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Unified Bounds for Quantum-Plank Integrals and Generalized Hermite-Hadamard Inequalities

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Abstract. This paper investigates the boundedness of quantum-Planck $(q-h)$-integral operators through novel extensions of classical inequalities, with particular emphasis on a broad spectrum of convexities, including standard convexity, (α, m)-convexity, (s, m)-convexity, and related generalizations. Employing the versatile $(\alpha, \hbar - m)$-convexity framework, we establish new Hermite-Hadamard type inequalities that not only unify existing approaches but also extend them to the setting of quantum calculus. Our results provide explicit upper and lower bounds for $(q-h)$-integrals, delivering sharper refinements for special subclasses such as $(\alpha, \hbar - m)$-p-convex functions. These developments encompass and generalize earlier quantum integral inequalities, thereby strengthening the theoretical foundation for stability, convergence, and approximation in quantum physics, combinatorics, and fractional modeling.

1. INTRODUCTION AND PRELIMINARIES

Convexity plays key role in various fields of mathematics and in their applications, from optimization and economics to machine learning and data analysis. Traditional convexity involves functions exhibiting specific inequalities that ensure desirable properties such as local minima coinciding with global minima, making it an essential concept for theoretical problem-solving. Recent developments in convex analysis have led to the exploration of generalized forms of convexity, which extend the classical notions to accommodate a broader range of functions and their applications [1,2].

In theory of inequalities convex functions are considered very important. Many classical integral

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and discrete mathematical inequalities are directly connected with Jensen's inequality (a very basic inclusion of convex functions). Also a lot of classical integral inequalities exist in the literature which had been published for quantum integrals. In [3], authors studied quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables are established for q-integrals. In [4], authors have studied Opial type inequality for quantum integrals, quantum and (p,q)-Hermite–Hadamard inequalities with quantum and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions are given [5,8]. For more quantum integral inequalities we refer the reader to [6,7] and references therein. At the same time convex and related functions are used in formulation of new and generalized versions of these quantum integral inequalities.

In the following, we comprehensively describe the notions and results which provide motivation in the formation of this article. Firstly, we write about convexity and its generalized as well as refined forms: A Convex function f satisfies, $f(\theta u + (1-\theta)v) \leq \theta f(u) + (1-\theta)f(v)$ with usual descriptions of θ and u, v see more detail [29]. The inequality obeying convex functions provides the opportunity to involve function and parameters provided it remains preserve. This have been performed by different authors and many new notions have been created which are known as $\hbar$-convex, m-convex, p-convex and their compositions lead to (s,m)-convex functions. In [17] a notion of $(\alpha-\hbar, m)$-convexity was introduced that contains almost all kinds of convexities. An $(\alpha-\hbar, m)$-convex function satisfies the inequality:

$$f(tx + m(1-t)y) \leq \hbar(t^\alpha)f(x) + m\hbar(1-t^\alpha)f(y), \tag{1.1}$$

with usual notations, see in detail [17]. Next, we are interested to give a definition which is key factor behind the formation of the results of this paper.

Definition 1.1. *[23] With all conditions as given in [23, Definition 4], a function f is called $(\alpha-h-m)$-convex with respect to Ψ, if we have the following inequality:*

$$fo\Psi^{-1}(tx + m(1-t)y) \leq \hbar(t^\alpha)fo\Psi^{-1}(x) + m\hbar(1-t^\alpha)fo\Psi^{-1}(y). \tag{1.2}$$

In the whole paper we will denote with $C^{\hbar}_{\alpha,m}(I)$. The above definition, more precisely the inequality (1.2) has many interesting consequences stated as follows:

Remark 1.1. *The upcoming inequality is obtained by setting $\Psi(x) = x^p \; p \in \mathbb{R} - 0, x \in (0,\infty)$*

$$f((tx + m(1-t)y)^{\frac{1}{p}}) \leq \hbar(t^\alpha)f(x^{\frac{1}{p}}) + m\hbar(1-t^\alpha)f(y^{\frac{1}{p}}). \tag{1.3}$$

From (1.3) the definitions of $(\alpha, \hbar - m) - p$-convexity, (p,h)-convexity, p-convexity, $(\alpha, \hbar - m)$-convexity, $(s-m)$-convexity can be reproduced, for more detail one can see [23].

In the following [23], we give an inequality for an $(\alpha, \hbar - m)$-convex function Ψ:

$$f\left(\Psi^{-1}\left(\frac{\Psi(a) + m\Psi(b)}{2}\right)\right) \leq \frac{\Gamma(\mu+1)}{(m\Psi(b) - \Psi(a))^\mu}\Bigg[\hbar\Big(\frac{1}{2^\alpha}\Big)J^{\mu}_{\Psi(a)^+} f(\Psi^{-1}(m\Psi(b))) \tag{1.4}$$

$$+ m^{\mu+1}H(\frac{1}{2})J^{\mu}_{\Psi(b)^-}f(\Psi^{-1}\left(\frac{\Psi(a)}{m}\right)\Big] \le \mu\Big[\hbar(\frac{1}{2^{\alpha}})f(a) + mH\Big(\frac{1}{2}\Big)f(b)\Big]$$

$$\int_0^1 \hbar(t^{\alpha})t^{\mu-1}dt + m\mu\Big[\hbar\Big(\frac{1}{2^{\alpha}}\Big)f(b) + mH\Big(\frac{1}{2}\Big)f\Big(\Psi^{-1}\Big(\frac{\Psi(a)}{m^2}\Big)\Big)\Big]\int_0^1 H(t^{\alpha})t^{\mu-1}dt,$$

where $J^{\mu}_{+}f$, $J^{\mu}_{-}f$ are left and right Riemann-Liouville fractional integrals and H is defined in [23]. After a brief description of convexity and related notions in the next, we speak about quantum calculus and give some definitions and results for quantum and quantum-plank integrals:

Quantum calculus, which is often called calculus without limits, offers a unique framework that builds on traditional differential and integral operators. Lately, quantum integral operators like the Jackson q-integral and the Hahn h-integral have gained a lot of attention because they're handy in fields like number theory, combinatorics, and quantum physics. These tools really shine when modeling processes that show discrete or non-local behaviors. So, incorporating convexity-based methods into quantum calculus has turned out to be a solid approach for getting useful inequalities and proving the stability and boundedness of quantum integral transforms [24–26].

The quotient $\frac{f(qx)-f(x)}{(q-1)x}$, $q \in (0,1)$ is well-known as quantum derivative or q-derivative, while quotient $\frac{f(x+h)-f(x)}{h}$ is known as the plank derivative or h-derivative respectively. q-integral and h-integral are given by

$$\int_a^x f(x)d_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a),\ x \in [a,b] \tag{1.5}$$

and

$$\int_a^b f(x)d_ht = \begin{cases} h(f(a)+f(a+h)+....+f(b-h)),\ if\ a<b, \\ 0,\ if\ a=b, \\ -h(f(a)+f(a+h)+....+f(a-h)),\ if\ a>b, \end{cases} \tag{1.6}$$

respectively.

The above notions of derivatives are combined in the quotient $\frac{f(x)-f(qx+(1-q)a+qh)}{(1-q)(x-a)-qh}$, $x \neq \frac{a(1-q)+qh}{1-q}$ called quantum-plank or $q-h$-derivative. In the next, we give definition of left and right $q-h$-integrals, which will be used frequently to formulate the results of this paper.

Definition 1.2. *[16] Let $0 < q < 1$ and function $f : I = [a,b] \to \mathbb{R}$ be continuous. Then the left $q-h$-integral and the right $q-h$-integral on I are denoted by $U^{a^+}_{q-h}f$ and $U^{b^-}_{q-h}f$ respectively, and defined as follows:*

$$U^{a^+}_{q-h}f(x) := \int_a^x f(t)_h d_qt \tag{1.7}$$

$$= ((1-q)(x-a)+qh)\sum_{k=0}^{\infty} q^k f(q^k a + (1-q^k)x + kq^k h),\ x > a,$$

$$U^{b^-}_{q-h}f(x) := \int_x^b f(t)_h d_qt \tag{1.8}$$

$$= ((1-q)(b-x)+qh)\sum_{k=0}^{\infty} q^k f(q^{nk}x+(1-q^k)b+kq^k h),\ x<b.$$

Bounding integral operators is a key topic within functional analysis and inequality theory. When looking at quantum calculus, figuring out the upper and lower limits for integrals that involve generalized convex functions helps us understand how operators behave across different types of functions. These limits are essential in fields like approximation theory, variational analysis, and even fractional modeling. The findings in this paper add to this ongoing work by establishing clear and precise inequalities for $q\breve{}h$-integrals under specific convexity conditions, which in turn strengthens the theoretical base for both discrete and quantum analysis [27,28].
The structure of this paper is as follows: Section 1 reviews the fundamental concepts of convexity and introduces $(\alpha,\hbar-m)$-convexity. Section 2 discusses the impact of function compositions on convexity and presents the main results related to $fo\varPsi^{-1}$ and its $(\alpha,\hbar-m)$-convexity for quantum calculus operators.

2. Implicit Form of q- and h-Hermite-Hadamard Inequalities

The initial result gives the upper bounds of $q-h$-integrals, stated and proved as follows:

Theorem 2.1. *Let $\xi o\varPsi^{-1}\in C^{\hbar}_{\alpha,m}$ and is decreasing. Let be a decreasing and $h_1=(x-a)h$, and $h_2=(b-x)h$ where $h\geq 0$ and $x\in[a,b]$, $0\leq a<b$. The following inequality holds for $q-h$-integrals:*

$$\frac{U^{a^+}_{q-h_1}\xi(\varPsi^{-1}(x))}{(1-q)(x-a)+qh_1}+\frac{U^{b^-}_{q-h_2}\xi(\varPsi^{-1}(x))}{(1-q)(b-x)+qh_2}\leq(\xi(\varPsi^{-1}(a))+\xi(\varPsi^{-1}(x))) \tag{2.1}$$
$$\sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})+m\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\bigg(\xi\bigg(\Psi^{-1}\bigg(\frac{x}{m}\bigg)\bigg)+\xi\bigg(\Psi^{-1}\bigg(\frac{b}{m}\bigg)\bigg)\bigg).$$

If $\xi o\Psi^{-1}$ is increasing and $(\alpha,\hbar-m)$-concave, then the reverse of inequality (2.1) *holds.*

Proof. The function $\xi(\Psi^{-1})$ is decreasing and belongs to the class $C^{\hbar}_{\alpha,m}$, therefore it must obey the following inequality,

$$\xi(\varPsi^{-1}(q^k a+(1-q^k)x+kq^k h))\leq\hbar(q^{k\alpha})\xi(\varPsi^{-1}(a))+m\hbar(1-q^{k\alpha})\xi\bigg(\varPsi^{-1}\bigg(\frac{x}{m}\bigg)\bigg). \tag{2.2}$$

This leads to the following inequality in between series of functions.

$$\sum_{k=0}^{\infty}q^k(\xi(\Psi^{-1}(q^k a+(1-q^k)x+kq^k h)))\leq\sum_{k=0}^{\infty}q^k(\hbar(q^{k\alpha})\xi(\Psi^{-1}(a))+m\sum_{k=0}^{\infty}q^k$$
$$\times(\hbar(1-q^{k\alpha})\xi\bigg(\Psi^{-1}\bigg(\frac{x}{m}\bigg)\bigg),$$

By using Definition 1.7, and doing some computation we have

$$\frac{U^{a^+}_{q-h_1}\xi(\Psi^{-1}(x))}{(1-q)(x-a)+qh_1}\leq\sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})\xi(\Psi^{-1}(a))+m\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\xi\bigg(\Psi^{-1}\bigg(\frac{x}{m}\bigg)\bigg). \tag{2.3}$$

Also, the following inequality can be obtained on the pattern of (2.2).

$$\xi(\Psi^{-1}(q^k x+(1-q^k)b+kq^k h))\leq \hbar(q^{k\alpha})\xi(\Psi^{-1}(x))+m\hbar(1-q^{k\alpha})\xi\bigg(\Psi^{-1}\bigg(\frac{b}{m}\bigg)\bigg).$$

Taking infinite sum over k on both sides, one can get

$$\sum_{k=0}^{\infty}q^k(\xi(\Psi^{-1}(q^k x+(1-q^k)b+kq^k h)))\leq \sum_{k=0}^{\infty}q^k(\hbar(q^{k\alpha})\xi(\Psi^{-1}(x))+m\sum_{k=0}^{\infty}q^k$$
$$\times(\hbar(1-q^{k\alpha})\xi\bigg(\Psi^{-1}\bigg(\frac{b}{m}\bigg)\bigg).$$

By using Definition 1.8, and doing some computation one can obtained

$$\frac{U^{b^-}_{q-h_1}\xi(\Psi^{-1}(x))}{(1-q)(b-x)+qh_2}\leq \sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})\xi(\Psi^{-1}(x))+m\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\xi\bigg(\Psi^{-1}\bigg(\frac{b}{m}\bigg)\bigg). \tag{2.4}$$

From (2.3) and (2.4), one can constitute the required inequality. □

Remark 2.1. *For $h\leq 0$. If $\xi o\Psi^{-1}$ is increasing and $(\alpha,\hbar-m)$-convex, then* (2.1) *also holds.*

Next, we give result for $(\alpha,\hbar-m)-p$-convex function.

Theorem 2.2. *Under the assumption of previous theorem, the following inequality holds:*

$$\frac{U^{a^+}_{q-h_3}(\xi o\eta(x))}{(1-q)(x^p-a^p)+qh_3}+\frac{U^{b^-}_{q-h_4}(\xi o\eta(x))}{(1-q)(b^p-x^p)+qh_4}\leq \sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})(\xi(a)+\xi(x))+m \tag{2.5}$$
$$\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\bigg(\xi o\eta\bigg(\frac{x^p}{m}\bigg)+\xi o\eta\bigg(\frac{b^p}{m}\bigg)\bigg),$$

where $h_3=(x^p-a^p)h$, $h_4=(b^p-x^p)h$ and $\eta(t)=t^{\frac{1}{p}}$.

Proof. Let us consider $\Psi(t)=t^p$, $p\in\mathbb{R}-0t\geq 0$. Then $\Psi^{-1}(t)=t^{\frac{1}{p}}$ and the inequality (2.3) and (2.4) takes the following form:

$$\frac{U^{a^+}_{q-h_3}\xi(x^{\frac{1}{p}})}{(1-q)(x-a)+qh_3}\leq \sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})\xi(a^{\frac{1}{p}})+m\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\xi\bigg(\bigg(\frac{x}{m}\bigg)^{\frac{1}{p}}\bigg). \tag{2.6}$$

$$\frac{U^{b^-}_{q-h_4}\xi(x^{\frac{1}{p}})}{(1-q)(b-x)+qh_4}\leq \sum_{k=0}^{\infty}q^k\hbar(q^{k\alpha})\xi(x^{\frac{1}{p}})+m\sum_{k=0}^{\infty}q^k\hbar(1-q^{k\alpha})\xi\bigg(\bigg(\frac{b}{m}\bigg)^{\frac{1}{p}}\bigg). \tag{2.7}$$

Now, replace x by x^p, a by a^p and b by b^p in inequalities (2.6) and (2.7) respectively, and adding after doing some computations we get the required inequality. □

Theorem 2.3. *Let Ψ satisfies the following equality $\Psi^{-1}\left(\frac{a+x-u}{m}\right) = \Psi^{-}(u)$, $u \in (a,x)$. Then under the assumptions of Theorem* (2.1) *excluding monotonicity the following $q-h$-integral inequality holds:*

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right) \leq \frac{1}{(x-a)+\frac{q}{1-q}h_1}\left(\hbar\left(\frac{1}{2^{\alpha}}\right)+m\hbar\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\int_a^x \xi(\Psi^{-1}(t))_{h_1}d_qt \tag{2.8}$$

$$\leq \xi(\Psi^{-1}(x))\int_0^1 \hbar(t^{\alpha})_h d_qt + m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)\int_0^1 \hbar(1-t^{\alpha})_h d_qt.$$

Proof. The function $\xi(\Psi^{-1})$ to the class $C^{\hbar}_{\alpha,m}$, therefore it must obey the following inequality,

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right) \leq \hbar\left(\frac{1}{2^{\alpha}}\right)\xi(\Psi^{-1}(ta+(1-t)x)) + m\hbar\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)$$
$$\times\,\xi\left(\Psi^{-1}\left(\frac{tx+(1-t)a}{m}\right)\right).$$

Taking $q-h$-integral we have

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right) \leq \frac{1-q}{(1-q)+qh}\left(\hbar\left(\frac{1}{2^{\alpha}}\right)\int_0^1 \xi(\Psi^{-1}(ta+(1-t)x))_h d_qt + m\right. \tag{2.9}$$
$$\left.\hbar\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 \xi\left(\Psi^{-1}\left(\frac{tx+(1-t)a}{m}\right)\right)_h d_qt.\right.$$

By using the condition $(\Psi^{-1}\left(\frac{a+x-u}{m}\right) = \Psi^{-}(u)$, one can have:

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right) \leq \frac{1-q}{(1-q)+qh}\left(\hbar\left(\frac{1}{2^{\alpha}}\right)+m\hbar\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right) \tag{2.10}$$
$$\int_0^1 \xi(\Psi^{-1}(tx+(1-t)a))_h d_qt.$$

Now, from Definition 1.7, one can get

$$\frac{qh+1-q}{qh_1+(x-a)(1-q)}\int_a^x \xi(\Psi^{-1}(t))_{h_1}d_qt \tag{2.11}$$
$$= (qh+(1-q))\sum_{k=0}^{\infty} q^k\xi(\Psi^{-1}(q^ka+(1-q^k)x+kq^k(x-a)h))$$
$$= \int_0^1 \xi(\Psi^{-1}(a+(x-a)t))_h d_qt.$$

Since, $\xi o\Psi^{-1} \in C^{\hbar}_{\alpha,m}([a,b])$ for $(\alpha,\hbar-m)$-convexity the forthcoming inequality is yielded:

$$\int_0^1 \xi(\Psi^{-1}(a+(x-a)t))_h d_qt \leq \xi(\Psi^{-1}(x))\int_0^1 \hbar(t^{\alpha})_h d_qt + m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)$$
$$\times\int_0^1 \hbar(1-t^{\alpha})_h d_qt.$$

The inequality (2.11) takes the following form:

$$\frac{qh+(1-q)}{qh_1+(1-q)(x-a)}\int_a^x \xi(\Psi^{-1}(t))_{h_1}d_qt=\int_0^1 \xi(\Psi^{-1}(a+(x-a)t))_hd_qt\le \xi(\Psi^{-1}(x)) \tag{2.12}$$
$$\int_0^1 \hbar(t^\alpha)_hd_qt+m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)\int_0^1 \hbar(1-t^\alpha)_hd_qt.$$

Inequalities (2.10), (2.11) and (2.12) constitute inequality (2.8), the proof is completed. ☐

Corollary 2.1. *Putting $h=0$ in* (2.8)*, it takes the form as follows:*

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right)\le\frac{\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)}{(x-a)}\int_a^x \xi(\Psi^{-1}(t))d_qt\le\xi(\Psi^{-1}(x))\int_0^1\hbar(t^\alpha)d_qt \tag{2.13}$$
$$+m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)\int_0^1\hbar(1-t^\alpha)d_qt.$$

Remark 2.2. *1. If $\hbar(t)=t^\alpha$ in 2.8, then the following inequality holds:*

$$\xi\left(\Psi^{-1}\left(\frac{a+x}{2}\right)\right)\le\frac{1}{(x-a)+\frac{q}{1-q}h_1}\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_a^x\xi(\Psi^{-1}(t))_{h_1}d_qt \tag{2.14}$$
$$\le(1-q)+qh)\left(\xi(\Psi^{-1}(x)-m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)\right)\sum_{k=0}^{\infty}q^k(1-q^k)^\alpha$$
$$+\sum_{k=0}^{\infty}q^{(1+\alpha)k}kh+m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right).$$

Theorem 2.4. *Under the assumption of Theorem 2.3, the following inequality holds:*

$$\xi o\eta\left(\frac{a^p+x^p}{2}\right)\le\frac{1}{(x^p-a^p)+\frac{q}{1-q}h_3}\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_{a^p}^{x^p}\xi o\eta(x)_{h_3}d_qt \tag{2.15}$$
$$\le\xi o\eta(x)\int_0^1\hbar(t^\alpha)_hd_qt+m\xi o\eta\left(\frac{a^p}{m}\right)\int_0^1\hbar(1-t^\alpha)_hd_qt.$$

Proof. Let us consider $\Psi(t)=t^p$, $p\in\mathbb{R}-0$ $t\ge0$. Then $\Psi^{-1}(t)=t^{\frac{1}{p}}$ and the inequality (2.10) and (2.11) takes the following form:

$$\xi\left(\frac{a+x}{2}\right)^{\frac{1}{p}}\le\frac{1-q}{(1-q)+qh}\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right) \tag{2.16}$$
$$\int_0^1\xi(tx+(1-t)a)^{\frac{1}{p}})_hd_qt.$$

$$\begin{aligned}
&\frac{qh+1-q}{qh_1+(x-a)(1-q)}\int_a^x \xi(t)_{h_3}^{\frac{1}{p}}d_qt \qquad (2.17)\\
&=(qh+(1-q))\sum_{k=0}^{\infty}q^k\xi(q^ka+(1-q^k)x+kq^k(x-a)h)^{\frac{1}{p}}\\
&=\int_0^1 \xi(\Psi^{-1}(a+(x-a)t))_hd_qt.
\end{aligned}$$

Since $\xi o\Psi^{-1}\in C^h_{\alpha,m}([a,b])$, we have

$$\begin{aligned}
&\int_0^1 \xi(a+(x-a)t)_h^{\frac{1}{p}}d_qt \le \xi(x)^{\frac{1}{p}}\int_0^1 \hbar(t^\alpha)_hd_qt+m\xi\left(\frac{a}{m}\right)^{\frac{1}{p}} \qquad (2.18)\\
&\times\int_0^1 \hbar(1-t^\alpha)_hd_qt.
\end{aligned}$$

Now, replace x by x^p, a by a^p and b by b^p in inequalities (2.16), (2.17) and (2.18) respectively, and after doing some computations we get the required inequality. □

Theorem 2.5. *Let assumptions of above Theorem 2.3 holds, excluding monotonicity. Moreover if* $\xi\left(\Psi^{-1}\left(\frac{x+b-v}{m}\right)\right)=\xi(\Psi^{-1}(v))$, $v\in(x,b)$, *then for right* $q-h$*-integrals we have*

$$\begin{aligned}
\xi\left(\Psi^{-1}\left(\frac{x+b}{2}\right)\right)&\le\frac{1}{(b-x)+\frac{q}{1-q}h_2\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)}\int_x^b \xi(\Psi^{-1}(t))_{h_2}d_qt \qquad (2.19)\\
&\le\xi(\Psi^{-1}(x))\int_0^1 \hbar(t^\alpha)_hd_qt+m\xi\left(\Psi^{-1}\left(\frac{b}{m}\right)\right)\int_0^1 \hbar(1-t^\alpha)_hd_qt.
\end{aligned}$$

Proof. The proof of above inequality is same as the proof of (2.8). □

Theorem 2.6. *Under the assumption of Theorem* (2.3)*, the following inequality holds:*

$$\begin{aligned}
\xi o\eta\left(\frac{x^p+b^p}{2}\right)&\le\frac{1}{(b^p-x^p)+\frac{q}{1-q}h_4\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)}\int_{x^p}^{b^p} \xi o\eta(x)_{h_4}d_qt \qquad (2.20)\\
&\le\xi o\eta(x)\int_0^1 \hbar(t^\alpha)_hd_qt+m\xi o\eta\left(\frac{b^p}{m}\right)\int_0^1 \hbar(1-t^\alpha)_hd_qt.
\end{aligned}$$

Proof. The proof of above inequality is same as the proof of (2.15). □

Theorem 2.7. *The following inequality also holds under the assumptions of Theorem 2.3:*

$$\begin{aligned}
\xi\left(\Psi^{-1}\left(\frac{a+b}{2}\right)\right)&\le\frac{1}{(b-a)+\frac{q}{1-q}h_5\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)}\int_a^b \xi(\Psi^{-1}(t))_{h_5}d_qt \qquad (2.21)\\
&\le\xi(\Psi^{-1}(b))\int_0^1 \hbar(t^\alpha)_hd_qt+m\xi\left(\Psi^{-1}\left(\frac{a}{m}\right)\right)\int_0^1 \hbar(1-t^\alpha)_hd_qt.
\end{aligned}$$

Proof. The inequality (2.21) can be obtained by putting; $x=b$ in (2.8) or $x=a$ in (2.19). □

Theorem 2.8. *Under the assumption of Theorem* (2.3), *the following inequality holds:*

$$\xi o\eta\left(\frac{a^p+b^p}{2}\right) \le \frac{1}{(b^p-a^p)+\frac{q}{1-q}h_6}\left(\hbar\left(\frac{1}{2^\alpha}\right)+m\hbar\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_{a^p}^{b^p}\xi o\eta(x)_{h_6}d_qt \tag{2.22}$$

$$\le \xi o\eta(b)\int_0^1 \hbar(t^\alpha)_h d_qt + m\xi o\eta\left(\frac{a^p}{m}\right)\int_0^1 \hbar(1-t^\alpha)_h d_qt.$$

Proof. The inequality (2.22) can be obtained by putting; $x = b$ in (2.15) or $x = a$ in (2.20). □

3. Conclusions

We investigated boundedness of quantum and plank integrals simultaneously in an implicit form. Upper bounds were calculated in Theorem 2.1 by applying a generalized convexity with respect to strictly monotone functions, also some examples were considered. In Theorem 2.3, upper and lower bounds were given in the form of Hadamard like inequality. This leads to inequalities in particular cases, that provide boundedness of quantum plank integrals for different kinds of convexities.

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