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Harmonic-Like Quasi Bifunction Equilibrium Inclusions

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Abstract. Some classes of harmonic-like quasi bifunction equilibrium inclusions are introduced and investigated. Using various techniques such as resolvent methods, auxiliary principle, dynamical systems coupled with finite difference approach, we suggest and analyze a number of new multistep iterative methods for solving harmonic-like quasi bifunction equilibrium inclusions. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis is also investigated. One can obtain a number of new classes of harmonic-like quasi function equlibrium problems by interchanging the role of operators. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

1. Introduction

Equilibrium problems theory, which is mainly due to Blum et al. [9] and Noor et al. [65], contains a wealth of new ideas and techniques. This theory can be viewed as a novel extension and generalization of the variational inequalities and variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics,geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas Equilibrium problems have been generalized and extended in several directions using novel and innovative ideas to handle complicated and complex problems, see [7, 9, 11, 12, 20–23, 28, 29, 31, 39, 45, 48, 49, 56, 58, 61–63, 65, 72] for more details.

It is worth mentioning that equlibrium problems and variational inequalities theory are closely related to the convexity theory, which contains a wealth of novel ideas and innovative techniques. Several new generalizations and extensions of the convex functions and convex sets have been

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introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions andconvex sets. Anderson et al. [2] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can be viewed as important generalization of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory, signal processing and semiconductor physics. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values and then taking the reciprocal of their total. More precisely, if η_1 and η_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula

$$(\frac{1}{\eta_1}+\frac{1}{\eta_2})^{-1}=\frac{\eta_1\eta_2}{\eta_1+\eta_2},$$

which is half the harmonic means. The harmonic mean is employed in finance to determine the average multiples like the price-income ratio. Al-Azemi et al. [5] studied the Asian options with harmonic average, which is a new direction in the study of the risk analysis stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [50–53] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions. The harmonic mean plays a crucial role in numerous scientific and engineering fields, particularly in semiconductor physics. In this context, the effective mass of charge carriers within a semiconductor is often computed using the harmonic mean of its crystallographic directions. Beyond electrical circuits and semiconductor physics, harmonic convexity also has implications in signal processing. Functions that exhibit harmonic convexity are associated with higher frequency components that can distort fundamental waveforms, leading to undesired oscillations. This property is particularly relevant in applications involving wave form stability and signal transmission, where minimizing undesirable frequency components is essential.

It has been established that the harmonic-like variational inequalities are equivalent to the fixed point problems. This equivalent formulations have played an important role to study the existence of the solution and to develop multistep iterative numerical methods for solving variational inequalities and related optimization problems. These multistep methods are novel generalizations of the Noor (three step) iterations [38] for solving the general variational inequalitiees. Ashish et. al. [3, 4], Cho et al. [14] and Kwuni et al. [26] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations. It is worth mentioning that Noor iterations have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. These three-step schemes are a natural generalization of the splitting methods of Ames for solving partial differential equations. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special cases.

The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [18]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [6, 18, 30, 40, 41, 46, 47, 52, 61–63, 73] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

We would like to mention that the sensitivity analysis provides useful information for designing or planning various equilibrium systems. Sensitivity analysis can provide new insight and stimulate new ideas and techniques for problem solving. Dafermos [17] studied the sensitivity analysis of the variational inequalities using the fixed point technique. This approach has strong geometrical flavour and has been investigated for various classes of quasi variational inequalities. Also see, [17, 40, 44, 47, 55, 56, 62, 63] and the references therein.

Motivated and inspired by ongoing recent research in harmonic-like variational inequalities, we consider and study the general harmonic-like quasi variational inequalities, involving two arbitrary operators. We establish the equivalence between the harmonic-likeequasi variational inequalities and fixed point problem, which has been used to consider an iterative method for solving harmonic-like quasi variational inequalities. We prove that the nonlinear programming problems and implicit second order obstacle boundary value problems can be studied via the harmonic-like quasi variational inequalities. Several special cases are discussed as applications of the harmonic-like quasi variational inequalities in Section 2. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along with the convergence analysis. The harmonic-like Wiener-Hopf equation technique is used to suggest some iterative methods in Section 4. We also apply the auxiliary principle technique involving an arbitrary operator is used to discuss some iterative schemes for solving the harmonic-like quasi variational inequalities in Section 5. Dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods exploring the finite difference idea. Our results in this section can be viewed as significant refinement of the known results.

Sensitivity analysis for variational inequalities has been studied by many authors using quite different techniques. In Section 7, we obtain some new results for the sensitivity analysis of the harmonic-like quasi variational inequalities.

One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the harmonic-like quasi variational inequalities and to point out that researchers in different field of variational inequalities and optimization. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of general harmonic-like quasi variational inequalities in other areas of sciences such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic, financial analysis and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and basic facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [9, 10, 30, 70–72] which are needed in the derivation of the main results.

Definition 2.1. *[71, 72] A set $\Omega_g \subseteq \mathcal{H}$ is said to be a general harmonic-like convex set, if there exists an arbitrary function $g : \mathcal{H} \longrightarrow \mathcal{H}$ such that*

$$g(\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu)) \in \Omega_g, \quad \forall \mu, w, \nu \in \Omega_g. \quad t \in [0,1], \quad w \in [\mu, \nu].$$

Note that every harmonic-like convex set is a general harmonic-like convex set, but the converse is not true. The element $w \in \Omega(\mu)$ is the weight function. It is worth mentioning that, for $w = \mu$, the general convex set was introduced Noor [60, 61], which is different than the other various general convex sets. For the applications of the general convex sets in information technology, railway systems, computer aided design, digital vector optimization and comparison with other concepts, see [19, 20]. If $g = I, w = \mu$, then the general convex set Ω_g is exactly the convex set Ω.

Definition 2.2. *The function $\Phi : \Omega_g \longrightarrow \mathcal{H}$ is said to be general harmonic-like convex, if there exists an arbitrary function g, such that*

$$\begin{aligned}&\Phi(g(\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu))) \\ &\quad \leq \Phi(g(\mu)) + t\{\Phi(g(\nu)) - \Phi(g(\mu))\}, \quad \forall \mu, w, \nu \in \Omega_g, \quad t \in [0,1].\end{aligned}$$

For $w = \mu$, the general harmonic-like convex function is exactly the general convex function, considered by Noor [60, 61].

For the differentiable general harmonic-like convex function, we have

Theorem 2.1. *[71, 72] Let Φ be a differentiable general harmonic-like convex function on the general harmonic-like convex set Ω_g. Then the minimum $\mu \in \Omega_g$ of the function Φ, if and only if, $\mu \in \Omega_g$ satisfies the inequality*

$$\langle \Phi'(g(\frac{2w\mu}{w+\mu})), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega_g, \tag{2.1}$$

where $\Phi'(.)$ is the differential of Φ, This motivated Noor [48] to introduce and investigate the problem of finding $\mu \in \Omega$, a closed convex set in $\mathcal{H}$ such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.2}$$

which is called the general harmonic-like variational inequalities. Note that the problem (2.1) is a special case of the problem (2.2) with $\mathcal{T}(\frac{2w\mu}{w+\mu}) = \Phi'(g(\frac{2w\mu}{w+\mu}))$.

It is known that the inequality of the type (2.2) may not arise as the optimality condition of the differentiable functions. We now consider some new classes of harmonic-like equilibrium problems, which include several new and known classes of variational inequalities and equilibrium inclusions as special cases.

For given a bifunction $F(.,.) = \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$, nonlinear operators $\mathcal{T}$, g and maximal bimonotone operator $A(.,.)$, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$0 \in F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) + A(g(\mu), g(\mu)), \quad \forall \nu, w \in \mathcal{H}. \tag{2.3}$$

which is called the general harmonic-like quasi bifunction equilibrium inclusion.
For $w = \mu$, the problem (2.3) reduces to finding $\mu \in \mathcal{H}$, such that

$$0 \in F(\mathcal{T}(\mu), \nu) + A(g(\mu), g(\mu)), \quad \forall \nu, w \in \mathcal{H}, \tag{2.4}$$

which is known as quasi equilibrium inclusion, introduced and studied by Noor [52].

Special Cases.

(1) If $A(\cdot, \mu) = \partial\varphi(\cdot, \mu) : \mathcal{H} \times \mathcal{H} \longrightarrow R \cup \{+\infty\}$, the subdifferential of a convex, proper and lower semi-continuous function $\varphi(\cdot, \mu)$ with respect to the first argument, then problem (2.3) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\begin{aligned} &\langle F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu), g(\nu) - g(\mu)\rangle \\ &\quad +\varphi(g(\nu), g(\mu)) - \varphi(g(\mu), g(\mu)) \geq 0, \quad \forall w, \nu \in \mathcal{H}, \end{aligned} \tag{2.5}$$

which is called the general mixed harmonic-like quasi bifunction equilibrium variational inequality.

(2) If $A(g(\mu), g(\nu)) = A(g(\mu))$, for all $\nu \in \mathcal{H}$, then problem (2.3) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$0 \in F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) + A(g(\mu)), \quad \forall \nu, w \in \mathcal{H}, \tag{2.6}$$

a harmonic-like equilibrium inclusion.

(3) If $g = I$, the identity operator, then problem (2.3) reduces to finding $\mu \in H$ such that

$$0 \in F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) + A(\mu, \mu), \forall \nu, w \in \mathcal{H}, \tag{2.7}$$

which is called the harmonic-like quasi equilibrium inclusion.

(4) If $\mathcal{A}(g(\mu)) \equiv \partial\varphi(g(\mu))$ is the subdifferential of a proper, convex and lower, semicontinuous function $\varphi : \mathcal{H} \longrightarrow R \cup \{+\infty\}$. then problem (2.5) reduces to: find $\mu \in H$ such that

$$\langle F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu), g(\nu) - g(\mu)\rangle + \varphi(g(\nu)) - \varphi(g(\mu)) \geq 0, \quad \forall \nu, w \in \mathcal{H}. \tag{2.8}$$

Problem (2.8) is known as the mixed harmonic-like general equilibrium variational inequality.

(5) If the function $\varphi(\cdot, \cdot)$ is the indicator function of a closed convex-valued set $\Omega(\mu)$ in H, that is,

$$\varphi(\mu, \mu) = \Omega(\mu)(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \Omega(\mu) \\ +\infty, & \text{otherwise} , \end{cases}$$

then problem (2.5) is equivalent to finding $\mu \in \Omega(\mu)$, such that

$$\langle F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \quad \nu \in \Omega(\mu), \tag{2.9}$$

is called the harmonic-like quasi bifunction equilibrium variational inequality.
For $w = \mu$, the problem (2.9) reduces to: find $\mu \in \Omega(\mu)$ such that

$$\langle F(\mathcal{T}(\mu), \nu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \quad \nu \in \Omega(\mu), \tag{2.10}$$

is called the general bofnction variational inequality, which, for $F(\mathcal{T}(\mu), \nu) = \langle \mathcal{T}(\mu)$, collapses to find $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}(\mu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \quad \nu \in \Omega(\mu), \tag{2.11}$$

is known as general variational inequality, considered and studied by Noor et al [48]. For formulation, motivation, numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi variational inequalities, see [**?**, **?**, 5, 10, 13–16, 18, 23, 26, 27, 30, 32, 38, 40, 42, 44, 49, 51] and the references therein.

(6) If $w = \mu$, $\Omega(\mu) = \Omega$, a convex set and $g = I$, the identity operator, then then problem (2.11) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \quad \nu \in \Omega, \tag{2.12}$$

which is called the variational inequality, introduced and studied by Stampacchia [52] in 1964. Variational inequalities can be viewed as novel extensions of the variational principles. Variational inequality has influenced several branches of mathematical, engineering, economics, transforation, regional and medical sciences and continue to inspire researchers to find its applications. For more details, see [5, 7, 8, 10–19, 21–24, 29, 30, 32, 34–44, 46–56].

(7) If $\Omega^*(\mu) = \{\mu \in H, \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar cone of the convex-valued cone $\Omega(\mu)$ in H, then problem (2.7) is equivalent to finding $\mu \in H$, such that

$$g(\mu) \in \Omega(\mu), F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) \in \Omega^*(\mu), \langle F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu), g(\mu)\rangle = 0, \tag{2.13}$$

which is called the generalized harmonic-like bifunction equilibriuml complementarity problem.

(8) If $\Omega^*(\mu) = \Omega^*$, then problem (2.13) reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \mathcal{K}, \quad F(\mathcal{T}(\frac{2w\mu}{w+\mu}, \nu) \in \Omega^*, \qquad \langle F(\mathcal{T}(\frac{2w\mu}{w+\mu}, \nu), g(\mu)\rangle = 0, \tag{2.14}$$

which is known as harmonic-like bifunction equilibrium complementarity problem. For $w = \mu$, the problem (2.14) is equvalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \geq 0, \quad F(\mathcal{T}(\mu), \nu) \in \Omega^*, \quad \langle F(\mathcal{T}(\mu), \nu), g(\mu)\rangle = 0.$$

is called the general bifunction equilibrium complemenatrity problem. For $g = I$, the identity operator, then the general complimentarity is called the nonlinear bifunction equilibrium complemenatrity problem. For the applications, formulations and generalizations of the complementarity problems, see [8,17,28,39,44].

Remark 2.1. *For special choices of the operators $\mathcal{T}$, g, $\mathcal{A}(.,.)$, the bifunction $F(.,.)$ and the convex valued set Ω, one can obtain a large number of implicit (quasi) complementarity problems and variational inequality problems, which are very special cases of problem (2.3). Thus it is clear that problem (2.3) is general and unifying one and has numerous applications in pure and applied sciences.*

Definition 2.3. *If A is a maximal monotone operator on $\mathcal{H}$, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by*

$$\mathcal{J}_A = (I + \rho A)^{-1}(\mu), \quad \forall \mu \in \mathcal{H},$$

where I is the identity operator.

It is known that the resolvent operator $\mathcal{J}_A$ is single-valued and nonexpansive, that is,

$$\|J_A(\mu) - J_A(\nu)\| \leq \|\mu - \nu\|, \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.2. *Since the operator $\mathcal{A}(.,.)$ is a maximal monotone operator with respect to the first argument, we denote by*

$$\mathcal{J}_{\mathcal{A}(\mu)} \equiv (I + \rho \mathcal{A}(\mu))^{-1}(\mu), \quad \forall \mu \in \mathcal{H},$$

the resolvent operator associated with $\mathcal{A}(., \mu) \equiv \mathcal{A}(\mu)$. For example, if $\mathcal{A}(., \mu) = \partial\varphi(., \mu)$, for all $\mu \in \mathcal{H}$, and $\varphi(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow R \cup \{\infty\}$ is a proper, convex and lower semicontinuous with respect to the first argument, then it is well known that $\partial\varphi(., \mu)$ is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator $\mathcal{J}_{\mathcal{A}(\mu)} = \mathcal{J}_{\varphi(\mu)}$ is defined as

$$\mathcal{J}_{\varphi(\mu)} = (I + \rho\partial\varphi(., \mu))^{-1}(\mu) = (I + \rho\partial\varphi(\mu))^{-1}, \quad \forall \mu \in \mathcal{H},$$

which is defined everywhere on the whole space H, where $\partial\varphi(\mu) \equiv \partial\varphi(., \mu)$.

Wee need the follwing assumption.

Assumption 1.

$$\|J_{A(\mu)}\omega - J_{A(\nu)}\omega\| \leq \xi\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \tag{2.15}$$

where $\xi > 0$ is a constant.

Assumption 1 has been used to prove the existence of a solution of general quasi variational and equilibrium inclusions as well as in analyzing convergence of the iterative methods. We use Assumption 1 to prove the existence of a solution of general harmonic-like quasi bifunction equilibrium inclusions as well as in analyzing the convergence of the iterative methods.

Definition 2.4. *The bifucntion $F.,.)$ with respect to the operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

(1) *Strongly harmonic-like joint monotone, if there exist a constant $\alpha > 0$, such that*

$$F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) - F(\mathcal{T}(\frac{2w\nu}{w+\eta}), \nu), \geq \alpha\|\mu - \eta\|^2, \quad \forall w, \mu, \nu, \eta \in \mathcal{H}.$$

(2) *harmonic-like joint Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) - F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)\| \leq \beta\|\mu - \eta\|, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(3) *harmonic-like joint monotone, if*

$$F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) - F(\mathcal{T}(\frac{2w\eta}{w+\eta}, \nu)), \mu - \eta\rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(4) *harmonic-like joint pseudo monotone, if*

$$F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) \geq 0 \quad \Rightarrow \quad F(\mathcal{T}(\frac{2w\nu}{w+\nu}), \mu) \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

Definition 2.5. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

(1) *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(2) *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

(3) *Monotone, if*

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu\rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

(4) *Pseudo monotone, if*

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\nu, \nu - \mu\rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.3. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3. Resolvent Methods

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general harmonic-like quasi bifunction equilibriuml inclusions.
We now establish the equivalence between the general quasi harmonic-like bifunction equilibriuml inclusions and the fixed point problem.

Lemma 3.1. *The function $\mu \in \mathcal{H}$ is a solution of the general harmonic-like quasi bifunction equilibriuml inclusion (2.3), if and only if, $\mu \in \mathcal{H}$ satisfies the relation*

$$g(\mu) = \mathcal{J}_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)], \tag{3.1}$$

where $\mathcal{J}_{\mathcal{A}(\leq)}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \mathcal{H}$ be a solution of (2.3), then, for a constant ρ,

$$\begin{aligned} \rho F(\mathcal{T}(\frac{2aw\mu}{w+\mu}), \nu) \quad &+ \quad \rho \mathcal{A}(g(\mu), g(\mu)) \ni 0, \\ &\Longleftrightarrow \\ -g(\mu) + \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) \quad &+ \quad g(\mu) + \rho A(g(\mu), g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ g(\mu) = \quad & \mathcal{J}_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)]. \end{aligned}$$

the required (3.1). □

Lemma 3.1 implies that the harmonic-like quasi bifunction equilibrium inclusion (2.3)is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) will play an important role in deriving the main results.

From the equation (3.1), we have

$$\mu = \mu - g(\mu) + J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu))].$$

We define the function Φ associated with (3.1) as

$$\Phi(\mu) = \mu - g(\mu) + J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu))], \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.3), it is enough to show that the map Φ defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. Lat the bifuncton $F(.,.)$ be jointly Lipschitz continuous with constant β. If the Assumption 1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \tag{3.3}$$

where

$$\theta = \rho\beta + k \tag{3.4}$$
$$k = \sqrt{1-2\sigma+\zeta^2} + \xi + \zeta. \tag{3.5}$$

then there exists a unique solution of the problem (2.3).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.3) are equivalent. Thus it is enough to show that the map $\Phi(u)$, defined by (3.2) has a fixed point.
For all $\eta \neq \mu \in \mathcal{H}$, we have

$$\begin{aligned}
& \|\Phi(\mu) - \Phi(\eta)\| = \|\mu - \eta - (g(\mu) - g(\eta))\| \\
& +A_{A(\mu)}\|[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] - J_{A(\eta)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)]\| \\
= \; & \|\nu - \mu - (g(\eta) - g(\mu))\| \\
& +\|A_{A(\mu)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)] - J_{A(\eta)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)]\| \\
+ \; & \|J_{A(\eta)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)] - J_{A(\eta)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)]\| \\
\leq \; & \|\mu - \nu - (g(\mu) - g(\eta))\| + \xi\|\eta - \mu\| \\
& +\|g(\eta) - g(\mu) - \rho(F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu) - F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu))\| \\
\leq \; & \|\mu - \eta - (g(\mu) - g(\eta)\| + \xi\|\eta - \mu\| + \zeta\|\eta - \mu\| + \rho\beta\|\eta - \mu\|.
\end{aligned} \tag{3.6}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu - \nu - (g(\mu) - g(\nu))\|^2 &\leq \|\mu - \nu\|^2 - 2\langle g(\mu) - g(\nu), \mu - \nu\rangle + \zeta^2\|g(\mu) - g(\nu)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu - \nu\|^2.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
\|F(\mu) - F(\nu)\| &\leq 2\left\{\sqrt{(1-2\sigma+\zeta^2)} + \xi + \zeta + \rho\beta\right\}\|\mu - \nu\| \\
&= \theta\|\mu - \nu\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \rho\beta + k \\
k &= 2\sqrt{1-2\sigma+\zeta^2} + \xi + \zeta.
\end{aligned}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $\Phi(\mu)$ defined by (3.2) has a fixed point, which is the unique solution of (2.3). □

The fixed point formulation (3.1) is applied to propose and suggest the iterative methods for solving the problem (2.3).

We now suggest and analyze the three step iterative methods for solving the quasi harmonic-like bifunction equilibrium inclusion(2.3).

Algorithm 3.1. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n \\ &\quad +\gamma_n\{\mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}),\nu)]\} \qquad (3.8) \\ z_n &= (1-\beta_n)\mu_n \\ &\quad +\beta_n\{y_n - g(y_n) + J_{A(y_n)}[g(y_n) - \rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}),\nu)]\} \qquad (3.9) \\ \mu_{n+1} &= (1-\alpha_n)\mu_n \\ &\quad +\alpha_n\{z_n - g(z_n) + J_{A(w_n)}[z_n - \rho F(\mathcal{T}(\frac{2wz_n}{w+z_n}),\nu)]\}. \qquad (3.10) \end{aligned}$$

which are known as modified Noor iterations.

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.1 converges to the exact solution $\mu \in \mathcal{H}$ of the harmonic-like quasi bifunction equilibrium inclusion (2.3) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \mathcal{H}$ of the harmonic-like quasi bifunction equilibrium inclusions (2.3). Let $\mu \in H$ be the unique solution of (2.3). Then, using Lemma 3.1, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)]\} \qquad (3.11) \\ &= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)]\} \qquad (3.12) \\ &= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)]\}. \qquad (3.13) \end{aligned}$$

From (3.10),(3.11) and Assumption 1, we have

$$\begin{aligned} \|\mu_{n+1}-\mu\| &= \|(1-\alpha_n)(\mu_n-\mu) + \alpha_n(z_n - \mu - (g(z_n) - g(\mu))) \\ &+ \alpha_n J_{A((z_n)}[g(z_n) - \rho F(\mathcal{T}(\frac{2wz_n}{w+z_n})w_n),\nu)] - J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)\}\| \\ &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\| \\ &+ \alpha_n J_{A((z_n)}[g(z_n) - \rho F(\mathcal{T}(\frac{2wz_n}{w+z_n})),\nu)] - J_{A(w_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)\}\| \\ &+ \alpha_n\{J_{(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)] - J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)\}\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|z_n-\mu-(g(z_n)-g(\mu))\| \\
&+ \alpha_n\|g(w_n)-g(\mu)-\rho(F(\mathcal{T}(\frac{2wz_n}{w+z_n}),\nu)-F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)\|+\alpha_n\eta\|w_n-\mu\| \\
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n(k+\rho\beta)\|w_n-\mu\| \\
&= (1-\alpha_n)\|u_n-\mu\|+\alpha_n\theta\|w_n-\mu\|, \qquad (3.14)
\end{aligned}
$$

where θ is defined by (3.4).
In a similar way, from (3.8) and (3.12), we have

$$
\begin{aligned}
\|z_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+2\beta_n\theta\|y_n-\mu-(g(y_n)-g(\mu))\| \\
&+ \beta_n\|g(y_n)-g(\mu)-\rho(y_n-\mu)\|+\beta_n\eta\|y_n-\mu\| \\
&\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n(k+\rho)\|y_n-\mu\|, \\
&\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|y_n-\mu\|, \qquad (3.15)
\end{aligned}
$$

where θ is defined by (3.3).
From (3.8) and (3.13), we obtain

$$
\begin{aligned}
\|y_n-\mu\| &\leq (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\| \\
&\leq (1-(1-\theta)\gamma_n)\|\mu_n-\mu\| \\
&\leq \|\mu_n-\mu\|. \qquad (3.16)
\end{aligned}
$$

From (3.15) and (3.16), we obtain

$$
\begin{aligned}
\|z_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\| \\
&= (1-(1-\theta)\beta_n)\|\mu_n-\mu\| \\
&\leq \|\mu_n-\mu\|. \qquad (3.17)
\end{aligned}
$$

Form the above equations, we have

$$
\begin{aligned}
\|\mu_{n+1}-\mu\| &\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\| \\
&= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\| \\
&\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.16), and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest some new perturbed iterative schemes for solving the harmonic-like quasi bifunction equilibrium inclusion (2.3).

Algorithm 3.2. *For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \nu)]\} + \gamma_n h_n \\ z_n &= (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + J_{A(y_n)}[g(y_n) - \rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}), \nu)]\} + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{z_n - g(z_n) + J_{A(z_n)}[g(z_n) - \rho F(\mathcal{T}(\frac{2wz_n}{w+z_n})\nu)\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $J_{A(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving harmonic-like quasi bifunction equilibrium inclusion(2.3).

Also, we can suggest the following iterative methods for solving the harmonic-like quasi bifunction equilibrium inclusion (2.3).

Algorithm 3.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \nu)], \tag{3.18}$$

which is known as the resolvent method.

Algorithm 3.4. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu)], \tag{3.19}$$

which is an implicit projection method and is equivalent to the following two-step method.

Algorithm 3.5. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(z_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2wz_n}{w+z_n}), \nu)], \end{aligned}$$

Algorithm 3.6. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}[g(\mu_{n+1}) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu)], \tag{3.20}$$

which is known as the modified resolvent method and is equivalent to the iterative method.

Algorithm 3.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2wu_n}{w+\mu_n}),\nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(z_n)}[g(z_n) - \rho F\mathcal{T}(\frac{2wz_n}{w+z_n}),\nu)], \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.3).
We can rewrite the equation (3.1) as:

$$\mu = \mu - g(\mu) + J_{A(\mu)}[g\Big(\frac{\mu+\mu}{2}\Big) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]. \tag{3.21}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 3.8. *[37]. For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}[g\Big(\frac{\mu_n+\mu_{n+1}}{2}\Big) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)]. \tag{3.22}$$

One can rewrite (3.1) as

$$\mu = \mu - g(\mu) + J_{A(\mu)}[(\frac{\mu+\mu}{2}) - \rho F(\mathcal{T}(\frac{2w(\mu+\mu)}{2w+\mu+\mu}),\nu)]. \tag{3.23}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.3).

Algorithm 3.9. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}[g(\frac{\mu_n+\mu_{n+1}}{2}) - \rho F(\mathcal{T}(\frac{2w(\mu_n+\mu_{n+1})}{2w+\mu_n+\mu_{n+1}}),\nu)].$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.6 as the predictor and Algorithm 3.9 as corrector. Thus, we obtain a new two-step method for solving the problem (2.3).

Algorithm 3.10. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}),\nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(\mu_n)}\Bigg[g(\Bigg(\frac{\omega_n+\mu_n}{2}\Bigg) - \rho F(\mathcal{T}(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}),\nu)], \end{aligned}$$

which is a new predictor-corrector two-step method.
For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + J_{A(\mu)}[g((1-\xi)\mu+\xi\mu) - \rho F(\mathcal{T}(\frac{2w((1-\xi)\mu+\xi\mu)}{w+(1-\xi)\mu+\xi\mu}),\nu)].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.3).

Algorithm 3.11. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_n)}[g((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho F(\mathcal{T}\frac{2w((1-\xi)\mu_n + \xi\mu_{n-1})}{w + (1-\xi)\mu_n + \xi\mu_{n-1}), \nu)}].$$

We now suggest a multi-step inertial iterative method for solving the harmonic-like quasi bifunction equilibrium inclusions. (2.3) is suggested.

Algorithm 3.12. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned}
z_n &= \mu_n - \theta_n (\mu_n - \mu_{n-1}), \\
y_n &= (1-\gamma_n) z_n \\
&\quad + \gamma_n \left\{ z_n - g(\omega_n) + J_{(A(z_n)} \left[g(\frac{z_n + \mu_n}{2}) - \rho F(\mathcal{T}(\frac{2w(z_n + \mu_n)}{2w + z_n + \mu_n}), \nu) \right] \right\}, \\
t_n &= (1-\beta_n) y_n + \beta_n \left\{ y_n - g(y_n) \right. \\
&\quad \left. + J_{A(y_n)} \left[g(\frac{y_n + \omega_n + \mu_n}{3}) - \rho F(\mathcal{T}(\frac{2w(y_n + z_n + \mu_n)}{3w + y_n + z_n + \mu_n}), \nu) \right] \right\}, \\
\mu_{n+1} &= (1-\alpha_n) z_n + \alpha_n \left\{ z_n - g(z_n) \right. \\
&\quad \left. + J_{A(z_n)} \left[g(\frac{z_n + y_n + z_n + \mu_n}{4}) - \rho F(\mathcal{T}(\frac{2w(y_n + z_n + t_n + \mu_n)}{4w + y_n + z_n + t_n + \mu_n}), \nu) \right] \right\},
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0, 1]$, $\quad \forall n \geq 1$.

Remark 3.1. *For different and suitable choice of the parameters ρ, η, α, operators $g, \mathcal{T}$, bifunctions $F(.,.)$, $\quad A(.,.)$ and convex-valued sets, one can recover new and known iterative methods for solving quasi harmonic-like equiblrium inclusions, harmonic-like complementarity problems and related optimization problems. Using the technique and ideas of Theorem 3.1 and Theorem 3.2, one can analyze the convergence of Algorithm 3.12 and its special cases.*

.

4. Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the harmonic-like quasi bifunction equilibrium inclusions. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [18]. It is worth mentioning that the dynamical systems are the initial value and boundary value problems. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems, see [6, 18, 30, 40, 41, 46, 47, 52, 61–63, 73]]. We consider some new iterative methods for solving the harmonic-like quasi bifunction equilibrium inclusions. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] - g(\mu)\}. \tag{4.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (2.3), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{4.2}$$

We now consider a dynamical system associated with the harmonic-like quasi bifunction equilibrium inclusions. Using this equivalent formulation (3.1), we suggest a class of resolvent dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{4.3}$$

where λ is a parameter. The system of type (4.3) is called the resolvent dynamical system associated with the problem (2.3). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.3) can be studied.
The equilibrium point of the dynamical system (4.14) is defined as follows.

Definition 4.1. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.14), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi bifunction equilibrium inclusion (2.3), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. This implies that $\mu \in \mathcal{H}$ is a solution of the quasi harmonic-like variational inclusion (2.3), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 4.2. *[15] The dynamical system is said to converge to the solution set S^* of (4.3), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{4.4}$$

where

$$dist(\mu, S^*) = inf_{\nu \in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (4.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 4.3. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 4.1. *(Gronwall Lemma) [15] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the resolvent dynamical system (4.3) converges to the unique solution of the harmonic-like quasi bifunction equilibrium inclusions (2.3).

Theorem 4.1. *Let the bifunction $F(.,.)$ be joint Lipschitz continuous with constant β and the operator $g : H \longrightarrow H$ be Lipschitz continuous with constant $\zeta > 0$ respectively. If $\lambda(\eta + \xi + \zeta + \rho\beta) < 1$ and Assumption 1 then, for each $\mu_0 \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (4.3) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] - g(\mu)\}, \quad \forall \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.
$\forall \mu, \eta \in H$, we have

$$\begin{aligned}
\|G(\mu) - G(\eta)\| &\leq \lambda\{\|J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] \\
&\quad - J_{A(\nu)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)]\|\} + \lambda\|g(\mu) - g(\eta)\| \\
&= \lambda\{\|g(\mu) - g(\eta)\| + \|J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] \\
&\quad - J_{A(\mu)}[g(\eta) - \rho\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)]\| \\
&\quad + \|J_{A(\mu)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)] - J_{A(\nu)}[g(\eta) - \rho F(\mathcal{T}(\frac{2w\eta}{w+\eta}), \nu)]\|\} \\
&\leq \lambda\{\|g(\mu) - g(\eta)\| + \xi\|\mu - \eta\|
\end{aligned}$$

$$
\begin{aligned}
& +\|g(\mu)-g(\eta)-\rho(F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)-F(\mathcal{T}(\frac{2w\eta}{w+\eta}),\nu))\|\} \\
\leq\ & \lambda\{\|g(\mu)-g(\eta)\|+\xi\|\mu-\eta\| \\
& +\{\|g(\mu)-g(\eta)\|+\rho\|F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)-F(\mathcal{T}(\frac{2w\eta}{w+\eta}),nu)\|\} \\
\leq\ & \lambda\{(\eta+\xi+\zeta+\beta\rho)\}\|\mu-\eta\|.
\end{aligned}
$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\eta+\xi+\zeta+\rho\beta)\}<1$ and for each $\mu\in\mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (4.3), defined on an interval $t_0\leq t<T_1$ with the initial condition $\mu(t_0)=\mu_0$. Let $[t_0,T_1)$ be its maximal interval of existence. Then we have to show that $T_1=\infty$. Consider , for any $\mu\in\Omega(\mu)$,

$$
\begin{aligned}
& \|G(\mu)\|=\|\frac{d\mu}{dt}\|=\lambda\|[g(\mu)-\rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)]-g(\mu)\| \\
\leq\ & \lambda\{\|J_{A(\mu)}[g(\mu)-\rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)]-J_{A(\mu)}[0]\|+\|J_{A(\mu)}[0]-g(\mu)\|\} \\
\leq\ & \lambda\{\delta\|\{g(\mu)-\rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}(\nu)\|+\|J_{A(\mu)}[g(\mu)]-J_{A(\mu)}[0]\|+\|J_{A(\mu)}[0]-g(u)\|\} \\
\leq\ & \lambda\{(\rho\beta+2+2\zeta)\|u\|+\|J_{A(\mu)}[0]\|\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\|+\int_{t_0}^{t}\|\mu(s)\|ds \\
&\leq (\|\mu_0\|+k_1(t-t_0))+k_2\int_{t_0}^{t}\|\mu(s)\|ds,
\end{aligned}
$$

where $k_1=\lambda\|J_{A(\mu)}[0]\|$ and $k_2=\delta\lambda(\rho\beta+2+2\zeta)$. Hence by the Gronwall Lemma 4.1, we have

$$
\|\mu(t)\|\leq\{\|u_0\|+k_1(t-t_0)\}e^{k_2(t-t_0)},\quad t\in[t_0,T_1).
$$

This shows that the solution is bounded on $[t_0,T_1)$. So $T_1=\infty$. □

Theorem 4.2. *If the operator $g:\mathcal{H}\longrightarrow\mathcal{H}$ is strongly monotone with constant $\sigma>0,\zeta>0$ and the bifunction $F(.,.)$ is jointly Lipschit continous with constan β, then the dynamical system (4.3) converges globally exponentially to the unique solution of the general harmonic-like quasi bifunction equilibrium inclusions. (2.3).*

Proof. Since the operator g is Lipschitz continuous and the bifunction $F(.,.)$ is jointly Lipschit continuous, it follows from Theorem 4.1 that the dynamical system (4.3) has unique solution $\mu(t)$ over $[t_0,T_1)$ for any fixed $\mu_0\in H$. Let $\mu(t)$ be a solution of the initial value problem (4.3). For a given $\mu^*\in H$ satisfying (2.3), consider the Lyapunov function

$$
L(\mu)=\lambda\|\mu(t)-\mu^*\|^2,\quad u(t)\in\mathcal{H}. \tag{4.5}
$$

From (4.3) and (4.5), we have

$$
\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle \mu(t)-\mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle \mu(t)-\mu^*, J_{A(\mu)}[g(\mu(t))-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu))]-g(\mu(t))\rangle \\
&= 2\lambda\langle \mu(t)-\mu^*, J_{A(\mu)}[g(\mu(t))-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu)]-g(\mu^*)+g(\mu^*)-g(\mu(t))\rangle \\
&= -2\lambda\langle \mu(t)-\mu^*, g(\mu(t))-g(\mu^*)\rangle \\
&\quad +2\lambda\langle \mu(t)-\mu^*, J_{A(\mu)}[g(\mu(t))-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t})),\nu))]-g(\mu^*)\rangle \\
&\leq -2\lambda\langle \rho(F(\mathcal{T}(\frac{2w\mu(t}{w+\mu(t}),\nu))-F(\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)}),\nu), g(\mu(t))-g(\mu^*)\rangle \\
&\quad +2\lambda\langle \mu(t)-\mu^*(t), J_{A(\mu)}[g(\mu(t))-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu)]-J_{A(\mu)}[g(\mu^*(t))-\rho\mu^*(t)]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t)-\mu^*\|^2+\lambda\|g(\mu(t))-g(\mu^*)\|^2 \\
&\quad +\lambda\|J_{A(\mu)}[\mu(t)-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu)] \\
&\qquad -J_{A(\mu)}[g(\mu^*(t))-\rho F(\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)}),\nu)\|^2 \qquad (4.6)
\end{aligned}
$$

Using the Lipschitz continuity of the operator g, and jointly Lipchitz cintinuity of the bifuncton $F(.,.)$, we have

$$
\begin{aligned}
&\|J_{A(\mu)}[g(\mu(t))-\rho F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu)]-J_{A(\mu)}[g(\mu^*(t))-\rho F(\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)}),\nu)]\| \\
&\leq \delta\|g(\mu(t))-g(\mu^*(t))-\rho(F(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)}),\nu)-F(\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)}),\nu))\| \\
&\leq \delta(\zeta+\rho\beta)\|\mu(t)-\mu^*(t)\|. \qquad (4.7)
\end{aligned}
$$

From (4.6) and (4.7), we have

$$\frac{d}{dt}\|\mu(t)-\mu^*(t)\| \leq 2\xi\lambda\|\mu(t)-\mu^*(t)\|,$$

where

$$\xi = (\delta(1+\rho\beta)-2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t)-\mu^*\| \leq \|\mu(t_0)-\mu^*\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (4.3) converges globally exponentially to the unique solution of the harmonic-like quasi bifunction equilibrium inclusions (2.3). □

We use the dynamical system (4.3) to suggest some iterative for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3). These methods can be viewed in the sense of Korpelevich [38] and Noor [50,56] involving the double resolvent.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (4.3) becomes

$$\frac{d\mu}{dt} + g(\mu) = J_{A(\mu)}\Big[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}),\nu)\Big], \quad \mu(t_0) = \alpha. \tag{4.8}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (4.8), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)], \tag{4.9}$$

where h is the step size.
Now, we can suggest the following implicit iterative method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3).

Algorithm 4.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}\Big[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu) - \frac{\mu_{n+1} - \mu_n}{h}\Big],$$

This is an implicit method. Algorithm 4.1 is equivalent to the following two-step method.

Algorithm 4.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}),\nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(\omega_n)}\Big[g(\mu_n) - \rho F(\mathcal{T}(\frac{2wy_n}{w+y_n})\nu) - \frac{y_n - \mu_n}{h}\Big], \end{aligned}$$

Discretizing (4.8), we now suggest an other implicit iterative method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3).

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = J_{A(\mu_{n+1})}[g(\mu_{n+1}) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)], \tag{4.10}$$

where h is the step size.
This formulation enables us to suggest the two-step iterative method.

Algorithm 4.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(y_n)}\Big[g(\omega_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n}) - \frac{y_n - \mu_n}{h}\Big]. \end{aligned}$$

Discretizing (4.8), we have

$$\frac{\mu_{n+1} - \mu_n}{h} = \mu_n - g(\mu_n) + J_{A(\mu_{n+1})}[g(\mu_{n+1}) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)], \tag{4.11}$$

where h is the step size.

This helps us to suggest the following implicit iterative method for solving the problem (2.3).

Algorithm 4.4. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A(y_n)}\Big[g(y_n) - \rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}), \nu)\Big]. \end{aligned}$$

Discretizing (4.8), we propose another implicit iterative method.

$$\begin{aligned} &\frac{\mu_{n+1}-\mu_n}{h} + g(\mu_n) \\ &\quad = J_{A(\mu_{n+1})}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu)] \end{aligned}$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.3).

Algorithm 4.5. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \mu_{n+1} &= \mu_n - g(\mu_n) \\ &\quad + J_{A(\mu_{n+1})}[g(\mu_n) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu)]. \end{aligned}$$

From (4.8), we have

$$\begin{aligned} &\frac{d\mu}{dt} + g(\mu) \\ &\quad = J_{A((1-\alpha)\mu+\alpha\mu)}[g((1-\alpha)\mu + \alpha\mu) \\ &\qquad - \rho F(\mathcal{T}(\frac{2w(1-\alpha)\mu+\alpha\mu)}{w+(1-\alpha)\mu+\alpha\mu}), \nu)], \end{aligned} \tag{4.12}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (4.12) and taking $h = 1$, we have

$$\begin{aligned} \mu_{n+1} &= \mu_n - g(\mu_n) + J_{A((1-\alpha)\mu_n+\alpha\mu_{n-1})}\Big[g\Big((1-\alpha)\mu_n \\ &\quad + \alpha\mu_{n-1}\Big) - \rho F(\mathcal{T}(\frac{2w((1-\alpha)\mu_n+\alpha\mu_{n-1})}{w+(1-\alpha)\mu_n+\alpha\mu_{n-1}}), \nu)\Big], \end{aligned} \tag{4.13}$$

which is an inertial type iterative method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3). Using the predictor-corrector techniques, we have

Algorithm 4.6. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_{Ay_n}\Big[g(y_n) - \rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}), \nu)\Big], \end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the harmonic-like quasi bifunction equilibrium inclusions (2.3). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\{J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)] - g(\mu)\}, \tag{4.14}$$
$$\mu(a) = \alpha, \mu(b) = \beta,$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (4.14) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (4.14) is defined as follows.

Definition 4.4. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.14), if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi bifunction equilibrium inclusions (2.3), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
From (4.14), we have

$$g(\mu) = J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)].$$

Thus, we can rewrite (4.14) as follows:

$$g(\mu) = J_{A(\mu)}\Big[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\Big]. \tag{4.15}$$

For $\lambda = 1$, the problem (4.14) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\begin{aligned} &\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} + g(\mu) \\ &\quad = J_{A(\mu)}\Big[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)\Big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \end{aligned} \tag{4.16}$$

The problem (4.16) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the harmonic-like quasi bifunction equilibrium inclusions . Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the harmonic-like quasi bifunction equilibrium inclusions and related optimization problems.

We discretize the second-order dynamical systems (4.16) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}$$
$$+g(\mu_n)=J_{A(\mu_n)}[\mu_n-\rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)], \tag{4.17}$$

where h is the step size.

If $\gamma=1, h=1$, then, from equation(4.17) we have

Algorithm 4.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n+g(\mu_n)+J_{A(\mu_n)}[g(\mu_n)-\rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)],$$

which is the extragradient method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3). Algorithm 4.7 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 4.8. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n-g(\mu_n)+J_{A(\mu_n)}[g(\mu_n)-\rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}),\nu)], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n\in[0,1]$ is a constant.

In a similar way, we have the following two-step method.

Algorithm 4.9. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n-g(\mu_n)+J_{A(y_n)}[g(y_n)-\rho F(\mathcal{T}(\frac{2wy_n}{w+y_n}),\nu)], \end{aligned}$$

which is also called the double resolvent method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3).

We discretize the second-order dynamical systems (4.3) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+g(\mu_{n+1})$$
$$=J_{A(\mu_{n+1})}\Big[g(\mu_n)-\rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)\Big],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3).

Algorithm 4.10. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) \\ +J_{A(\mu_n)}\Big[g(\mu_{n+1}) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu) - \gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h}\Big].$$

Algorithm 4.10 is called the hybrid inertial proximal method for solving the quasi harmonic-like quasi bifunction equilibrium inclusions and related optimization problems. This is a new proposed method.

Note that, for $\gamma = 0$, Algorithm 4.10 reduces to the following iterative method.

Algorithm 4.11. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) \\ +J_{A(\mu_{n+1})}\Big[g(\mu_{n+1}) - \rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu) + \frac{\mu_n - \mu_{n-1}}{h}\Big],$$

which is called the inertial double resolvent method.

We now consider the third order dynamical systems associated with the harmonic-like quasi bifunction equilibrium inclusions of the type (2.3). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} + g(\mu) = J_{A(\mu)}[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu)], \tag{4.18}$$
$$u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = \beta_1,$$

where $\gamma > 0, \zeta, \xi, \beta, \alpha, \beta_1$ and $\rho > 0$ are constants. Problem (4.18) is called third order dynamical system associated with harmonic-like quasi bifunction equilibrium inclusions (2.3).

The equilibrium point of the dynamical system (4.18) is defined as follows.

Definition 4.5. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.14), if,*

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi bifunction equilibrium inclusions (2.3), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (4.3) can be written as

$$g(\mu) = J_{A(\mu)}\Big[g(\mu) - \rho F(\mathcal{T}(\frac{2w\mu}{w+\mu}), \nu) + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\Big]. \tag{4.19}$$

We discretize the third-order dynamical systems (4.18) using central finite difference and backward difference schemes to have

$$\begin{aligned}
&\gamma\frac{\mu_{n+2}-2\mu_{n+1}+2\mu_{n-1}-\mu_{n-2}}{2h^3}+\zeta\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}\\
&\quad+\xi\frac{3\mu_n-4\mu_{n-1}+\mu_{n-2}}{2h}\\
&\quad+g(\mu_n)=J_{A(\mu_n)}[g(\mu_n)-\rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)], \qquad (4.20)
\end{aligned}$$

where h is the step size.
If $\gamma=1, h=1, \zeta=1, \xi=1$, then, from equation(4.20) after adjustment, we have

Algorithm 4.12. *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$u_{n+1} \quad = \quad \mu_n - g(\mu_n)+\Pi_{\Omega(\mu_n)}\Big[g(\mu_n)-\rho F(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}),\nu)+\frac{\mu_{n-1}-3\mu_n}{2}\Big],$$

which is an inertial type hybrid iterative methods for solving the harmonic-like quasi bifunction equilibrium inclusions (2.3).

Remark 4.1. *For appropriate and suitable choice of the operators $\mathcal{T}$, g, bifunction $F(.,.)$,convex-valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving harmonic-like quasi bifunction equilibrium inclusions and related optimization problems.*

5. Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behaviour of such problems as a result of changes in the problem data is always of concern. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving the problems due to these and other reasons. Dafermos [17] studied the sensitivity analysis of the variational inequalities uisng the fixed point approach. These concepts have been stuided using quiate different techniaues for variational inequalitied, equilibrium and optimization problems. For recent actvities, see [17, 40, 44, 47, 55, 56, 62] and the references therein. In this section, we study the sensitivity analysis of the inverse quasi variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed.
We now consider the parametric versions of the problem (2.3). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $g(\mu,\lambda)$ be given identity operator

defined on $\mathcal{H}\times\mathcal{H}\times M$ and take value in $\mathcal{H}\times\mathcal{H}$. From now onward, we denote $g_\lambda(.)\equiv g(.,\lambda)$ and $\mathcal{T}_\lambda(.)\equiv\mathcal{T}(.,\lambda)$, respectively, unless otherwise specified.

The parametric harmonic-like variational inequality problem is to find $(\mu,\lambda)\in\mathcal{H}\times M$ such that

$$\langle\rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu}{w+\mu}),\nu))+g_\lambda(\mu)-g_\lambda(\mu),\nu-g_\lambda(\mu)\rangle\geq 0,\quad\forall w,\nu\in\Omega(\mu).\tag{5.1}$$

We also assume that, for some $\overline{\lambda}\in M$, problem (5.1) has a unique solution $\overline{\mu}$. From Lemma 3.1, we see that the parametric harmonic-like quasi bifunction equilibrium inclusions

$$g_\lambda(\mu)=\Pi_{\Omega(\mu)}[g_\lambda(\mu)-\rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu}{w+\mu}),\nu))],$$

or equivalently

$$\mu=\mu-g_\lambda(\mu)+\Pi_{\Omega(\mu)}[g_\lambda(\mu)-\rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu}{w+\mu}),\nu))].$$

We now define the mapping Φ_λ associated with the problem (5.1) as

$$\begin{aligned}\Phi_\lambda(\mu)&=\mu-g_\lambda(\mu)\\&+\Pi_{\Omega(\mu)}[g_\lambda(\mu)-\rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu}{w+\mu}),\nu))],\quad\forall(\mu,\lambda)\in X\times M.\end{aligned}\tag{5.2}$$

We use this equivalence to study the sensitivity analysis of the harmonic-like quasi bifunction equilibrium inclusion. We assume that for some $\overline{\lambda}\in M$, problem (5.1) has a solution $\overline{\mu}$ and X is a closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (5.1) has a unique solution $z(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 5.1. *Let $I_\lambda(.)$ be an operator on $X\times M$. Then, the bifunction $F_\lambda(.)$ with respecto the operator $\mathcal{T}_\lambda$ is said to :*

(a) Locally strongly jointly monotone with constant $\sigma>0$, if

$$\langle F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu}{w+\mu}),\nu))-F_\lambda(\mathcal{T}_\lambda(\frac{2w\eta}{w+\eta}),\nu)),\mu-\eta\rangle\geq\sigma\|\mu-\eta\|^2,\quad\forall\lambda\in\eta,w,\mu,\nu\in X.$$

(b) jointly Locally Lipschitz continuous with constant $\zeta>0$, if

$$\|F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu)}{w+\mu}),\nu))-F_\lambda(\mathcal{T}_\lambda(\frac{2w\eta}{w+\eta}),\nu)))\|\leq\zeta\|\mu-\nu\|,\quad\forall\lambda\in w,\mu,\nu\in X.$$

We consider the case, when the solutions of the parametric harmonic-like quasi bifunction equilibrium inclusion (5.1) lie in the interior of X. Following the ideas of Dafermos [17] and Noor et al. [44], we consider the map $F_\lambda(\mu)$ as defined by (5.2). We have to show that the map $\Phi_\lambda(\mu)$ has a fixed point, which is a solution of the parametric harmonic-like quasi bifunction equilibrium inclusion (5.1). First of all, we prove that the map $\Phi_\lambda(\mu)$, defined by (5.2), is a contraction map with respect to μ uniformly in $\lambda\in M$.

Lemma 5.1. *Let the operator $g_\lambda(.,.)$ be a locally strongly monotone with constants $\sigma > 0$ and locally Lipschitz continuous with constants $\zeta > 0$ respectively. If Assumption 1 holds and the bifunction $F_\lambda(.,.)$ is Locally Lipchitz continous with constant $\beta > 0$, we have*

$$\|\Phi_\lambda(\mu_1) - \Phi_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

for

$$\rho < \frac{1-k}{\beta} \quad k < 1, \tag{5.3}$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta + \rho\beta\right\} = \{k + \rho\beta\} \tag{5.4}$$

and

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta. \tag{5.5}$$

Proof. In order to prove the existence of a solution of (5.1), it is enough to show that the mapping $\Phi_\lambda(\mu)$, defined by (5.2), is a contraction mapping.
For $\mu_1 \neq \mu_2 \in \mathcal{H}$, and using Assumption 1, we have

$$\begin{aligned}
&\|\Phi_\lambda(\mu_1) - \Phi_\lambda(\mu_2)\| \leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&+\|J_{A(\mu_1)}[g_\lambda(\mu_1) - \rho F(\mathcal{T}_\lambda(\frac{2w\mu_1}{w+\mu_1}), \nu))] - J_{A(u_2)}[g_\lambda(\mu_2) - \rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu_2}{w+\mu_2}), \nu))]\| \\
&+\|J_{A(u_1)}[g_\lambda(\mu_1) - \rho F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu_1}{w+\mu_1}), \nu))] - J_{A(u_2)}[g_\lambda(\mu_1) - \rho F_\lambda \mathcal{T}_\lambda(\frac{2w\mu_1}{w+\mu_1}), \nu))]\| \\
\leq\ & \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&+\eta\|\mu_1 - \mu_2\| + \|g_\lambda(\mu_) - g_\lambda(\mu_2) - \rho(F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu_1}{w+\mu_1}, \nu))) - F_\lambda \mathcal{T}_\lambda(\frac{2w\mu_2}{w+\mu_2}), \nu))\| \\
\leq\ & \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + \rho\|F_(\mathcal{T}_\lambda(\frac{2w\mu_1}{w+\mu_1}), \nu)) - F_\lambda(\mathcal{T}_\lambda(\frac{2w\mu_2}{w+\mu_2}), \nu))\| \\
&+\|g_\lambda(\mu_1) - g_\lambda(\mu_2)\| + \rho\beta\|\mu_1 - \mu_2)\|.
\end{aligned} \tag{5.6}$$

Since the operator g is a locally strongly monotone with constant $\sigma > 0$ and locally Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2)\|^2 &\leq \|u_1 - u_2\|^2 - 2\langle g_\lambda(\mu_1) - g_\lambda(\mu_2), \mu_1 - \mu_2\rangle \\
&\quad +\|g_\lambda(\mu_1) - g_\lambda(\mu_2)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu_1 - \mu_2\|^2.
\end{aligned} \tag{5.7}$$

From (5.5), (5.6), (5.7) and using the locally Lipschitz continuity of the operator g_λ, we have

$$\begin{aligned}
\|\Phi_\lambda(\mu_1) - \Phi_\lambda(\mu_2)\| &\leq \left\{\eta + \zeta + \sqrt{(1 - 2\sigma + \zeta^2)} + \rho\beta\right\}\|\mu_1 - \mu_2\| \\
&= \theta\|\mu_1 - \mu_2\|,
\end{aligned}$$

where

$$\theta = k + \rho\beta.$$

From (5.3), it follows that $\theta < 1$. Thus it follows that the mapping $\Phi_\lambda(\mu)$, defined by (5.2), is a contraction mapping and consequently it has a fixed point, which belongs to $\mathcal{H}$ satisfying the harmonic-like quasi bifunction equilibrium inclusion. (5.1), the required result. □

Remark 5.1. *From Lemma 3.1, we see that the map $\Phi_\lambda(\mu)$ defined by (5.2) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = \Phi_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric harmonic-like quasi bifunction equilibrium inclusion. (5.1). Again using Lemma 3.1, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $\Phi_\lambda(\mu)$ and it is also a fixed point of $\Phi_{\overline{\lambda}}(\mu)$. Consequently, we conclude that*

$$\mu(\overline{\lambda}) = \overline{\mu} = \Phi_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric harmonic-like quasi bifunction equilibrium inclusion (5.1) using the technique of Noor [54].

Lemma 5.2. *Assume that all the assumptions of Lemma 5.1 hold, then the function $u(\lambda)$ satisfying (5.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

We now state and prove the main result of this paper and is the motivation of our next result.

Theorem 5.1. *Let $\overline{\mu}$ be the solution of the parametric harmonic-like quasi bifunction equilibrium inclusion(5.1) for $\lambda = \overline{\lambda}$. Let $\mathcal{T}_\lambda, g_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operator for all $\mu, \nu \in X$. If the map $\lambda \rightarrow J_{A_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric harmonic-like quasi bifunction equilibrium inclusion (5.2) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma5.1, Lemma 5.2 and Remark 5.1. □

Conclusion. Some new classes of harmonic-like quasi bifunction equilibrium inclusions are introduced and studied. Resolvent operator technique is used to establish the equivalence between the bifunction equlibrium inclusion and the fixed poitn ptoblems. This alernative formulation is used to suggest some new multi step multi-step iterative methods for solving the quasi harmonic-like quasi bifunction equilibrium inclusions using the techniques of resolvent method, auxiliary techniques and dynamical systems. Convergence analysis of the proposed method is discussed under suitable weaker conditions. These new multistep methods can be viewed as important generalization of the well known iterations [38,40]. It is an open problem to compare these proposed methods with other methods. Sensitivity analysis is also investigated for harmonic-like quasi bifunction equilibrium inclusions using the equivalent fixed point approach. We have only considered the theoretical aspects of these multistep steps and their specail cases. Applying the technique and ideas of Ashish et. al. [3,4], Cho et al. [14], Kwuni et al. [26] and Negi et al. [33], one can explore the applications of Julia set and Mandelbrot set in Noor orbit using these new iterative methods in various fields of pure and applied sciences. It is well known that the applications of the fuzzy set theory, stochastic control [7, 72], quantum calculus, fractal, logistic map [75], fractional and random traffic equilibrium [12], Secure data transmission through fractal-based cryptosystem [33] and traffic assignment [72] can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research, green energy [32], machine larning, data science, artificial intelligence and variational inequalities. Comparison of these multistep methods with other techniques and development of efficient implementable numerical methods for solving quasi equilibrium inclusions are the open problems.

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