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Fixed Point Theorems in Interpolative $\mathfrak{G}$-Metric Spaces: A Novel Approach

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Abstract. In this paper, we have introduced a new notion of interpolative $\mathfrak{G}$-metric space. We establish a fixed point theorem for a contractive mapping in interpolative $\mathfrak{G}$-metric space. Some examples are also provided to illustrate the validity of the result. The presented theorem extends, generalizes and refines various existing results from the literature. As an application we present a model to establish convergence in group decision making.

1. Introduction

Metric fixed point theory plays a crucial role and it has many applications in sciences, pure and applied mathematics. In 1922, Banach [1] proved one of the very interesting fixed point results known as the Banach Contraction principle. In 2005, the concept of $\mathfrak{G}$-metric space was introduced by Mustafa and Sims [2]. In addition to this they proved some of the interesting fixed point results for the same (see [3]). A spade of research in the field of metric fixed point theory and its applications using various types of metric and metric like spaces using various generalisations of Banach contraction, see [5–11].

In 2024, Karapinar [12] introduced the notion of an interpolative metric space which is natural extension of metric space as follows:

Definition 1.1 ([12]). *Let $\mho$ be a nonempty set. A mapping $d : \mho \times \mho \to [0, +\infty)$ is said to be (α, c)-interpolative metric if*

$K1$: $d(\mathfrak{x}, \mathfrak{y}) = 0$, *if and only if*, $\mathfrak{x} = \mathfrak{y}$, *for all* $\mathfrak{x}, \mathfrak{y} \in \mho$;

$K2$: $d(\mathfrak{x}, \mathfrak{y}) = d(\mathfrak{y}, x)$, *for all* $\mathfrak{x}, \mathfrak{y} \in \mho$;

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*K*3: *There exists an* $\alpha \in (0,1)$ *and* $c \geq 0$ *such that*

$$d(\mathfrak{x},\mathfrak{y}) \leq d(\mathfrak{x},\mathfrak{z}) + d(\mathfrak{z},\mathfrak{y}) + c[(d(\mathfrak{x},\mathfrak{z}))^{\alpha}(d(\mathfrak{z},\mathfrak{y}))^{1-\alpha}]$$

for all $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mho$.

The pair $(\mho, d)$ *is called* (α, c)*-interpolative metric space.*

Example 1.1. *Let* $\mho$ *be a nonempty set and define a function* $d : \mho \times \mho \to [0,\infty]$ *as follows:*

$$d(\mathfrak{x},\mathfrak{y}) := |\mathfrak{x}-\mathfrak{y}|^2, \quad \text{for all } \mathfrak{x},\mathfrak{y} \in \mho.$$

The conditions $(k1)$, $(k2)$ *are satisfied trivially. For the condition* $(k3)$, *we have*

$$\begin{aligned} d(\mathfrak{x},\mathfrak{y}) &= |\mathfrak{x}-\mathfrak{y}|^2 \\ &= |\mathfrak{x}-\mathfrak{z}+\mathfrak{z}-\mathfrak{y}|^2 \\ &\leq |\mathfrak{x}-\mathfrak{z}|^2 + |\mathfrak{z}-\mathfrak{y}|^2 + 2|\mathfrak{x}-\mathfrak{z}|\,|\mathfrak{z}-\mathfrak{y}| \\ &\leq d(\mathfrak{x},\mathfrak{z}) + d(\mathfrak{z},\mathfrak{y}) + 2[d(\mathfrak{x},\mathfrak{z})^{\frac{1}{2}} d(\mathfrak{z},\mathfrak{y})^{\frac{1}{2}}]\,. \end{aligned}$$

Therefore, the condition $(K3)$ *is satisfied. Here* $\alpha = \frac{1}{2} \in (0,1)$ *and* $c \geq 2$.
Hence $(\mho, d)$ *is* $\left(\frac{1}{2}, 2\right)$*-interpolative metric space.*

Assume that $r > 0$ *and* $\mathfrak{x} \in \mho$. *Denote*

$$\mathfrak{B}(\mathfrak{x}, r) = \{y \in \mho : d(\mathfrak{x},\mathfrak{y}) < r\},$$

an open ball in (α, c)*-interpolative metric space* $(\mho, d)$.

In addition to this, some of the interesting fixed point results are also proved by Karapinar and Aggarwal [12] in the interpolative metric spaces. Inspired, in this article we introduce the notion of interpolative $\mathfrak{G}$-metric space and establish the fixed point theorem in the setting of this space. The rest of the paper is organised as follows: In Section-2, we establish fixed point results in the setting of interpolative $\mathfrak{G}$-metric space We also supplement the derived result through non trivial examples. In Section-3, we present an application to decision making based on derived results and in Section-4, we present an application to find analytical solution to Integral equations. We conclude the manuscript with open problems.

Throughout the paper, N_0 denotes the sets of whole numbers.

2. Main Result

In this section, we shall introduce a new concept of (α, c)-interpolative $\mathfrak{G}$-metric space in which rectangle inequality is replaced by a new inequality in the broader sense.

Definition 2.1. *Let* $\mho$ *be a nonempty set. A function* $\mathfrak{G} : \mho^3 \to [0,+\infty)$ *is said to be* (α, c)*-interpolative* $\mathfrak{G}$*-metric if*

*G*1: $\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = 0$ *if* $\mathfrak{x} = \mathfrak{y} = \mathfrak{z}$;

*G*2: $\mathfrak{G}(\mathfrak{x},\mathfrak{x},\mathfrak{y}) > 0$; *for all* $\mathfrak{x},\mathfrak{y} \in \mho$, *with* $\mathfrak{x} \neq \mathfrak{y}$;

*G*3: $\mathfrak{G}(\mathfrak{x},\mathfrak{x},\mathfrak{y}) \le \mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z})$, *for all* $\mathfrak{x}, y, \mathfrak{z} \in \mho$, *with* $z \neq y$;

*G*4: $\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \mathfrak{G}(x,\mathfrak{z},\mathfrak{y}) = \mathfrak{G}(\mathfrak{y},\mathfrak{z},\mathfrak{x}) = \ldots$, *(symmetry in all three variables);*

*G*5: $\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) \le \mathfrak{G}(\mathfrak{x},a,a) + \mathfrak{G}(a,\mathfrak{y},\mathfrak{z}) + c[(\mathfrak{G}(\mathfrak{x},a,a))^{\alpha}(\mathfrak{G}(a,\mathfrak{y},\mathfrak{z}))^{1-\alpha}]$,

for all $\mathfrak{x},\mathfrak{y},\mathfrak{z},a \in \mho$.

The pair $(\mho,\mathfrak{G})$ *is called an* (α,c)*-interpolative* $\mathfrak{G}$*-metric space.*

Example 2.1. *Let* $(\mho,\delta)$ *be a standard metric space. Define a function* $\mathfrak{G}:\mho\times\mho\times\mho\to[0,\infty)$ *as follow:*

$$\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) := \delta(\mathfrak{x},\mathfrak{y},\mathfrak{z})(\delta(\mathfrak{x},\mathfrak{y},\mathfrak{z})+A), \quad \textit{for all } \mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mho.$$

Since δ *is a metric on* $\mho$, *the conditions* $(G1)$-$(G4)$ *are satisfied. For* $(G5)$, *it is adequate to assume* $c \ge 2$ *for any* $\alpha \in (0,1)$. *Thus,* $(\mho,\mathfrak{G})$ *is* $\left(\frac{1}{2},2\right)$*-interpolative* $\mathfrak{G}$*-metric space.*

We have

$$\begin{aligned}
\mathfrak{G}(\mathfrak{x},y,\mathfrak{z}) &= \delta(\mathfrak{x},\mathfrak{y},\mathfrak{z})(\delta(\mathfrak{x},\mathfrak{y},\mathfrak{z})+A)\\
&\le (\delta(\mathfrak{x},a,a)+\delta(a,\mathfrak{y},\mathfrak{z}))(\delta(\mathfrak{x},a,a)+\delta(a,\mathfrak{y},\mathfrak{z})+A)\\
&\le [\delta(\mathfrak{x},a,a)(\delta(\mathfrak{x},a,a)+A)+\delta(\mathfrak{x},a,a)\delta(a,\mathfrak{y},\mathfrak{z})]\\
&\quad + [\delta(a,\mathfrak{y},\mathfrak{z})(\delta(a,\mathfrak{y},\mathfrak{z})+A)+\delta(a,\mathfrak{y},\mathfrak{z})\delta(\mathfrak{x},a,a)]\\
&\le [\delta(\mathfrak{x},a,a)(\delta(\mathfrak{x},a,a)+A)]+[\delta(a,\mathfrak{y},\mathfrak{z})(\delta(a,\mathfrak{y},\mathfrak{z})+A)]+2\delta(\mathfrak{x},a,a)\delta(a,\mathfrak{y},\mathfrak{z})\\
&\le \mathfrak{G}(\mathfrak{x},a,a)+\mathfrak{G}(a,\mathfrak{y},\mathfrak{z})+2(\delta(\mathfrak{x},a,a))^{\frac{1}{2}}(\delta(\mathfrak{x},a,a))^{\frac{1}{2}}(\delta(a,\mathfrak{y},\mathfrak{z}))^{\frac{1}{2}}(\delta(a,\mathfrak{y},\mathfrak{z}))^{\frac{1}{2}}\\
&\le \mathfrak{G}(\mathfrak{x},a,a)+\mathfrak{G}(a,\mathfrak{y},\mathfrak{z})+2(\delta(\mathfrak{x},a,a))^{\frac{1}{2}}[\delta(\mathfrak{x},a,a)+A]^{\frac{1}{2}}(\delta(a,\mathfrak{y},\mathfrak{z}))^{\frac{1}{2}}[\delta(a,\mathfrak{y},\mathfrak{z})+A]^{\frac{1}{2}}\\
&\le \mathfrak{G}(\mathfrak{x},a,a)+\mathfrak{G}(a,\mathfrak{y},\mathfrak{z})+2(\mathfrak{G}(\mathfrak{x},a,a)^{\frac{1}{2}}(\mathfrak{G}(a,\mathfrak{y},\mathfrak{z}))^{\frac{1}{2}}.
\end{aligned}$$

Thus, the function $\mathfrak{G}\,(\mathfrak{x},\mathfrak{y},\mathfrak{z})$ *does not form a* $\mathfrak{G}$*-metric.*

Example 2.2. *Let* $\mho$ *be a nonempty set and define a function* $\mathfrak{G}:\mho\times\mho\times\mho\to[0,\infty]$ *as follows:*

$$\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = |\mathfrak{x}-\mathfrak{y}|^2+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-x|^2, \quad \textit{for all } \mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mho.$$

The conditions $(G1)$-$(G4)$ *are satisfied trivially. As for the condition* $(G5)$, *we have*

$$\begin{aligned}
\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) &= |(\mathfrak{x}-a)-(\mathfrak{y}-a)|^2+|\mathfrak{y}-\mathfrak{z}|^2+|(\mathfrak{z}-a)-(\mathfrak{x}-a)|^2\\
&\le |\mathfrak{x}-a|^2+|\mathfrak{y}-a|^2+2|\mathfrak{x}-a|\,|\mathfrak{y}-a|+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-a|^2+|\mathfrak{x}-a|^2\\
&\quad +2|\mathfrak{z}-a|\,|\mathfrak{x}-a|\\
&\le 2|\mathfrak{x}-a|^2+|\mathfrak{y}-a|^2+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-a|^2+2|\mathfrak{x}-a|\,|\mathfrak{y}-a|+2|\mathfrak{z}-a|\,|\mathfrak{x}-a|\\
&\le 2|\mathfrak{x}-a|^2+|\mathfrak{y}-a|^2+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-a|^2+2|\mathfrak{x}-a|\,(|\mathfrak{y}-a|+|\mathfrak{z}-a|)\\
&\le \mathfrak{G}(\mathfrak{x},a,a)+\mathfrak{G}(a,\mathfrak{y},\mathfrak{z})+2[\mathfrak{G}(\mathfrak{x},a,a)^{\frac{1}{2}}G(a,\mathfrak{y},\mathfrak{z})^{\frac{1}{2}}.
\end{aligned}$$

Therefore, the condition $(G5)$ *is satisfied.*

Here $\alpha=\frac{1}{2}\in(0,1)$ *and* $c\ge 2$.

Hence $(\mho,G)$ *is* $\left(\frac{1}{2},2\right)$*-interpolative* $\mathfrak{G}$*-metric space.*

Definition 2.2. *Let $(\mho, \mathfrak{G})$ be a (α, c)-interpoloative $\mathfrak{G}$-metric space and let $\{\mathfrak{x}_n\}$ be a sequence in $\mho$. We say that $\{\mathfrak{x}_n\}$ is $\mathfrak{G}$-converges to x in $\mho$, if $\mathfrak{G}(\mathfrak{x}, \mathfrak{x}_n, \mathfrak{x}_m) \to 0$ as $n, m \to \infty$.*

Definition 2.3. *Let $(\mho, \mathfrak{G})$ be a (α, c)-interpolative $\mathfrak{G}$-metric space and let $\{\mathfrak{x}_n\}$ be a sequence in $\mho$. We say that $\{\mathfrak{x}_n\}$ is $\mathfrak{G}$-Cauchy sequence in $\mho$, if and only if, $\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_m, \mathfrak{x}_l) \to 0$ as $n, m, l \to \infty$.*

Definition 2.4. *Let $(\mho, \mathfrak{G})$ be a (α, c)-interpolative $\mathfrak{G}$-metric space. We say that $(\mho, \mathfrak{G})$ is a complete (α, c)-interpolative $\mathfrak{G}$-metric space if every $\mathfrak{G}$-Cauchy sequence converges in $\mho$.*

Theorem 2.1. *Let $(X, \mathfrak{G})$ be a complete (α, c)-interpolative $\mathfrak{G}$-metric space and let $\sqsupset$ be a self map on $\mho$. Suppose that there exists k with $k \in [0, 1)$ such that*

$$\mathfrak{G}(\sqsupset\mathfrak{x}, \sqsupset\mathfrak{y}, \sqsupset\mathfrak{z}) \leq k\mathfrak{G}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}), \tag{2.1}$$

for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mho$.

Then, $\sqsupset$ has unique fixed point in $\mho$.

Proof. Let $\mathfrak{x}_0 \in \mho$ be a arbitrary point and consider a sequence $\{\mathfrak{x}_n\}$. As follows $\mathfrak{x}_{n+1} = \sqsupset\mathfrak{x}_n$ for all $n \in \mathbb{N}_0$. If $\mathfrak{x}_{n_0} = \mathfrak{x}_{n_{0+1}}$ for any $n_0 \in \mathbb{N}_0$, then $\mathfrak{x}_{n_0} = \mathfrak{x}_{n_{0+1}} = \sqsupset\mathfrak{x}_{n0}$. In other words $\mathfrak{x}_{n_0}$ turns into fixed point that completes the proof. We shall assume $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$ for all $n \in \mathbb{N}_0$.

Thus, it is deduced that $\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) > 0$, for all $n \in \mathbb{N}_0$.

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) \leq k\mathfrak{G}(\mathfrak{x}_{n-1}, \mathfrak{x}_n, \mathfrak{x}_n). \tag{2.2}$$

By successive iterations, we obtain

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) \leq k^n\mathfrak{G}(\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_1), \quad \text{for all } n \in \mathbb{N}_0. \tag{2.3}$$

By applying the limit (2.3) to both sides, we get

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) = 0. \tag{2.4}$$

In the view of (2.4), the limit converges to zero, we conclude that

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) \leq 1, \quad \text{for all } n \geq q \tag{2.5}$$

for some large enough $q \in \mathbb{N}$.

In what follows, we aim to establish the constructed sequence $\{\mathfrak{x}_n\}$ is $\mathfrak{G}$-Cauchy. To accomplish this we may assume that $m, n \in N$ with $m > n > q$. Before we prove that sequence is $\mathfrak{G}$-Cauchy, we shall omit the simple case:

If $\mathfrak{x}_n = \mathfrak{x}_m$, we have $\sqsupset^m(\mathfrak{x}_0) = \sqsupset^n(\mathfrak{x}_0)$.

Thus, we obtain

$$\sqsupset^{m-n}(\sqsupset^n(\mathfrak{x}_0)) = \sqsupset^n(\mathfrak{x}_0).$$

Therefore, we conclude that $\sqsupset^n(\mathfrak{x}_0)$ is the fixed point of $\sqsupset^{m-n}$. Moreover, we have

$$\sqsupset(\sqsupset^{m-n}(\sqsupset^n(\mathfrak{x}_0))) = \sqsupset^{m-n}(\sqsupset(\sqsupset^n(\mathfrak{x}_0))) = \sqsupset(\sqsupset^n(\mathfrak{x}_0)).$$

Now, we prove that constructive successive sequence is Cauchy. For this purpose, we presume that

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r+1}, \mathfrak{x}_{n+r+1}) = 0. \tag{2.6}$$

For $r \in \mathbb{N}$, put $r = 1$,

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+2}, \mathfrak{x}_{n+2}) \leq \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}\mathfrak{x}_{n+1}) + \mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) \\ + c[(\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}))^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}, \mathfrak{x}_{n+2}))^{1-\alpha}. \tag{2.7}$$

Taking limit as $n \to \infty$ on both sides (2.7), we have

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+2}, \mathfrak{x}_{n+2}) = 0. \tag{2.8}$$

Also, we have

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+3}, \mathfrak{x}_{n+3}) \leq \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+2}, \mathfrak{x}_{n+2}) + \mathfrak{G}(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+3}, \mathfrak{x}_{n+3}) \\ + c[(\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+2}, \mathfrak{x}_{n+2}))^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+3}, \mathfrak{x}_{n+3}))^{1-\alpha} \tag{2.9}$$

From (2.4) and (2.9) and by taking limit in the above inequality as $n \to \infty$ we find that

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+3}, \mathfrak{x}_{n+3}) = 0. \tag{2.10}$$

Let us consider that the statement holds in general case, we derive

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r}, \mathfrak{x}_{n+r}) = 0, \quad \text{for some } r \in \mathbb{N}. \tag{2.11}$$

Using (2.7), we have

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r+1}, \mathfrak{x}_{n+r+1}) \leq \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r}, \mathfrak{x}_{n+r}) + \mathfrak{G}(\mathfrak{x}_{n+r}, \mathfrak{x}_{n+r+1}, \mathfrak{x}_{n+r+1}) \\ + c[(\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r}, \mathfrak{x}_{n+r}))^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+r}, \mathfrak{x}_{n+r+1}, \mathfrak{x}_{n+r+1}))^{1-\alpha}. \tag{2.12}$$

From (2.4) and (2.11), by taking the limit of the above inequality as $n \to \infty$, we find that

$$\lim_{n\to\infty} \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+r+1}, \mathfrak{x}_{n+r+1}) = 0. \tag{2.13}$$

Then, we get

$$\mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m) < 1, \tag{2.14}$$

for $m > n > q$ for some $q \in \mathbb{N}$. We shall consider

$$\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_m, \mathfrak{x}_m) \leq \mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}) + \mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m) + c(\mathfrak{G}(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+1}))^{\alpha} \\ (\mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m))^{1-\alpha}] \\ \leq k^{n-q}\mathfrak{G}(\mathfrak{x}_q, \mathfrak{x}_{q+1}, \mathfrak{x}_{q+1}) + \mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m) + c[(k^{n-q}(\mathfrak{G}(\mathfrak{x}_q, \mathfrak{x}_{q+1}, \mathfrak{x}_{q+1}))^{\alpha} \\ (\mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m))^{1-\alpha}. \tag{2.15}$$

Employing from the fact (2.14), we have

$$(\mathfrak{G}(\mathfrak{x}_{n+1}, \mathfrak{x}_m, \mathfrak{x}_m))^{1-\alpha} < 1 \tag{2.16}$$

Hence, the right-hand side of the inequality (2.15) becomes

$$\begin{aligned}&\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m)\\ &\quad+[c(k^{n-q})^{\alpha}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m))^{1-\alpha}]\\ &\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+[1+c(k^{n-q})^{\alpha}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m))^{-\alpha}]\\ &\quad\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m).\end{aligned}\tag{2.17}$$

Thus, we have

$$\mathfrak{G}(\mathfrak{x}_n,\mathfrak{x}_m,\mathfrak{x}_m)\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+[1+c(k^{n-q})^{\alpha}]\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m).\tag{2.18}$$

We notice that

$$\begin{aligned}\mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_m,\mathfrak{x}_m)&\leq \mathfrak{G}(\mathfrak{x}_{n+1},\mathfrak{x}_{n+2},\mathfrak{x}_{n+2})+\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m)\\ &\quad+c[k^{n-q+1}\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1})^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m))^{1-\alpha}]\\ &\leq k^{n-q+1}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m)\\ &\quad+c[k^{n-q+1}\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1})^{\alpha}(\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m)(\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m))^{-\alpha}]\\ &\leq k^{n-q+1}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+[1+ck^{n-q+1}]\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m).\end{aligned}\tag{2.19}$$

Combining the inequalities (2.18) and (2.19), we get that

$$\begin{aligned}\mathfrak{G}(\mathfrak{x}_n,\mathfrak{x}_m,\mathfrak{x}_m)&\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))+[1+c(k^{n-q})^{\alpha}]k^{n-q+1}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))\\ &\quad+[1+c(k^{n-q})^{\alpha}][(1+ck^{n-q+1})^{\alpha}]\mathfrak{G}(\mathfrak{x}_{n+2},\mathfrak{x}_m,\mathfrak{x}_m)\end{aligned}\tag{2.20}$$

keeping all these observations in mind, we deduced that

$$\begin{aligned}\mathfrak{G}(\mathfrak{x}_n,\mathfrak{x}_m,\mathfrak{x}_m)&\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))\sum_{i=0}^{m-n-1}k^i\prod_{j=0}^{i-1}(1+ck^{n-q+j})^{\alpha}\\ &\leq k^{n-q}(\mathfrak{G}(\mathfrak{x}_q,\mathfrak{x}_{q+1},\mathfrak{x}_{q+1}))\sum_{i=0}^{m-n-1}k^i\prod_{j=0}^{i-1}(1+ck^{j})^{\alpha}.\end{aligned}\tag{2.21}$$

The right-hand side of the above inequality is dominated by the sequence $\sum_{i=0}^{\infty} s^i$, which is convergent by letting $n,m\to\infty$ where,

$$S_i=\prod_{j=0}^{i-1}(1+ck^j)^{\alpha}.$$

Thus $\{\mathfrak{x}_n\}$ is $\mathfrak{G}$-Cauchy sequence in $\mho$.

As $(\mho,d)$ is a complete (a,c)-interpolative $\mathfrak{G}$-metric space,the sequence $\{\mathfrak{x}_n\}$ converges to $u\in\mho$. We state that u is the fixed point of $\beth$. Assume, to the contrary that $\mathfrak{G}(u,\beth u,\beth u)>0$.

Note that

$$\mathfrak{G}(\mathfrak{x}_{n+1},\beth u,\beth u)=\mathfrak{G}(\beth\mathfrak{x}_n,\beth u,\beth u)\leq k\mathfrak{G}(\mathfrak{x}_n,u,u).\tag{2.22}$$

Applying the lim on both sides of the inequalities (2.22), we have $k<1$ which is contradiction.

Therefore, $\daleth u = u$ is the fixed point of $\daleth$ in $\mho$. □

Example 2.3. *Let $\mho$ be a nonempty set and define a function $\mathfrak{G} : \mho \times \mho \times \mho \to [0, \infty]$ as follows:*

$$\mathfrak{G}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = |\mathfrak{x} - \mathfrak{y}|^2 + |\mathfrak{y} - \mathfrak{z}|^2 + |\mathfrak{z} - \mathfrak{x}|^2, \quad \textit{for all } \mathfrak{x}, y \in \mho.$$

Clearly $(\mho, G)$ is $\left(\frac{1}{2}, 2\right)$-interpolative $\mathfrak{G}$-metric space (see Example 2.2).

Define a mapping $\daleth : \mho \to \mho$ by $\daleth\mathfrak{x} = \frac{\mathfrak{x}}{2}$.

$$\mathfrak{G}(\daleth\mathfrak{x}, \daleth\mathfrak{y}, \daleth\mathfrak{z}) = \left|\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{2}\right|^2 + \left|\frac{\mathfrak{y}}{2} - \frac{\mathfrak{z}}{2}\right|^2 + \left|\frac{\mathfrak{z}}{2} - \frac{\mathfrak{x}}{2}\right|^2,$$
$$\mathfrak{G}(\daleth\mathfrak{x}, \daleth\mathfrak{y}, \daleth\mathfrak{z}) \le \frac{1}{4}[|\mathfrak{x} - \mathfrak{y}|^2 + |\mathfrak{y} - \mathfrak{z}|^2 + |\mathfrak{z} - \mathfrak{x}|^2].$$

So, all the assumptions of Theorem 2.1 are satisfied with $k = \frac{1}{4}$.

Consequently $\daleth$ has a fixed point.

Clearly $\daleth$ has a unique fixed point 0.

Corollary 2.1. *Let $(X, \mathfrak{G})$ be a complete (α, c)-interpolative $\mathfrak{G}$-metric space and let $\daleth$ be a self map on $\mho$. Suppose that there exists $\psi \in \Psi$ such that*

$$\mathfrak{G}(\daleth\mathfrak{x}, \daleth\mathfrak{y}, \daleth\mathfrak{z}) \le \psi\mathfrak{G}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}). \tag{2.23}$$

Then, $\daleth$ has unique fixed point in $\mho$.

Proof. By putting $\psi(t) = kt, k \in [0, 1)$ in Theorem 2.1, we get the result. □

3. Consensus Model in Decision Making

As an application, we present a model to show how group consensus in decision-making can be formalized using fixed point theory in interpolative $\mathfrak{G}$-metric space. The structure of interpolative $\mathfrak{G}$-metric space enables the system to take into consideration both the existence of a fixed point and three-way interactions between decision makers, ensures convergence to a stable consensus.

Suppose that three decision makers A, B, C evaluate a single alternative and assign initial ratings $x_A^{(0)} = 0.6, x_B^{(0)} = 0.8, x_C^{(0)} = 0.4 \in \mho = [0, 1]$.

Define a consensus operator $\daleth : \mho \to \mho$ by:

$$\daleth x = \frac{1}{3}(\mathfrak{x} + f_B\mathfrak{x} + f_c\mathfrak{x}), \tag{3.1}$$

where $f_B, f_c : \mho \to \mho$ represent influence functions of agents B and C, respectively.

Let us suppose that

$$f_B\mathfrak{x} = \frac{1}{2}\mathfrak{x} + \frac{1}{4}, \tag{3.2}$$
$$f_c\mathfrak{x} = \frac{1}{2}\mathfrak{x} + \frac{1}{5} \tag{3.3}$$

and

$$\begin{aligned}\mathfrak{G}(\beth\mathfrak{x},\beth\mathfrak{y},\beth\mathfrak{z}) &= |\beth\mathfrak{x}-\beth\mathfrak{y}|^2+|\beth\mathfrak{y}-\beth\mathfrak{z}|^2+|\beth\mathfrak{z}-\beth\mathfrak{x}|^2\\
&= \frac{1}{9}|(\mathfrak{x}-\mathfrak{y})+(f_B\mathfrak{x}-f_B\mathfrak{y})+(f_c\mathfrak{x}-f_c\mathfrak{y})|^2\\
&\quad+\frac{1}{9}|(\mathfrak{y}-\mathfrak{z})+(f_B\mathfrak{y}-f_B\mathfrak{z})+(f_c\mathfrak{y}-f_c\mathfrak{z})|^2\\
&\quad+\frac{1}{9}|(\mathfrak{z}-\mathfrak{x})+(f_B\mathfrak{z}-f_B\mathfrak{x})+(f_c\mathfrak{z}-f_c\mathfrak{x})|^2\\
&= \frac{1}{9}\left|(\mathfrak{x}-\mathfrak{y})+\left(\frac{1}{2}\mathfrak{x}-\frac{1}{2}\mathfrak{y}\right)+\left(\frac{1}{2}\mathfrak{x}-\frac{1}{2}\mathfrak{y}\right)\right|^2\\
&\quad+\frac{1}{9}\left|(\mathfrak{y}-\mathfrak{z})+\left(\frac{1}{2}\mathfrak{y}-\frac{1}{2}\mathfrak{z}\right)+\left(\frac{1}{2}\mathfrak{y}-\frac{1}{2}\mathfrak{z}\right)\right|^2\\
&\quad+\frac{1}{9}\left|(\mathfrak{z}-\mathfrak{x})+\left(\frac{1}{2}\mathfrak{z}-\frac{1}{2}\mathfrak{x}\right)+\left(\frac{1}{2}\mathfrak{z}-\frac{1}{2}\mathfrak{x}\right)\right|^2\\
&= \frac{4}{9}[|\mathfrak{x}-\mathfrak{y}|^2+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-\mathfrak{x}|^2\\
&< \frac{5}{9}[|\mathfrak{x}-\mathfrak{y}|^2+|\mathfrak{y}-\mathfrak{z}|^2+|\mathfrak{z}-\mathfrak{x}|^2\\
&< k\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}), \quad \text{where } k=\frac{5}{9}<1.\end{aligned}$$

So, $\beth$ satisfies inequality (2.7).

Hence using Theorem 2.1, we can say that $\beth$ has a unique fixed point.

To find the fixed point

$$\begin{aligned}\beth\mathfrak{x} &= \frac{1}{3}(\mathfrak{x}+f_B\mathfrak{x}+f_c\mathfrak{x}),\\
\beth\mathfrak{x} &= \frac{1}{3}\left(\mathfrak{x}+\frac{1}{2}\mathfrak{x}+\frac{1}{4}+\frac{1}{2}\mathfrak{x}+\frac{1}{5}\right),\\
\beth\mathfrak{x} &= \frac{1}{3}\left(2\mathfrak{x}+\frac{9}{20}\right).\end{aligned}$$

Using, $\beth\mathfrak{x}=\mathfrak{x}$ we get

$$\frac{1}{3}\left(2\mathfrak{x}+\frac{9}{20}\right)=\mathfrak{x},$$

that is, $\mathfrak{x}=\frac{9}{20}=0.45$, which is the stable consensus value by three decision makers.

4. Application to Solve an Integral Equation

In this section, we shall solve an integral equation of the type

$$\mathfrak{x}(t)=\int_0^t k(t,s,\mathfrak{x}(s))ds, \quad t\in[0,2], \tag{4.1}$$

where $k:[0,2]\times[0,2]\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies a Lipschitz condition.

Let $\mho=C([0,2],\mathbb{R})$, the space of continuous real valued function on $[0,2]$.

Define the $\mathfrak{G}$-metric space as:

$$\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = |\mathfrak{x}-\mathfrak{y}|^2 + |\mathfrak{y}-\mathfrak{z}|^2 + |\mathfrak{z}-\mathfrak{x}|^2, \quad \text{for all } \mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mho. \tag{4.2}$$

Define the operator

$$\beth\mathfrak{x}(t) = \int_0^2 k(t,s,\mathfrak{x}(s))ds \tag{4.3}$$

Now, we show that $\beth$ is contractive mapping in the interpolative $\mathfrak{G}$-metric space.

We suppose that $k(t,s,\mathfrak{x})$ satisfies:

$$|k(t,s,\mathfrak{x}_1) - k(t,s,\mathfrak{x}_2)| \le L|\mathfrak{x}_1 - \mathfrak{x}_2|, \quad \text{for all } t,s \in [0,2], \mathfrak{x}_1,\mathfrak{x}_2 \in \mathbb{R}. \tag{4.4}$$

Let $\mathfrak{x},\mathfrak{y} \in \mho$. Then for all $t \in [0,2]$:

$$\begin{aligned}
|\beth\mathfrak{x}(t) - \beth\mathfrak{y}(t)|^2 &= \left|\int_0^2 [k(t,s,\mathfrak{x}(s)) - k(t,s,\mathfrak{y}(s))]ds\right|^2 \\
&\le \int_0^2 [|k(t,s,\mathfrak{x}(s)) - k(t,s,\mathfrak{y}(s))]ds|^2 \\
&\le L^2 \int_0^2 |\mathfrak{x}(s) - \mathfrak{y}(s)|^2 ds \\
&\le 4L^2 ||\mathfrak{x}-\mathfrak{y}||_\infty^2 .
\end{aligned} \tag{4.5}$$

Similarly,

$$|\beth\mathfrak{y}(t) - \beth\mathfrak{z}(t)|^2 \le 4L^2||\mathfrak{y}-\mathfrak{z}||_\infty^2, \tag{4.6}$$

$$|\beth\mathfrak{z}(t) - \beth\mathfrak{x}(t)|^2 \le 4L^2||\mathfrak{z}-\mathfrak{x}||_\infty^2, \tag{4.7}$$

$$\begin{aligned}
\mathfrak{G}(\beth\mathfrak{x},\beth\mathfrak{y},\beth\mathfrak{z}) &= |\beth\mathfrak{x}-\beth\mathfrak{y}|^2 + |\beth\mathfrak{y}-\beth\mathfrak{z}|^2 + |\beth\mathfrak{z}-\beth\mathfrak{x}|^2 \\
&\le 4L^2||\mathfrak{x}-\mathfrak{y}||_\infty^2 + 4L^2||\mathfrak{y}-\mathfrak{z}||_\infty^2 + 4L^2||\mathfrak{z}-\mathfrak{x}||_\infty^2 \\
&\le 4L^2[||\mathfrak{x}-\mathfrak{y}||_\infty^2 + ||\mathfrak{y}-\mathfrak{z}||_\infty^2 + ||\mathfrak{z}-\mathfrak{x}||_\infty^2] \\
&\le 4L^2\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}).
\end{aligned} \tag{4.8}$$

Let $k = 2L$ then

$$L < \frac{1}{2} \tag{4.9}$$

$$\le k^2\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z})$$

$$\mathfrak{G}(\beth\mathfrak{x},\beth\mathfrak{y},\beth\mathfrak{z}) \le k\mathfrak{G}(\mathfrak{x},\mathfrak{y},\mathfrak{z}). \tag{4.10}$$

Therefore, by the Theorem 2.1 has a unique solution on $[0,2]$.

5. Conclusion

A new notion of interpolative $\mathfrak{G}$-metric space is introduced in the present manuscript and a fixed point theorem for a contractive mapping is proved for the same.Validity of the result is also shown by some suitable examples. A decision making model is also established to show the convergence to a stable consensus. Further, an integral equation is also solved by making use of the main result. It will be an open problem to extend and generalise our results in the setting of more generalised spaces and generalised contractive conditions.

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References

[1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[2] Z. Mustafa, A New Structure for Generalized Metric Spaces: With Applications to Fixed Point Theory, Ph.D. Thesis, The University of Newcastle, Australia, 2005.

[3] Z. Mustafa, B. Sims, A New Approach to Generalized Metric Spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.

[4] P.P. Murthy, P. Sahu, A. Elsonbaty, K.H. Khan, R. Ramaswamy, et al., Fixed Points for (ψ,ω)-Weakly Cyclic Type Generalized Contraction Condition in Metric Spaces with an Application, Mathematics 11 (2022), 166. https://doi.org/10.3390/math11010166.

[5] R. Ramaswamy, G. Mani, A.J. Gnanaprakasam, O.A.A. Abdelnaby, S. Radenović, Solving an Integral Equation via Tricomplex-Valued Controlled Metric Spaces, Axioms 12 (2023), 56. https://doi.org/10.3390/axioms12010056.

[6] M. Kumar, N. Bhardwaj, G. Mani, R. Ramaswamy, K. Hayat Khan, et al., Fixed Point Theorems for (β,ϕ)-Expansive Mappings in Controlled Metric Space, Eur. J. Pure Appl. Math. 18 (2025), 7124. https://doi.org/10.29020/nybg.ejpam.v18i4.7124.

[7] R. Ramaswamy, Fixed Point Theorems on Multiplicative Cone Bipolar Metric Space and Their Applications, Eur. J. Pure Appl. Math. 18 (2025), 6802. https://doi.org/10.29020/nybg.ejpam.v18i4.6802.

[8] G. Mani, A.J. Gnanaprakasam, Y. Li, Z. Gu, The Existence and Uniquenes Solution of Nonlinear Integral Equations via Common Fixed Point Theorems, Mathematics 9 (2021), 1179. https://doi.org/10.3390/math9111179.

[9] G. Mani, R. Ramaswamy, A.J. Gnanaprakasam, V. Stojiljković, Z.M. Fadail, et al., Application of Fixed Point Results in the Setting of $\mathcal{F}$-Contraction and Simulation Function in the Setting of Bipolar Metric Space, AIMS Math. 8 (2023), 3269–3285. https://doi.org/10.3934/math.2023168.

[10] G. Mani, A.J. Gnanaprakasam, N. Kausar, M. Munir, S. Khan, et al., Solving an Integral Equation via Intuitionistic Fuzzy Bipolar Metric Spaces, Decis. Mak.: Appl. Manag. Eng. 6 (2023), 536–556. https://doi.org/10.31181/dmame622023624.

[11] E. Karapınar, On Interpolative Metric Spaces, Filomat 38 (2024), 7729–7734. https://doi.org/10.2298/fil2422729k.

[12] E. Karapınar, R.P. Agarwal, Some Fixed Point Results on Interpolative Metric Spaces, Nonlinear Anal.: Real World Appl. 82 (2025), 104244. https://doi.org/10.1016/j.nonrwa.2024.104244.