments we prove the exponential decay of the solutions of the (K) model.

1. Introduction

In this paper we study the stabilization of global solutions of the linear Kawahara equation (K) with periodic boundary conditions under the effect of a localized damping mechanism, that is, we consider the following problem:

$$\begin{cases} u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} + a(x)u = 0 & x \in (0,2\pi),\ t>0 \\ u(0,t) = u(2\pi,t), & t>0 \\ u_x(0,t) = u_x(2\pi,t), & t>0 \\ u_{xx}(0,t) = u_{xx}(2\pi,t), & t>0 \\ u_{xxx}(0,t) = u_{xxx}(2\pi,t), & t>0 \\ u_{xxxx}(0,t) = u_{xxxx}(2\pi,t), & t>0 \\ u(x,0) = u_0(x), & x \in (0,2\pi) \end{cases} \tag{1.1}$$

The parameter η is a negative real number, $\kappa \neq 0$, β is a real number and $a \in L^\infty(0,2\pi)$, $a \geq 0$ a.e. in $(0,2\pi)$ and we assume that $a(x) \geq a_0 > 0$ a.e. in an open subinterval ω of $(0,2\pi)$, where the damping is effectively acting .
In the Kawahara equation

$$u_t + u_x + \kappa u_{xxx} + \eta u_{xxxxx} + uu_x = 0, \tag{1.2}$$

the conservative dispersive effect is represented by the term $(\kappa u_{xxx} + \eta u_{xxxxx})$. This equation is a model for plasma wave, capilarity-gravity water waves and other dispersive phenomena when the cubic KdV-type equation is weak. Kawahara [10] pointed out that it happens when the coefficient of the third order derivative in the KdV equation becomes very small or even zero. It is then necessary to take into account the higher order effect of dispersion in order to balance the nonlinear effect. Kakutani and Ono [9] showed that for a critical value of angle between the magneto-acoustic wave in a cold collision-free plasma and the external magnetic

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field, the third order derivative term in the KdV equation vanishes and may be replaced by the fifth order derivative term. Following this idea, Kawahara [10] studied a generalized nonlinear dispersive equation which has a form of the KdV equation with an additional fifth order derivative term. This equation has also been obtainded by Hasimoto [8] for the shallow wave near critical values of surface tension. More precisely, in this work Hasimoto found these critical values when the Bond number is near to one third.

While analyzing the evolution of solutions of the water wave-problem, Schneider and Wayne [19] also showed that the coefficient of the third order dispersive term in nondimensionalized statements of the KdV equation vanishes when the Bond number is equal to one third. The Bond number is proportional to the strength of the surface tension and in the KdV equation it is related to the leading order dispersive effects in the water-waves problem. With its disappearance, the resulting equation is just Burger's equation whose solutions typically form shocks in finite time. Thus, if we wish to model interesting behavior in the water-wave problem it is necessary to include higher order terms. That is, it is necessary to consider the Kawahara equation. In any case, the inclusion of the fifth order derivative term takes into account the comparative magnitude of the coefficients of the third and fifth power terms in the linearized dispersion relation.

Berloff and Howard [3] presented the Kawahara equation as the purely dispersive form of the following nonlinear partial differential equation

$$u_t + u^r u_x + au_{xx} + bu_{xxx} + cu_{xxxx} + du_{xxxxx} = 0.$$

The above equation describes the evolution of long waves in various problems in fluid dynamics. The Kawahara equation corresponds to the choice $a = c = 0$ and $r = 1$ and describes water waves with surface tension. Bridges and Derks [6] presented the Kawahara equation – or fifth-order KdV-type equation – as a particular case of the general form

$$u_t + \kappa u_{xxx} + \eta u_{xxxxx} = \frac{\partial}{\partial x} f(u, u_x, u_{xx}) \tag{1.3}$$

where $u(x,t)$ is a scalar real valued function, κ and $\eta \neq 0$ are real parameters and $f(u, u_x, u_{xx})$ is some smooth function. The form (1.2) occurs most often in applications and corresponds to the choice of f in (1.3) with the form $f(u, u_x, u_{xx}) = -\frac{u^2}{2}$.

As noted by Kawahara [10], we may assume without loss of generality that $\eta < 0$ in (1.2). In fact, if we introduce the following simple transformations

$$u \to -u, \quad x \to -x \quad \text{and} \quad t \to t$$

we can obtain an equation of the form of equation (1.2) in which κ and η are replaced, respectively, by $-\kappa$ and $-\eta$.

Hagarus *et al.* pointed out that the Kawahara equation

$$u_t = u_{xxxxx} - \varepsilon u_{xxx} + uu_x \tag{1.4}$$

in which ε is a real parameter models water waves in the long-wave regime for moderate values of surface tension, Weber numbers close to 1/3; and that for such Weber numbers the usual description of long water waves via the Korteweg-de Vries (KdV) equation fails since the cubic term in the linear dispersion relation vanishes and fifth order dispersion becomes relevant at leading order, $\omega(k) = k^5 + \varepsilon k^3$.

Positive (resp. negative) values of the parameter ε in (1.4) correspond to Weber numbers larger (resp. smaller) than $1/3$.
Dispersive problems have been object of intensive research (see, for instance, the classical paper of Benjamin-Bona-Mahoni [2], Biagioni-Linares [4], Bona-Chen [5], Menzala *et al.* [15], Rosier [16], and references therein). Recently global stabilization of the generalized KdV system have been obtained by Rosier-Zhang [17] and Linares-Pazoto[12] with critical exponents. For the stabilization of global solutions of the Kawahara under the effect of a localized damping mechanism, see Vasconcellos and Silva [20, 21].
For controllability problems involving dispersive systems, we can consider the works of Russel-Zhang [18] and Laurent *et al.* [12] about the KdV system; the paper by Linares-Ortega [14], where the Benjamin-Ono equation has been analyzed and the paper of Zhang and Zhao [22] for the Kawahara equation.
The total energy associated with the (1.1) system is defined by

$$E(t) = \frac{1}{2}\int_0^{2\pi} |u(x,t)|^2 dx = \frac{1}{2}\|u(t)\|^2.$$

Using the above boundary conditions we prove that

$$\frac{dE}{dt} = \frac{\eta}{2}|u_{xx}(0,t)|^2 - \int_0^{2\pi} a(x)|u(x,t)|^2 dx \le 0, \quad \forall t > 0.$$

So, $E(t)$ is a nonincreasing function of time. This paper is devoted to analyze the following questions: Does the energy $E(t) \to 0$ as $t \to \infty$? Is it possible to find a rate of decay of the energy?
Then, we can state our main result:

Theorem 1.1. *There exist $C > 0$ and $\gamma > 0$ such that the energy $E(t)$ associated to the problem (1.1) satisfies*

$$E(t) \le Ce^{-\gamma t}\|u_0\|^2_{L^2(0,2\pi)}$$

for all $u_0 \in L^2(0,2\pi)$.

To prove the above theorem we need some generalizations of Ingham inequality (see for instance [1],[7] and [11]), multiplier techniques and compactness arguments.
We organize this work as follows.
In Section 2, we present some auxiliary lemmas, useful to demonstrate our main result. In Section 3, we prove Theorem 1.1 and in Section 4, we present our final remarks.

2. Auxiliary Lemmas

Lemma 2.1. *Consider the problem:*

$$\begin{cases} v_t + \beta v_x + \kappa v_{xxx} + \eta v_{xxxxx} = 0 & x \in (0,2\pi),\ t > 0 \\ v(0,t) = v(2\pi,t), & t > 0 \\ v_x(0,t) = v_x(2\pi,t), & t > 0 \\ v_{xx}(0,t) = v_{xx}(2\pi,t), & t > 0 \\ v_{xxx}(0,t) = v_{xxx}(2\pi,t), & t > 0 \\ v_{xxxx}(0,t) = v_{xxxx}(2\pi,t), & t > 0 \\ v(x,0) = u_0(x), & x \in (0,2\pi) \end{cases} \tag{2.5}$$

The parameter η is a negative real number, $\kappa \neq 0$ and β is a real number. Then, for $T > 0$, there exists a constant $C_1 = C_1(T) > 0$ such that

$$\|u_0\|^2_{L^2(0,2\pi)} \leq C_1 \int_0^T \int_\omega |v(x,t)|^2 dxdt,$$

where ω is an open subinterval of $(0, 2\pi)$.

Proof. We assume a solution v of the system (2.5) can be written as $v(x,t) = X(x)T(t)$. Then

$$XT' + \beta TX' + \kappa TX''' + \eta TX''''' = 0$$

that is

$$\frac{T'}{T} = -\frac{\beta X' + \kappa X''' + \eta X'''''}{X} = \lambda$$

for some constant λ. Thus, we obtain

$$\begin{cases} \beta X' + \kappa X''' + \eta X''''' + \lambda X = 0 \quad x \in (0, 2\pi), \\ X(0) = X(2\pi), \\ X'(0) = X'(2\pi), \\ X''(0) = X''(2\pi), \\ X'''(0) = X'''(2\pi), \\ X''''(0) = X''''(2\pi), \end{cases} \tag{2.6}$$

and

$$T' - \lambda T = 0 \tag{2.7}$$

To solve (2.6), we use the characteristic equation

$$\eta r^5 + \kappa r^3 + \beta r + \lambda = 0.$$

We can show that the eigenvalues λ are pure imaginary numbers. Notice that for each $k \in \mathbb{Z}$, the function

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

is an eigenfunction of (2.6) associated with the eigenvalue

$$\lambda_k = (-\eta k^5 + \kappa k^3 - \beta k)i.$$

Furthermore, for any $l \in \mathbb{Z}$, let

$$m_l = \#\{k \in \mathbb{Z}, \quad \lambda_k = \lambda_l\}.$$

Then, $m_l \leq 5$ for any l and in particular $m(l) = 1$, if $|l|$ is large enough. Moreover,

$$\lim_{|k| \to \infty} |\lambda_k - \lambda_{k+1}| = \infty. \tag{2.8}$$

We have

$$X(x) = C_k e^{ikx}, \quad k \in \mathbb{Z}. \tag{2.9}$$

Then, by (2.7) and (2.9), it follows that

$$v(x,t) = \sum_{k \in \mathbb{Z}} c_k e^{i(kx + \sigma_k t)}, \qquad \sigma_k = -\eta k^5 + \kappa k^3 - \beta k \tag{2.10}$$

where

$$u_0(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}.$$

As pointed out by, Jaffard and Micu [7],

$$\limsup_{n} |\lambda_{n+1} - \lambda_n| > \frac{2\pi}{T}$$

gives a sufficient condition for the validity of an Ingham type inequality For each T, since we have (2.8), from an Ingham inequality (see for instance Theorem 3.5 in Baiocchi, Komornik and Loreti [1] for Ingham inequalities for sequences with repeated eigenvalues and with weak gap conditions.), it follows that there exists a constant $C = C(T) > 0$ such that

$$\|u_0\|^2_{L^2(0,2\pi)} = \sum_{k\in\mathbb{Z}} |c_k|^2 \leq C(T) \int_0^T \left| \sum_{k\in\mathbb{Z}} c_k e^{i\sigma_k} t \right|^2 dt \tag{2.11}$$

Therefore, using (2.11) and the Fubini Theorem, we have

$$\begin{aligned}
\int_0^T \int_\omega |v(x,t)|^2 dxdt &= \int_\omega \int_0^T \left| \sum_{k\in\mathbb{Z}} c_k e^{ikx} e^{i\sigma_k t} \right|^2 dtdx \\
&\geq \frac{1}{C(T)} \int_\omega \sum_{k\in\mathbb{Z}} \left| c_k e^{ikx} \right|^2 dx = \frac{1}{C(T)} \int_\omega \sum_{k\in\mathbb{Z}} |c_k|^2 dx \\
&= \frac{l(\omega)}{C(T)} \sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{l(\omega)}{C(T)} \|u_0\|^2_{L^2(0,2\pi)}.
\end{aligned}$$

□

Here, we denote by $l(\omega)$ the length of subset ω

Lemma 2.2. *Let w be a solution of the following problem:*

$$\begin{cases}
w_t + \beta w_x + \kappa w_{xxx} + \eta w_{xxxxx} = -a(x)u(x,t) & x \in (0,2\pi),\ t > 0 \\
w(0,t) = w(2\pi,t), & t > 0 \\
w_x(0,t) = v_w(2\pi,t), & t > 0 \\
w_{xx}(0,t) = w_{xx}(2\pi,t), & t > 0 \\
w_{xxx}(0,t) = w_{xxx}(2\pi,t), & t > 0 \\
w_{xxxx}(0,t) = w_{xxxx}(2\pi,t), & t > 0 \\
w(x,0) = 0, & x \in (0,2\pi)
\end{cases} \tag{2.12}$$

where $a = \chi_\omega$, $\omega \subset (0,2\pi)$ and u is the solution of (1.1). The parameter η is a negative real number, $\kappa \neq 0$ and β is a real number. Then, for $T > 0$, there exists a constant $C_2 = C_2(T) > 0$ such that

$$\|w(t)\|^2_{L^2(0,2\pi)} \leq C_2 \int_0^T \int_\omega |u(x,t)|^2 dxdt.$$

Proof. If we multiply the equation (2.12) by w, integrate in $(0,2\pi)$ and use the periodic boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(0,2\pi)} = -\int_\omega u(x,t)w(x,t)dx, \qquad t > 0.$$

Thus

$$\frac{d}{dt}\|w(t)\|^2_{L^2(0,2\pi)} \leq \int_\omega |u(x,t)|^2 dx + \int_0^{2\pi} |w(x,t)|^2 dx.$$

Now, if $\gamma(t) = \|w(t)\|^2_{L^2(0,2\pi)}$, we obtain

$$\begin{cases} \gamma'(t) \leq g(t) + \gamma(t) \\ \gamma(0) = 0 \end{cases}$$

where $g(t) = \int_\omega |u(x,t)|^2 dx$. Hence, by Gronwall inequality, there exists a constant $C_2 = C_2(T) > 0$, such that

$$\gamma(t) \leq C_2(T) \int_0^T g(t)dt, \qquad t \in (0,T)$$

and the Lemma follows. □

Lemma 2.3. *For each $T > 0$, there exists a constant $C_3 = C_3(T) > 0$ such that*

$$\frac{1}{2}\|u_0\|^2_{L^2(0,2\pi)} \leq C_3 \int_0^T \int_\omega |u(x,t)|^2 dxdt,$$

where u is the solution of (1.1).

Proof. Let v and w be respectively the solutions of the problems (2.5) and (2.12). So we have $u = v + w$ (or $v = u - w$). Now using Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} \|u_0\|^2_{L^2(0,2\pi)} &\leq C_1 \int_0^T \int_\omega |v(x,t)|^2 dxdt \\ &\leq 2C_1 \left[\int_0^T \int_\omega |u(x,t)|^2 dxdt + \int_0^T \int_\omega |w(x,t)|^2 dxdt \right] \\ &\leq 2C_1 \left[\int_0^T \int_\omega |u(x,t)|^2 dxdt + C_2 T \int_0^T \int_\omega |u(x,t)|^2 dxdt \right] \\ &= 2C_1(1 + C_2 T) \int_0^T \int_\omega |u(x,t)|^2 dxdt. \end{aligned}$$

The inequality stated in the lemma holds with $C_3 = C_1(1 + C_2 T)$. □

3. Proof of Theorem 1.1

Now, we are able to prove Theorem 1.1. In fact, if we multiply the equation in system (1.1) by u and integrate in $(0, 2\pi)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,2\pi)} = -\int_\omega |u(x,t)|^2 dxdt \leq 0$$

So, $E(t) = \frac{1}{2}\|u(t)\|^2_{L^2(0,2\pi)}$ is a decreasing function of time and moreover

$$E(T) - E(0) = -\int_0^T \int_\omega |u(x,t)|^2 dxdt.$$

Thus

$$(1 + C_3)E(T) = -C_3 \int_0^T \int_\omega |u(x,t)|^2 dxdt + C_3 E(0) + E(T).$$

Since

$$E(T) \leq E(0) = \frac{1}{2}\|u_0\|^2_{L^2(0,2\pi)},$$

it follows, by Lemma 2.3, that:

$$(1 + C_3)E(T) \leq C_3 E(0).$$

Therefore

$$E(T) \leq \frac{C_3}{1+C_3}E(0), \qquad T > 0.$$

Finally, we use the semigroup property to obtain Theorem 1.1.

Remark 3.1. *In the Lemma 2.2 and in the Theorem 1.1, we can consider* $a \in L^\infty(0,2\pi)$*,* $a \geq 0$ *a.e. in* $(0,2\pi)$ *and assume that* $a(x) \geq a_0 > 0$ *a.e. in an open subinterval* ω *of* $(0,2\pi)$ *and the proofs follow in the same way.*

4. Final Remarks

We can observe that, if we consider the parameter $\beta = 0$ in the system (1.1) the Theorem 1.1 follows similarly.

Now, we will make some comments concerning the exact controllability for Kawahara system:

In the linear case, boundary exact controllability is proved, using HUM method and multipliers techniques, by Vasconcellos-Silva [20].

In the nonlinear case, internal exact controllability can be found in Zhang-Zhao [22], where was considered periodic domain with an internal control acting on an arbitrary small nonempty subdomain of $[0,2\pi]$. Aided by the Bourgain smoothing property of the Kawahara equation on a periodic domain, it was showed that the system is locally exactly controllable.

We believe that it is possible to show the boundary exact controllability for linear Kawahara system in periodic domain

That is, we consider the following problem:

Given u_0 and u_T in $L^2(0,L)$, find $h_j \in L^2(0,L)$, $\quad j = 0,1,2,3,4$ such that the solution of the bellow system:

$$\begin{cases} u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} = 0 & x \in (0,2\pi),\ t > 0 \\ u(0,t) - u(2\pi,t) = h_0, & t > 0 \\ u_x(0,t) - u_x(2\pi,t) = h_1, & t > 0 \\ u_{xx}(0,t) - u_{xx}(2\pi,t) = h_2, & t > 0 \\ u_{xxx}(0,t) - u_{xxx}(2\pi,t) = h_3, & t > 0 \\ u_{xxxx}(0,t) - u_{xxxx}(2\pi,t) = h_4, & t > 0 \\ u(.,0) = u_0 \end{cases} \tag{4.13}$$

satisfies $u(\cdot,T) = u_T$.

As proved by Rosier in [16] for linear KdV system, we could use HUM method and generalizations of Ingham inequalities to obtain the above problem.

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