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Characterizations for a Wright Distribution Series on Certain Subfamilies of Univalent Functions

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Abstract. In this paper, we investigate some sufficient conditions and inclusion relations for the Wright distribution series to be in the subfamilies $\mathcal{US}(\Upsilon,\Theta)$ and $\mathcal{UC}(\Upsilon,\Theta)$. Also, sufficient criteria for these subfamilies to contain the integral operator $IO(\tau,\pounds;n;\varsigma)$ described by the Wright distribution series. Our findings will also lead to other corollaries.

1. Introduction

The Wright function is a special function first introduced by E. M. Wright within the context of asymptotic partition theory. Its use has expanded to become a powerful tool in mathematical statistics and fractional analysis. It is known for its ability to generate probability distributions, such as the M-Wright distribution, capable of representing fractional behaviors and long-term memory. Thus, the Wright function is a bridge between mathematical analysis and modern statistical modeling in the study of fractional random processes and heavy-tailed distributions [1].

In Geometric Function Theory, the Wright distribution series offers a versatile analytical framework that can create and examine new families of functions when paired with fractional derivatives. By extending traditional univalent function results to fractional, nonlocal, and extended analytic contexts, this relationship connects complicated geometric analysis, probability distributions, and special functions under a single, cohesive framework [2,3].

Let F be the family of analytic functions in $\Delta = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, of the form:

$$A(\varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} a_{\daleth}\varsigma^{\daleth}, \tag{1.1}$$

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Furthermore, let F^- be a subfamily of functions in F of the form:

$$A(\varsigma) = \varsigma - \sum_{\daleth=2}^{\infty} |a_{\daleth}|\, \varsigma^{\daleth}, \qquad \varsigma \in \Delta. \tag{1.2}$$

Definition 1.1. *[4] A function A of the form (1.1) lies in the subfamily $\mathcal{US}(\Upsilon, \Theta)$ if it verifies*

$$\Re\left\{ e^{-i\Upsilon}\left(\frac{\varsigma A'(\varsigma)}{A(\varsigma)}\right)\right\} > \left|\frac{\varsigma A'(\varsigma)}{A'(\varsigma)} - 1\right| + \Theta \ \ (\varsigma \in \Delta, |\Upsilon| < \pi/2\,, 0 \leq \Theta < 1)$$

and $A \in \mathcal{UC}(\Upsilon, \Theta)$ if and only if $\varsigma A'(\varsigma) \in \mathcal{US}(\Upsilon, \Theta)$.

We write

$$\mathcal{US}F^-(\Upsilon, \Theta) = \mathcal{US}(\Upsilon, \Theta) \cap F^-$$

and

$$\mathcal{UC}F^-(\Upsilon, \Theta) = \mathcal{UC}(\Upsilon, \Theta) \cap F^-.$$

In particular, Ravichandran et al. [5] introduced the families of uniformly spirallike $\mathcal{US}(\Upsilon, 0) \equiv \mathcal{US}(\Upsilon)$ and uniformly convex spirallike $\mathcal{UC}(\Upsilon, 0) \equiv \mathcal{UC}(\Upsilon)$. For $\Upsilon = 0$, the families $\mathcal{US}(\Upsilon)$ and $\mathcal{UC}(\Upsilon)$, respectively, reduces to the families $\mathcal{US}(0,0) \equiv \mathcal{US}$ and $\mathcal{UC}(0,0) \equiv \mathcal{UC}$ established by Rønning [6].

For more intriguing advancements of various related families of consistently spirallike and uniformly convex spirallike, we can find it in Al-Hawary et al. [7,8], Bharati et al. [9], Frasin et al. [10], Goodman [11,12], Kanas and Wisniowska [13], and Yousef et al. [14,15].

By providing a power series with coefficients that are probabilities of the Poisson distribution, Porwal [16] opened up new research avenues in geometric function theory and a connection between complex analysis and probability distribution. Additional significant distribution series, including the Borel distribution [17], binomial distribution [18], generalized distribution [19], generalized hypergeometric distribution [20], Pascal distribution [21,22], Poisson distribution [23], and Miller-Ross Poisson distribution [24] have recently been used by several authors based on his research. The geometric properties of several special functions have actually been covered by a number of authors (see [25–29]).

Wright [30] introduced the following Wright functions:

$$\Psi(\tau, \pounds; \varsigma) = \sum_{\daleth=0}^{\infty} \frac{1}{\urcorner(\tau\daleth + \pounds)} \frac{\varsigma^{\daleth}}{\daleth!}, \tag{1.3}$$

where $\tau > -1, \pounds, \varsigma \in \mathbb{C}$ and $\urcorner(.)$ the usual Gamma function.

Wright functions are important in many fields, such as partial differential equations of fractional order, Green functions, Mikusinski operational calculus, and the asymptotic theory of partitions (see, for instance, [31–33]).

In particular, when $\tau = 1$ and $\pounds = r + 1$, the functions $\Psi(1, r + 1; -\varsigma^2/4)$ are described using the Bessel functions $J_r(\varsigma)$, given as:

$$J_r(\varsigma) = \left(\frac{\varsigma}{2}\right)^r \Psi(1, r+1; \frac{-\varsigma^2}{4}) = \sum_{\daleth=0}^{\infty} \frac{1}{\gimel(\daleth + r + 1)} \frac{(\varsigma/2)^{2\daleth+r}}{\daleth!}.$$

Furthermore, for $\tau > 0, r > -1$, the function $\Psi(\tau, r+1; -\varsigma) \equiv J_r^{\tau}(\varsigma)$ is the generalized Bessel function. For the details, see [34].

Note that $\Psi(\tau, £, \varsigma) \notin F$. So, it is natural to normalized it as the following:

$$\mathcal{D}(\tau, £; \varsigma) \equiv \gimel(£)\varsigma\Psi(\tau, £; \varsigma) = \sum_{\daleth=0}^{\infty} \frac{\gimel(£)}{\gimel(\tau\daleth + £)} \frac{\varsigma^{\daleth+1}}{\daleth!}, \ \tau > -1, £ > 0, \varsigma \in \Delta.$$

Easily, we can write

$$\mathcal{D}(\tau, £; \varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} \frac{\gimel(£)}{\gimel(\tau(\daleth - 1) + £)} \frac{\varsigma^{\daleth}}{(\daleth - 1)!}, \ \tau > -1, £ > 0, \varsigma \in \Delta.$$

Observe that, $\mathcal{D}(\tau, £; \varsigma)$ is satisfied in the equation:

$$-\mathcal{D}(1, r+1; -\varsigma) \equiv \overline{J}_r(\varsigma) \equiv \gimel(r+1)\varsigma^{1-r/2} J_r(2\sqrt{\varsigma}), \tag{1.4}$$

were $\overline{J}_r(\varsigma)$ is the Bessel function.

We now present the Wright distribution in the manner described below: first, we define the series

$$\mathcal{D}(\tau, £; n) = \sum_{\daleth=0}^{\infty} \frac{n^{\daleth+1}\gimel(£)}{\daleth!\gimel(\tau\daleth + £)},$$

which converges for every $\tau, £, n > 0$.

The probability mass function of Wright distribution is given by

$$q(\daleth) = \frac{n^{\daleth+1}\gimel(£)}{\daleth!\gimel(\tau\daleth + £)\mathcal{D}(\tau, £; n)}, \ \tau, £, n > 0, \ \daleth = 0, 1, 2, \cdots.$$

Note that for $\tau = 0$ it reduces to the Poisson distribution.

Now, we introduce the Wright distribution series using the definition of Wright distribution, as follows

$$\mathcal{WD}(\tau, £; n; \varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\gimel(£)}{(\daleth - 1)!\gimel(\tau(\daleth - 1) + £)\mathcal{D}(\tau, £; n)} \varsigma^{\daleth}. \tag{1.5}$$

The convolution of two series $A(\varsigma)$ of the form (1.1) and $B(\varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} b_{\daleth}\varsigma^{\daleth}$ is defined as:

$$(A * J)(\varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} a_{\daleth} j_{\daleth} \varsigma^{\daleth}.$$

Now, we introduce the linear operator $\mathcal{WD}(\tau, £; n; \varsigma) : F \to F$ as:

$$\mathcal{WD}(\tau, £; n; \varsigma)A = A(\varsigma) * \mathcal{WD}(\tau, £; n; \varsigma) = \varsigma + \sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\gimel(£)}{(\daleth - 1)!\gimel(\tau(\daleth - 1) + £)\mathcal{D}(\tau, £; n)} a_{\daleth}\varsigma^{\daleth}.$$

In this paper, we investigate the Wright distribution series $\mathcal{WD}(\tau, £; n; \varsigma)$ in the subfamilies $\mathcal{US}(\Upsilon, \Theta)$ and $\mathcal{UC}(\Upsilon, \Theta)$. Several sufficient conditions for the Wright distribution series $\mathcal{WD}(\tau, £; n; \varsigma)$ and $\mathcal{WD}(\tau, £; n; \varsigma)A$ to be in these families are given. Also, we define the integral operator $IO(\tau, £; n; \varsigma)$ using the Wright distribution series and give sufficient conditions for this operator to be in these subfamilies. From the obtained results, using specific values for Υ and Θ will follow several corollaries from the main results.

Lemma 1.1. *(see [4]) (i) A sufficient condition for a function A of the form (1.1) to be in the subfamily $\mathcal{US}(\Upsilon,\Theta)$ and a necessary and sufficient for a function A of the form (1.2) to be in the subfamily $\mathcal{US}F^{-}(\Upsilon,\Theta)$ is that*

$$\sum_{ⅎ=2}^{\infty}(2ⅎ-\cos\Upsilon-\Theta)\,|a_{ⅎ}| \le \cos\Upsilon-\Theta \qquad (|\Upsilon|<\pi/2\,;0\le\Theta<1). \tag{1.6}$$

(ii) A sufficient condition for a function A of the form (1.1) to be in the subfamily $\mathcal{UC}(\Upsilon,\Theta)$ and a necessary and sufficient for a function A of the form (1.2) to be in the subfamily $\mathcal{UC}F^{-}(\Upsilon,\Theta)$ is that

$$\sum_{ⅎ=2}^{\infty}ⅎ(2ⅎ-\cos\Upsilon-\Theta)\,|a_{ⅎ}| \le \cos\Upsilon-\Theta \qquad (|\Upsilon|<\pi/2\,;0\le\Theta<1). \tag{1.7}$$

2. Conditions for the Wright Series $\mathcal{WD}(\tau,£;n;\varsigma)$

In this segment, we will provide sufficient conditions for the Wright series $\mathcal{WD}(\tau,£;n;\varsigma)$ to be in the subfamilies $\mathcal{US}(\Upsilon,\Theta)$ and $\mathcal{UC}(\Upsilon,\Theta)$.

Theorem 2.1. *If*

$$\frac{2n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)+2(1-\cos\Upsilon)\mathcal{D}(\tau,£;n)\le(2-\cos\Upsilon-\Theta)n \tag{2.1}$$

for $\tau,£,n>0$, $|\Upsilon|<\pi/2$ and $0\le\Theta<1$, then $\mathcal{WD}(\tau,£;n;\varsigma)\in\mathcal{US}(\Upsilon,\Theta)$.

Proof. Since

$$\mathcal{WD}(\tau,£;n;\varsigma)=\varsigma+\sum_{ⅎ=2}^{\infty}\frac{n^{ⅎ}\daleth(£)}{(ⅎ-1)!\daleth(\tau(ⅎ-1)+£)\mathcal{D}(\tau,£;n)}\varsigma^{ⅎ}$$

by (1.6), it suffices to show that

$$\sum_{ⅎ=2}^{\infty}\frac{n^{ⅎ}\daleth(£)(2ⅎ-\cos\Upsilon-\Theta)}{(ⅎ-1)!\daleth(\tau(ⅎ-1)+£)\mathcal{D}(\tau,£;n)}\le\cos\Upsilon-\Theta.$$

Setting $ⅎ=(ⅎ-1)+1$, we get

$$\begin{aligned}
&\sum_{ⅎ=2}^{\infty}\frac{n^{ⅎ}\daleth(£)(2ⅎ-\cos\Upsilon-\Theta)}{(ⅎ-1)!\daleth(\tau(ⅎ-1)+£)\mathcal{D}(\tau,£;n)}\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(\sum_{ⅎ=2}^{\infty}\frac{2n^{ⅎ}\daleth(£)}{(ⅎ-2)!\daleth(\tau(ⅎ-1)+£)}+\sum_{ⅎ=2}^{\infty}\frac{n^{ⅎ}\daleth(£)(2-\cos\Upsilon-\Theta)}{(ⅎ-1)!\daleth(\tau(ⅎ-1)+£)}\right)\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(2\sum_{ⅎ=0}^{\infty}\frac{n^{ⅎ+2}\daleth(£)}{ⅎ!\daleth(\tau(ⅎ+1)+£)}+\sum_{ⅎ=1}^{\infty}\frac{n^{ⅎ+1}\daleth(£)(2-\cos\Upsilon-\Theta)}{ⅎ!\daleth(\tauⅎ+£)}\right)\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(\frac{2n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)+(2-\cos\Upsilon-\Theta)\left(\mathcal{D}(\tau,£;n)-n\right)\right)\le\cos\Upsilon-\Theta.
\end{aligned}$$

Hence,

$$\frac{2n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)+2(1-\cos\Upsilon)\mathcal{D}(\tau,£;n)\leq(2-\cos\Upsilon-\Theta)n.$$

□

Theorem 2.2. *If*

$$\begin{aligned}&\frac{2n^2\daleth(£)}{\daleth(\tau+2£)}\mathcal{D}(\tau,\tau+2£;n)+\frac{(6-\cos\Upsilon-\Theta)n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)\\&+2(1-\cos\Upsilon)\mathcal{D}(\tau,£;n)\\&\leq(2-\cos\Upsilon-\Theta)n\end{aligned}$$

for $\tau, £, n>0$, $|\Upsilon|<\pi/2$ *and* $0\leq\Theta<1$, *then* $\mathcal{WD}(\tau,£;n;\varsigma)\in\mathcal{UC}(\Upsilon,\Theta)$.

Proof. Since $\mathcal{WD}(\tau,£;n;\varsigma)$ given by (1.5) and by (1.7), it suffices to show that

$$\sum_{\Finv=2}^{\infty}\frac{\Finv(2\Finv-\cos\Upsilon-\Theta)n^{\Finv}\daleth(£)}{(\Finv-1)!\daleth(\tau(\Finv-1)+£)\mathcal{D}(\tau,£;n)}\leq\cos\Upsilon-\Theta.$$

Setting $\Finv^2=(\Finv-1)(\Finv-2)+3(\Finv-1)+1$ and $\Finv=(\Finv-1)+1$, we get

$$\begin{aligned}&\sum_{\Finv=2}^{\infty}\frac{\Finv(2\Finv-\cos\Upsilon-\Theta)n^{\Finv}\daleth(£)}{(\Finv-1)!\daleth(\tau(\Finv-1)+£)\mathcal{D}(\tau,£;n)}\\
&=\sum_{\Finv=2}^{\infty}\frac{(2\Finv^2-(\cos\Upsilon+\Theta)\,\Finv)n^{\Finv}\daleth(£)}{(\Finv-1)!\daleth(\tau(\Finv-1)+£)\mathcal{D}(\tau,£;n)}\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(\sum_{\Finv=3}^{\infty}\frac{2n^{\Finv}\daleth(£)}{(\Finv-3)!\daleth(\tau(\Finv-1)+£)}+\sum_{\Finv=2}^{\infty}\frac{(6-\cos\Upsilon-\Theta)n^{\Finv}\daleth(£)}{(\Finv-2)!\daleth(\tau(\Finv-1)+£)}\right.\\
&\left.+\sum_{\Finv=2}^{\infty}\frac{(2-\cos\Upsilon-\Theta)n^{\Finv}\daleth(£)}{(\Finv-1)!\daleth(\tau(\Finv-1)+£)}\right)\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(\sum_{\Finv=0}^{\infty}\frac{2n^{\Finv+3}\daleth(£)}{\Finv!\daleth(\tau(\Finv+2)+£)\mathcal{D}(\tau,£;n)}+\sum_{\Finv=0}^{\infty}\frac{(6-\cos\Upsilon-\Theta)n^{\Finv+2}\daleth(£)}{\Finv!\daleth(\tau(\Finv+1)+£)\mathcal{D}(\tau,£;n)}\right.\\
&\left.+\sum_{\Finv=1}^{\infty}\frac{(2-\cos\Upsilon-\Theta)n^{\Finv+1}\daleth(£)}{\Finv!\daleth(\tau\Finv+£)\mathcal{D}(\tau,£;n)}\right)\\
&=\frac{1}{\mathcal{D}(\tau,£;n)}\left(\frac{2n^2\daleth(£)}{\daleth(\tau+2£)}\mathcal{D}(\tau,\tau+2£;n)+\frac{(6-\cos\Upsilon-\Theta)n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)\right.\\
&\left.+\,(2-\cos\Upsilon-\Theta)\left(\mathcal{D}(\tau,£;n)-n\right)\right)\\
&\leq\cos\Upsilon-\Theta.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{2n^2\daleth(£)}{\daleth(\tau+2£)}\mathcal{D}(\tau,\tau+2£;n)+\frac{(6-\cos\Upsilon-\Theta)n\daleth(£)}{\daleth(\tau+£)}\mathcal{D}(\tau,\tau+£;n)\\&+2(1-\cos\Upsilon)\mathcal{D}(\tau,£;n)\end{aligned}$$

$$\leq (2 - \cos \Upsilon - \Theta) n.$$

□

3. Inclusion Specifications

For $\wp_1 \in (0,1]$, $\wp_2 < 1$ and $\wp_3 \in \mathbb{C} - \{0\}$, a function $A \in \Delta$ is said to be in the subfamily $\mathcal{G}^{\wp_3}(\wp_1, \wp_2)$ if it satisfies

$$\left| \frac{(1-\wp_1)\frac{A(\varsigma)}{\varsigma} + \wp_1 A'(\varsigma) - 1}{2\wp_3(1-\wp_2) + (1-\wp_1)\frac{A(\varsigma)}{\varsigma} + \wp_1 A'(\varsigma) - 1} \right| < 1, \ |\varsigma| < 1.$$

Lemma 3.1. *[33] If $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$ of the form (1.1), then*

$$|a_{\daleth}| \leq \frac{2\,|\wp_3|\,(1-\wp_2)}{\wp_1(\daleth - 1) + 1}, \ \ \daleth \in \mathbb{N} - \{1\}.$$

Theorem 3.1. *For $\tau, £, n > 0$, $|\Upsilon| < \pi/2$, $0 \leq \Theta < 1$ and $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, if the following inequality is satisfied*

$$2\mathcal{D}(\tau, £; n) - 2n - (\cos\Upsilon + \Theta)\left(\mathcal{D}(\tau, £ - \tau; n)\frac{\urcorner(£)}{n\urcorner(£ - \tau)} - \frac{\urcorner(£)}{\urcorner(£-\tau)} - n\right)$$
$$\leq \frac{(\cos\Upsilon - \Theta)\,\wp_1 \mathcal{D}(\tau, £; n)}{2\,|\wp_3|\,(1-\wp_2)},$$

then $\mathcal{WD}(\tau, £; n; \varsigma)A \in \mathcal{US}(\Upsilon, \Theta)$.

Proof. Let $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, by (1.6) it suffices to show that

$$\sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(2\daleth - \cos\Upsilon - \Theta)}{(\daleth-1)!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau, £; n)}\,|a_{\daleth}| \leq \cos\Upsilon - \Theta.$$

Since $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, then by Lemma 3.1

$$\sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(2\daleth - \cos\Upsilon - \Theta)}{(\daleth-1)!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau, £; n)}\,|a_{\daleth}|$$
$$\leq 2\,|\wp_3|\,(1-\wp_2)\sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(2\daleth - \cos\Upsilon - \Theta)}{(\daleth-1)!\urcorner(\tau(\daleth-1)+£)\,(\wp_1(\daleth-1)+1)\,\mathcal{D}(\tau, £; n)}.$$

Also, since $\wp_1(\daleth - 1) + 1 \geq \daleth\wp_1$, we get

$$\sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(2\daleth - \cos\Upsilon - \Theta)}{(\daleth-1)!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau, £; n)}$$
$$\leq \frac{2\,|\wp_3|\,(1-\wp_2)}{\wp_1}\sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(2\daleth - \cos\Upsilon - \Theta)}{\daleth!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau, £; n)}$$
$$= \frac{2\,|\wp_3|\,(1-\wp_2)}{\wp_1\mathcal{D}(\tau, £; n)}\left(\sum_{\daleth=1}^{\infty} \frac{2n^{\daleth+1}\urcorner(£)}{\daleth!\urcorner(\tau\daleth+£)} - \sum_{\daleth=2}^{\infty} \frac{n^{\daleth}\urcorner(£)(\cos\Upsilon + \Theta)}{\daleth!\urcorner(\tau(\daleth-1)+£)}\right)$$

$$
\begin{aligned}
&= \frac{2\,|\wp_3|\,(1-\wp_2)}{\wp_1 \mathcal{D}(\tau,£;n)}\,[2\mathcal{D}(\tau,£;n) - 2n \\
&-(\cos \Upsilon + \Theta)\left(\mathcal{D}(\tau,£-\tau;n)\frac{\urcorner(£)}{n\urcorner(£-\tau)} - \frac{\urcorner(£)}{\urcorner(£-\tau)} - n\right)\Bigg] \\
&\le \cos \Upsilon - \Theta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&2\mathcal{D}(\tau,£;n) - 2n - (\cos \Upsilon + \Theta)\left(\mathcal{D}(\tau,£-\tau;n)\frac{\urcorner(£)}{n\urcorner(£-\tau)} - \frac{\urcorner(£)}{\urcorner(£-\tau)} - n\right) \\
&\le \frac{(\cos \Upsilon - \Theta)\,\wp_1 \mathcal{D}(\tau,£;n)}{2\,|\wp_3|\,(1-\wp_2)}.
\end{aligned}
$$

□

Theorem 3.2. *For $\tau, £, n > 0$, $|\Upsilon| < \pi/2$, $0 \le \Theta < 1$ and $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, if the following inequality is satisfied*

$$
\begin{aligned}
&\frac{2n\urcorner(£)}{\urcorner(\tau+£)}\mathcal{D}(\tau,\tau+£;n) + (2 - \cos \Upsilon - \Theta)\,(\mathcal{D}(\tau,£;n) - n) \\
&\le \frac{(\cos \Upsilon - \Theta)\,\wp_1 \mathcal{D}(\tau,£;n)}{2\,|\wp_3|\,(1-\wp_2)},
\end{aligned}
$$

then $\mathcal{WD}(\tau,£;n;\varsigma)A \in \mathcal{UC}(\Upsilon,\Theta)$.

Proof. Let $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, by (1.7) it suffices to show that

$$
\sum_{\daleth=2}^{\infty} \frac{\daleth(2\daleth - \cos \Upsilon - \Theta)n^{\daleth}\urcorner(£)}{(\daleth-1)!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau,£;n)}\,|a_{\daleth}| \le \cos \Upsilon - \Theta.
$$

Since $A \in \mathcal{G}^{\wp_3}(\wp_1, \wp_2)$, then by Lemma 3.1 and $\wp_1(\daleth-1) + 1 \ge \daleth\wp_1$, and since the proof comparable to that of Theorem 2.1. We therefore leave out the specifics. □

4. An Operator $IO(\tau,£;n;\varsigma)$

Theorem 4.1. *For $\tau, £, n > 0$, $|\Upsilon| < \pi/2$ and $0 \le \Theta < 1$, if the integral operator $IO(\tau,£;n;\varsigma)$ is given by*

$$
IO(\tau,£;n;\varsigma) = \int_0^{\varsigma} \frac{\mathcal{WD}(\tau,£;n;s)}{s}ds,\ |\varsigma| < 1, \tag{4.1}
$$

then $IO(\tau,£;n;\varsigma) \in \mathcal{US}(\Upsilon,\Theta)$ if

$$
\begin{aligned}
&2\mathcal{D}(\tau,£;n) - 2n - (\cos \Upsilon + \Theta)\left(\mathcal{D}(\tau,£-\tau;n)\frac{\urcorner(£)}{n\urcorner(£-\tau)} - \frac{\urcorner(£)}{\urcorner(£-\tau)} - n\right) \\
&\le (\cos \Upsilon - \Theta)\,\Upsilon_1 \mathcal{D}(\tau,£;n).
\end{aligned}
$$

Proof. Since

$$
IO(\tau,£;n;\varsigma) = \varsigma + \frac{n^{\daleth}\urcorner(£)}{\daleth!\urcorner(\tau(\daleth-1)+£)\mathcal{D}(\tau,£;n)}\varsigma^{\daleth}, \tag{4.2}
$$

then by (1.6) it suffices to show that

$$\sum_{\daleth=2}^{\infty}\frac{(2\daleth-\cos\Upsilon-\Theta)n^{\daleth}\urcorner(\pounds)}{\daleth!\urcorner(\tau(\daleth-1)+\pounds)\mathcal{D}(\tau,\pounds;n)}\leq\cos\Upsilon-\Theta.$$

The proof is comparable to that of Theorem 3.1. We therefore leave out the specifics. □

Theorem 4.2. *For $\tau,\pounds,n>0$, $|\Upsilon|<\pi/2$ and $0\leq\Theta<1$, the integral operator $IO(\tau,\pounds;n;\varsigma)$ given by (4.1) be in the subfamily $\mathcal{UC}(\Upsilon,\Theta)$ if it verifies the condition (2.1).*

Proof. Since $IO(\tau,\pounds;n;\varsigma)$ given by *(4.2)* and by *(1.7)* it suffices to show that

$$\sum_{\daleth=2}^{\infty}\frac{\daleth(2\daleth-\cos\Upsilon-\Theta)n^{\daleth}\urcorner(\pounds)}{\daleth!\urcorner(\tau(\daleth-1)+\pounds)\mathcal{D}(\tau,\pounds;n)}\leq\cos\Upsilon-\Theta.$$

The proof is comparable to that of Theorem 2.1. We therefore leave out the specifics. □

5. Corollaries

Many subresults we can obtain from our main results by specializing the parameters Υ and Θ, especially if $\Theta=0$; we get the following corollaries related to subfamilies $\mathcal{US}(\Upsilon)$ and $\mathcal{UC}(\Upsilon)$.

Corollary 5.1. *If*

$$\frac{2n\urcorner(\pounds)}{\urcorner(\tau+\pounds)}\mathcal{D}(\tau,\tau+\pounds;n)+2(1-\cos\Upsilon)\mathcal{D}(\tau,\pounds;n)\leq(2-\cos\Upsilon)n \tag{5.1}$$

for $\tau,\pounds,n>0$, and $|\Upsilon|<\pi/2$, then $\mathcal{WD}(\tau,\pounds;n;\varsigma)\in\mathcal{US}(\Upsilon)$.

Corollary 5.2. *If*

$$\begin{aligned}&\frac{2n^2\urcorner(\pounds)}{\urcorner(\tau+2\pounds)}\mathcal{D}(\tau,\tau+2\pounds;n)+\frac{(6-\cos\Upsilon)n\urcorner(\pounds)}{\urcorner(\tau+\pounds)}\mathcal{D}(\tau,\tau+\pounds;n)\\&+2(1-\cos\Upsilon)\mathcal{D}(\tau,\pounds;n)\\&\leq(2-\cos\Upsilon)n\end{aligned}$$

for $\tau,\pounds,n>0$, and $|\Upsilon|<\pi/2$, then $\mathcal{WD}(\tau,\pounds;n;\varsigma)\in\mathcal{UC}(\Upsilon)$.

Corollary 5.3. *For $\tau,\pounds,n>0$, $|\Upsilon|<\pi/2$ and $A\in\mathcal{G}^{\wp_3}(\wp_1,\wp_2)$, if*

$$\begin{aligned}&2\mathcal{D}(\tau,\pounds;n)-2n-\cos\Upsilon\left(\mathcal{D}(\tau,\pounds-\tau;n)\frac{\urcorner(\pounds)}{n\urcorner(\pounds-\tau)}-\frac{\urcorner(\pounds)}{\urcorner(\pounds-\tau)}-n\right)\\&\leq\frac{\wp_1\mathcal{D}(\tau,\pounds;n)\cos\Upsilon}{2\,|\wp_3|\,(1-\wp_2)},\end{aligned}$$

then $\mathcal{WD}(\tau,\pounds;n;\varsigma)A\in\mathcal{US}(\Upsilon)$.

Corollary 5.4. *For $\tau,\pounds,n>0$, $|\Upsilon|<\pi/2$ and $A\in\mathcal{G}^{\wp_3}(\wp_1,\wp_2)$, if*

$$\frac{2n\urcorner(\pounds)}{\urcorner(\tau+\pounds)}\mathcal{D}(\tau,\tau+\pounds;n)+(2-\cos\Upsilon)\left(\mathcal{D}(\tau,\pounds;n)-n\right)\leq\frac{\wp_1\mathcal{D}(\tau,\pounds;n)\cos\Upsilon}{2\,|\wp_3|\,(1-\wp_2)},$$

then $\mathcal{WD}(\tau,\pounds;n;\varsigma)A\in\mathcal{UC}(\Upsilon)$.

Corollary 5.5. *For $\tau, £, n > 0$ and $|\Upsilon| < \pi/2$, the integral operator $IO(\tau, £; n; \varsigma)$ given by (4.1) be in the subfamily $\mathcal{US}(\Upsilon, \Theta)$ if*

$$2\mathcal{D}(\tau, £; n) - 2n - \cos\Upsilon\left(\mathcal{D}(\tau, £ - \tau; n)\frac{\daleth(£)}{n\daleth(£-\tau)} - \frac{\daleth(£)}{\daleth(£-\tau)} - n\right)$$
$$\leq \Upsilon_1\mathcal{D}(\tau, £; n)\cos\Upsilon.$$

Corollary 5.6. *For $\tau, £, n > 0$ and $|\Upsilon| < \pi/2$, the integral operator $IO(\tau, £; n; \varsigma)$ given by (4.1) be in the subfamily $\mathcal{UC}(\Upsilon)$ if it verifies the condition (5.1).*

Remark 5.1. *If we take $\Upsilon = \Theta = 0$, we get the results due to Porwal et al. [35] in case $\gamma = 1$.*

6. Conclusions

In the present work, we investigated sufficient conditions for the Wright distribution series $\mathcal{WD}(\tau, £; n; \varsigma)$, $\mathcal{WD}(\tau, £; n; \varsigma)A$, and the integral operator $IO(\tau, £; n; \varsigma)$ defined by the Wright distribution series to be in the inclusive subfamilies $\mathcal{US}(\Upsilon, \Theta)$ and $\mathcal{UC}(\Upsilon, \Theta)$. Additionally, this study encourages future researchers to discover new requirements for belonging to these subfamilies, applicable not only to the Wright distribution series but also to several other specific functions or distributions.

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