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BEST APPROXIMATION OF THE DUNKL MULTIPLIER OPERATORS $T_{k,\ell,m}$

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Abstract. We study some class of Dunkl multiplier operators $T_{k,\ell,m}$; and we give for them an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,\ell,m}$ on a Hilbert spaces $H^s_{k\ell}$.

1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha x := x - \frac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k, \ell : \Re \to \mathbb{C}$ be two multiplicity functions on $\Re$ (a functions which are constants on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$ and $\gamma_\ell := \sum_{\alpha\in\Re_+} \ell(\alpha)$.

Throughout this paper, we will assume that $k(\alpha), \ell(\alpha) \geq 0$ for all $\alpha \in \Re$, and $\gamma_\ell \geq \gamma_k$. Moreover, let w_k denote the weight function $w_k(x) := \prod_{\alpha\in\Re_+} |\langle \alpha, x\rangle|^{2k(\alpha)}$, for all $x \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree $2\gamma_k$.

Let c_k be the Mehta-type constant given by

$$c_k := \left(\int_{\mathbb{R}^d} e^{-|x|^2/2} w_k(x) \mathrm{d}x\right)^{-1}.$$

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We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(x) := c_k w_k(x)\mathrm{d}x$; and by $L^p(\mu_k)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}\mu_k(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [2]) by

$$\mathcal{F}_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x)\mathrm{d}\mu_k(x), \quad y \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Let $s > 0$. We consider the Hilbert $H^s_{k\ell}$ consisting of functions $f \in L^2(\mu_\ell)$ such that $e^{s|z|^2/2}\mathcal{F}_\ell(f) \in L^2(\mu_k)$. The space $H^s_{k\ell}$ is endowed with the inner product

$$\langle f, g\rangle_{H^s_{k\ell}} := \int_{\mathbb{R}^d} e^{s|z|^2} \mathcal{F}_\ell(f)(z)\overline{\mathcal{F}_\ell(g)(z)}\mathrm{d}\mu_k(z).$$

Let m be a function in $L^2(\mu_k)$. The Dunkl multiplier operators $T_{k,\ell,m}$, are defined for $f \in H^s_{k\ell}$ by

$$T_{k,\ell,m} f(x, a) := \mathcal{F}_k^{-1}(m(a.)\mathcal{F}_\ell(f))(x), \quad (x, a) \in \mathbb{K} := \mathbb{R}^d \times (0, \infty).$$

These operators are studied in [14] where the author established some applications (Calderón's reproducing formulas, best approximation formulas, extremal functions....). In particular, when $k = \ell$ these operators are studied in [13].

For $m \in L^2(\mu_k)$ satisfying the admissibility condition: $\int_0^\infty |m(ax)|^2 \frac{\mathrm{d}a}{a} = 1$, a.*e*. $x \in \mathbb{R}^d$, then the operators $T_{k,\ell,m}$ satisfy, for $f \in H^s_{k\ell}$:

$$\|T_{k,\ell,m} f\|^2_{L^2(\Omega_k)} = \|\mathcal{F}_\ell(f)\|^2_{L^2(\mu_k)},$$

where Ω_k is the measure on $\mathbb{K}$ given by $\mathrm{d}\Omega_k(x, a) := \frac{\mathrm{d}a}{a}\mathrm{d}\mu_k(x)$.

Building on the ideas of Matsuura et al. [5], Saitoh [9, 11] and Yamada et al. [18], and using the theory of reproducing kernels [8], we give best approximation of the operator $T_{k,\ell,m}$ on the Hilbert spaces $H^s_{k\ell}$. More precisely, for all $\lambda > 0$, $g \in L^2(\Omega_k)$, the infimum

$$\inf_{f\in H^s_{k\ell}} \left\{\lambda\|f\|^2_{H^s_{k\ell}} + \|g - T_{k,\ell,m} f\|^2_{L^2(\Omega_k)}\right\},$$

is attained at one function $f^*_{\lambda,g}$, called the extremal function, and given by

$$F^*_{\lambda,g}(y) = \int_{\mathbb{R}^d} \frac{E_\ell(iy, z)}{1 + \lambda e^{s|z|^2}} \left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(., b))(z)\frac{\mathrm{d}b}{b}\right] \mathrm{d}\mu_\ell(z).$$

Next we show for $F^*_{\lambda,g}$ the following properties.

(i) $\|F^*_{\lambda,g}\|_{H^s_{k\ell}} \le \dfrac{1}{2\sqrt{\lambda}}\|g\|_{L^2(\Omega_k)}$.

(ii) $T_{k,\ell,m}F^*_{\lambda,g}(y, a) = \displaystyle\int_{\mathbb{R}^d} \frac{m(az)E_k(iy, z)}{1 + \lambda e^{s|z|^2}} \left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(., b))(z)\frac{\mathrm{d}b}{b}\right] \mathrm{d}\mu_k(z)$.

In the Dunkl setting, the extremal functions are studied in several directions [12, 13, 14, 15, 16].

This paper is organized as follows. In section 2 we define and study the Dunkl multiplier operators $T_{k,\ell,m}$ on the Hilbert space $H^s_{k\ell}$. The last section of this paper is devoted to give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,\ell,m}$ on the Hilbert space $H^s_{k\ell}$.

2. Dunkl type multiplier operators

The Dunkl operators $\mathcal{D}_j$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \Re_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(., y)(x) = y_j u(x, y)$, $j = 1, ..., d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [1, 3]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [7]). In our case (see [1, 2]),

$$|E_k(ix, y)| \leq 1, \quad x, y \in \mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [2], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [3]. The Dunkl transform of a function f in $L^1(\mu_k)$, is defined by

$$\mathcal{F}_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x) \mathrm{d}\mu_k(x), \quad y \in \mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(y) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(x) \mathrm{d}x, \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [2, 3]).

Theorem 2.1 (i) $L^1 - L^\infty$-boundedness. *For all $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ and*

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}.$$

(ii) Inversion theorem. *Let $f \in L^1(\mu_k)$, such that $\mathcal{F}_k(f) \in L^1(\mu_k)$. Then*

$$f(x) = \mathcal{F}_k(\mathcal{F}_k(f))(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d.$$

(iii) Plancherel theorem. *The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,*

$$\|\mathcal{F}_k(f)\|_{L^2(\mu_k)} = \|f\|_{L^2(\mu_k)}.$$

Let $s > 0$. We define the Hilbert space $H^s_{k\ell}$, as the set of all $f \in L^2(\mu_\ell)$ such that $e^{s|z|^2/2}\mathcal{F}_\ell(f) \in L^2(\mu_k)$. The space $H^s_{k\ell}$ provided with the inner product

$$\langle f, g\rangle_{H^s_{k\ell}} := \int_{\mathbb{R}^d} e^{s|z|^2} \mathcal{F}_\ell(f)(z)\overline{\mathcal{F}_\ell(g)(z)} \mathrm{d}\mu_k(z),$$

and the norm $\|f\|_{H^s_{k\ell}} = \sqrt{\langle f, f\rangle_{H^s_{k\ell}}}$. The space $H^s_{k\ell}$ satisfies the following properties.

(i) The $H^s_{k\ell}$ has the reproducing kernel

$$h^s_{k\ell}(x, y) = \frac{c_\ell}{c_k} \int_{\mathbb{R}^d} e^{-s|z|^2} E_\ell(ix, z) E_\ell(-iy, z) w_{\ell-k}(z) \mathrm{d}\mu_\ell(z).$$

If $k=\ell$, then h^s_{kk} is the Dunkl-type heat kernel [6, 12] and this kernel is given by

$$h^s_{kk}(x,y)=\frac{1}{(2s)^{\gamma_k+d/2}}e^{-(|x|^2+|y|^2)/4s}E_k\Big(\frac{x}{\sqrt{2s}},\frac{y}{\sqrt{2s}}\Big).$$

(ii) The space $H^s_{k\ell}$ is continuously contained in $L^2(\mu_\ell)$ and

$$\|f\|^2_{L^2(\mu_\ell)}\le\frac{c_\ell}{c_k}\Big(\frac{2}{e}\Big)^{\gamma_\ell-\gamma_k}\Big(\frac{\gamma_\ell-\gamma_k}{s}\Big)^{\gamma_\ell-\gamma_k}\|f\|^2_{H^s_{k\ell}}.$$

(iii) If $f\in H^s_{k\ell}$ then $\mathcal{F}_\ell(f)\in L^1(\mu_\ell)$ and $\|\mathcal{F}_\ell(f)\|_{L^1(\mu_\ell)}\le C_{k,\ell}\|f\|_{H^s_{k\ell}}$, where

$$C_{k,\ell}=\Big(\frac{c_\ell}{c_k}\int_{\mathbb{R}^d}e^{-s|z|^2}w_{\ell-k}(z)\mathrm{d}\mu_\ell(z)\Big)^{1/2}. \tag{2.2}$$

(iv) If $f\in H^s_{k\ell}$, then $\mathcal{F}_\ell(f)\in L^1\cap L^2(\mu_\ell)$ and

$$f(x)=\int_{\mathbb{R}^d}E_\ell(ix,z)\mathcal{F}_\ell(f)(z)\mathrm{d}\mu_\ell(z),\quad \text{a.e. } x\in\mathbb{R}^d.$$

Let $\lambda>0$. We denote by $\langle.,.\rangle_{\lambda,H^s_{k\ell}}$ the inner product defined on the space $H^s_{k\ell}$ by

$$\langle f,g\rangle_{\lambda,H^s_{k\ell}}:=\lambda\langle f,g\rangle_{H^s_{k\ell}}+\langle\mathcal{F}_\ell(f),\mathcal{F}_\ell(g)\rangle_{L^2(\mu_k)}, \tag{2.3}$$

and the norm $\|f\|_{\lambda,H^s_{k\ell}}:=\sqrt{\langle f,f\rangle_{\lambda,H^s_{k\ell}}}$. On $H^s_{k\ell}$ the two norms $\|.\|_{H^s_{k\ell}}$ and $\|.\|_{\lambda,H^s_{k\ell}}$ are equivalent. This $(H^s_{k\ell},\langle.,.\rangle_{\lambda,H^s_{k\ell}})$ is a Hilbert space with reproducing kernel given by

$$K^s_{k\ell}(x,y)=\frac{c_\ell}{c_k}\int_{\mathbb{R}^d}\frac{E_\ell(ix,z)E_\ell(-iy,z)}{1+\lambda e^{s|z|^2}}w_{\ell-k}(z)\mathrm{d}\mu_\ell(z). \tag{2.4}$$

Let m be a function in $L^2(\mu_k)$. The Dunkl multiplier operators $T_{k,\ell,m}$, are defined for $f\in H^s_{k\ell}$ by

$$T_{k,\ell,m}f(x,a):=\mathcal{F}_k^{-1}(m(a.)\mathcal{F}_\ell(f))(x),\quad (x,a)\in\mathbb{K}. \tag{2.5}$$

We denote by Ω_k the measure on $\mathbb{K}$ given by $\mathrm{d}\Omega_k(x,a):=\frac{\mathrm{d}a}{a}\mathrm{d}\mu_k(x)$; and by $L^2(\Omega_k)$, the space of measurable functions F on $\mathbb{K}$, such that

$$\|F\|_{L^2(\Omega_k)}:=\Big(\int_{\mathbb{R}^d}\int_0^\infty|F(x,a)|^2\mathrm{d}\Omega_k(x,a)\Big)^{1/2}<\infty.$$

Let m be a function in $L^2(\mu_k)$ satisfying the admissibility condition

$$\int_0^\infty|m(ax)|^2\frac{\mathrm{d}a}{a}=1,\quad a.e.\ x\in\mathbb{R}^d. \tag{2.6}$$

Then from Theorem 2.1 (iii) , for $f\in H^s_{k\ell}$, we have

$$\|T_{k,\ell,m}f\|_{L^2(\Omega_k)}=\|\mathcal{F}_\ell(f)\|_{L^2(\mu_k)}\le\|f\|_{H^s_{k\ell}}. \tag{2.7}$$

3. Extremal functions for the operators $T_{k,\ell,m}$

In this section, by using the theory of extremal function and reproducing kernel of Hilbert space [8, 9, 10, 11] we study the extremal function associated to the Dunkl multiplier operators $T_{k,\ell,m}$. In the particular case when $k = \ell$ this function is studied in [16, 17]. The main result of this section can be stated as follows.

Theorem 3.1. *Let $m \in L^2(\mu_k)$ satisfying (2.6). For any $g \in L^2(\Omega_k)$ and for any $\lambda > 0$, there exists a unique function $F^*_{\lambda,g}$, where the infimum*

$$\inf_{f\in H^s_{k\ell}} \left\{ \lambda \|f\|^2_{H^s_{k\ell}} + \|g - T_{k,\ell,m} f\|^2_{L^2(\Omega_k)} \right\} \tag{3.1}$$

is attained. Moreover, the extremal function $F^*_{\lambda,g}$ is given by

$$F^*_{\lambda,g}(y) = \int_{\mathbb{R}^d} \int_0^\infty g(x,a) Q_s(x,y,a) \mathrm{d}\Omega_k(x,a),$$

where

$$Q_s(x,y,a) = \int_{\mathbb{R}^d} \frac{\overline{m(az)} E_k(-ix,z) E_\ell(iy,z)}{1 + \lambda e^{s|z|^2}} \mathrm{d}\mu_\ell(z).$$

Proof. Let $s, \lambda > 0$. Since $m \in L^2(\mu_k)$ and satisfying (2.6), then by (2.7), the inner product $\langle ., . \rangle_{\lambda, H^s_{k\ell}}$ defined by (2.3) is written by

$$\langle f, g \rangle_{\lambda, H^s_{k\ell}} = \lambda \langle f, g \rangle_{H^s_{k\ell}} + \langle T_{k,\ell,m} f, T_{k,\ell,m} g \rangle_{L^2(\Omega_k)}.$$

Then, the existence and unicity of the extremal function $F^*_{\lambda,g}$ satisfying (3.1) is obtained in [4, 5, 10]. Especially, $F^*_{\eta,g}$ is given by the reproducing kernel of $H^s_{k\ell}$ with $\|.\|_{\lambda, H^s_{k\ell}}$ norm as

$$F^*_{\lambda,g}(y) = \langle g, T_{k,\ell,m}(K^s_{k\ell}(., y)) \rangle_{L^2(\Omega_k)}, \tag{3.2}$$

where $K^s_{k\ell}$ is the kernel given by (2.4). Then, we obtain the result by Theorem 2.1 (ii) and the fact that

$$\mathcal{F}_\ell(K^s_{k\ell}(., y))(z) = \frac{c_\ell}{c_k} \frac{E_\ell(-iy, z)}{1 + \lambda e^{s|z|^2}} w_{\ell-k}(z), \quad z \in \mathbb{R}^d. \tag{3.3}$$

□

Theorem 3.2. *Let $\lambda > 0$ and $g \in L^2(\Omega_k)$. The extremal function $F^*_{\lambda,g}$ satisfies*

(i) $|F^*_{\lambda,g}(y)| \le \dfrac{C_{k,\ell}}{2\sqrt{\lambda}} \|g\|_{L^2(\Omega_k)}$,

where $C_{k,\ell}$ is the constant given by (2.2).

(ii) $\|F^*_{\lambda,g}\|^2_{L^2(\mu_\ell)} \le \dfrac{D_{k,\ell}}{\lambda} \|m\|^2_{L^2(\mu_k)} \displaystyle\int_{\mathbb{R}^d} \int_0^\infty |g(x,a)|^2 \frac{e^{(|x|^2+a^2)/2}}{a^{2\gamma_k+d+1}} \mathrm{d}\Omega_k(x,a)$,

where

$$D_{k,\ell} = \frac{c_k \sqrt{\pi}}{4 c_\ell \sqrt{2} a^{2\gamma_k+d}} \left(\frac{2}{e}\right)^{\gamma_\ell - \gamma_k} \left(\frac{\gamma_\ell - \gamma_k}{s}\right)^{\gamma_\ell - \gamma_k}.$$

Proof. (i) From (2.7) and (3.2), we have

$$\begin{aligned} |F^*_{\lambda,g}(y)| &\le \|g\|_{L^2(\Omega_k)} \|T_{k,\ell,m}(K^s_{k\ell}(., y))\|_{L^2(\Omega_k)} \\ &\le \|g\|_{L^2(\Omega_k)} \|\mathcal{F}_\ell(K^s_{k\ell}(., y))\|_{L^2(\mu_k)}. \end{aligned}$$

Then, by (3.3) we deduce

$$|F^*_{\lambda,g}(y)| \le \|g\|_{L^2(\Omega_k)} \left(\frac{c_\ell}{c_k} \int_{\mathbb{R}^d} \frac{w_{\ell-k}(z) \mathrm{d}\mu_\ell(z)}{[1 + \lambda e^{s|z|^2}]^2} \right)^{1/2}.$$

Using the fact that $[1+\lambda e^{s|z|^2}]^2 \geq 4\lambda e^{s|z|^2}$, we obtain the result.

(ii) We write

$$F^*_{\lambda,g}(y)=\int_{\mathbb{R}^d}\int_0^\infty \sqrt{a}e^{-(|x|^2+a^2)/4}\frac{e^{(|x|^2+a^2)/4}}{\sqrt{a}}g(x,a)Q_s(x,y,a)\mathrm{d}\Omega_k(x,a).$$

Applying Hölder's inequality, we obtain

$$|F^*_{\lambda,g}(y)|^2\leq\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^d}\int_0^\infty |g(x,a)|^2\frac{e^{(|x|^2+a^2)/2}}{a}|Q_s(x,y,a)|^2\mathrm{d}\Omega_k(x,a).$$

Thus and from Fubini-Tonnelli's theorem, we get

$$\|F^*_{\lambda,g}\|^2_{L^2(\mu_\ell)}\leq\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^d}\int_0^\infty |g(x,a)|^2\frac{e^{(|x|^2+a^2)/2}}{a}\|Q_s(x,.,a)\|^2_{L^2(\mu_\ell)}\mathrm{d}\Omega_k(x,a). \tag{3.4}$$

The function $z\to\frac{\overline{m(az)}E_k(-ix,z)}{1+\lambda e^{s|z|^2}}$ belongs to $L^1\cap L^2(\mu_\ell)$, then by Theorem 2.1 (ii),

$$Q_s(x,y,a)=\mathcal{F}_\ell^{-1}\Big(\frac{\overline{m(az)}E_k(-ix,z)}{1+\lambda e^{s|z|^2}}\Big)(y).$$

Thus, by Theorem 2.1 (iii) we deduce that

$$\|Q_s(x,.,a)\|^2_{L^2(\mu_\ell)}=\int_{\mathbb{R}^d}|\mathcal{F}_\ell(Q_s(x,.,a))(z)|^2\mathrm{d}\mu_\ell(z)\leq\int_{\mathbb{R}^d}\frac{|m(az)|^2\mathrm{d}\mu_\ell(z)}{[1+\lambda e^{s|z|^2}]^2}.$$

Then

$$\begin{aligned}\|Q(x,.,a)\|^2_{L^2(\mu_\ell)} &\leq \frac{c_k}{4\lambda c_\ell}\int_{\mathbb{R}^d}e^{-s|z|^2}|m(az)|^2w_{\ell-k}(z)\mathrm{d}\mu_k(z)\\ &\leq \frac{c_k}{4\lambda c_\ell a^{2\gamma_k+d}}\Big(\frac{2}{e}\Big)^{\gamma_\ell-\gamma_k}\Big(\frac{\gamma_\ell-\gamma_k}{s}\Big)^{\gamma_\ell-\gamma_k}\|m\|^2_{L^2(\mu_k)}.\end{aligned}$$

From this inequality we deduce the result. □

Theorem 3.3. *Let $s,\lambda>0$. For every $g\in L^2(\Omega_k)$, we have*

(i) $F^*_{\lambda,g}(y)=\displaystyle\int_{\mathbb{R}^d}\frac{E_\ell(iy,z)}{1+\lambda e^{s|z|^2}}\Big[\int_0^\infty\overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\Big]\mathrm{d}\mu_\ell(z)$.

(ii) $\mathcal{F}_\ell(F^*_{\lambda,g})(z)=\displaystyle\frac{1}{1+\lambda e^{s|z|^2}}\Big[\int_0^\infty\overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\Big]$.

(iii) $\|F^*_{\lambda,g}\|_{H^s_{k\ell}}\leq\displaystyle\frac{1}{2\sqrt{\lambda}}\|g\|_{L^2(\Omega_k)}$.

Proof. (i) From (3.2) we have

$$F^*_{\lambda,g}(y)=\int_{\mathbb{R}^d}\int_0^\infty g(x,b)\overline{T_{k,\ell,m}(K^s_{k\ell}(.,y))}(x,b)\mathrm{d}\Omega_k(x,b).$$

Since

$$\int_{\mathbb{R}^d}\int_0^\infty|g(x,b)\overline{T_{k,\ell,m}(K^s_{k\ell}(.,y))}(x,b)|\mathrm{d}\Omega_k(x,b)\leq\|g\|_{L^2(\Omega_k)}\|\mathcal{F}_\ell(K^s_{k\ell}(.,y))\|_{L^2(\mu_k)}<\infty,$$

then, by Fubini's theorem, Theorem 2.1 (iii) and (3.3) we obtain

$$\begin{aligned} F^*_{\lambda,g}(y) &= \int_0^\infty \int_{\mathbb{R}^d} g(x,b)\overline{T_{k,\ell,m}(K^s_{k\ell}(.,y))}(x,b)\mathrm{d}\mu_k(x)\frac{\mathrm{d}b}{b} \\ &= \int_0^\infty \int_{\mathbb{R}^d} \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\overline{\mathcal{F}_\ell(K^s_{k\ell}(.,y))}(z)\mathrm{d}\mu_k(z)\frac{\mathrm{d}b}{b} \\ &= \int_0^\infty \int_{\mathbb{R}^d} \frac{\overline{m(bz)}\mathcal{F}_k(g(.,b))(z)E_\ell(iy,z)}{1+\lambda e^{s|z|^2}}\mathrm{d}\mu_\ell(z)\frac{\mathrm{d}b}{b}. \end{aligned}$$

Since

$$\int_0^\infty \int_{\mathbb{R}^d} \Big|\frac{\overline{m(bz)}\mathcal{F}_k(g(.,b))(z)E_\ell(iy,z)}{1+\lambda e^{s|z|^2}}\Big|\mathrm{d}\mu_\ell(z)\frac{\mathrm{d}b}{b} \leq \frac{C_{k,\ell}}{2\sqrt{\lambda}}\|g\|_{L^2(\Omega_k)} < \infty,$$

then, by Fubini's theorem we deduce that

$$F^*_{\lambda,g}(y) = \int_{\mathbb{R}^d} \frac{E_\ell(iy,z)}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right]\mathrm{d}\mu_\ell(z).$$

(ii) The function $z \to \frac{1}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right]$ belongs to $L^1 \cap L^2(\mu_\ell)$. Then by Theorem 2.1 (ii) and (iii), it follows that $F^*_{\lambda,g}$ belongs to $L^2(\mu_\ell)$, and

$$\mathcal{F}_\ell(F^*_{\lambda,g})(z) = \frac{1}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right].$$

(iii) From (ii), Hölder's inequality and (2.6) we have

$$|\mathcal{F}_\ell(F^*_{\eta,g})(z)|^2 \leq \frac{1}{[1+\eta e^{s|z|^2}]^2}\left[\int_0^\infty |\mathcal{F}_k(g(.,b))(z)|^2\frac{\mathrm{d}b}{b}\right].$$

Thus,

$$\begin{aligned} \|F^*_{\lambda,g}\|^2_{H^s_{k\ell}} &\leq \int_{\mathbb{R}^d} \frac{e^{s|z|^2}}{[1+\lambda e^{s|z|^2}]^2}\left[\int_0^\infty |\mathcal{F}_k(g(.,b))(z)|^2\frac{\mathrm{d}b}{b}\right]\mathrm{d}\mu_k(z) \\ &\leq \frac{1}{4\lambda}\int_{\mathbb{R}^d}\left[\int_0^\infty |\mathcal{F}_k(g(.,b))(z)|^2\frac{\mathrm{d}b}{b}\right]\mathrm{d}\mu_k(z) = \frac{1}{4\lambda}\|g\|^2_{L^2(\Omega_k)}, \end{aligned}$$

which ends the proof. □

Theorem 3.4. *Let $s, \lambda > 0$. For every $g \in L^2(\Omega_k)$, we have*

$$T_{k,\ell,m}F^*_{\lambda,g}(y,a) = \int_{\mathbb{R}^d} \frac{m(az)E_k(iy,z)}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right]\mathrm{d}\mu_k(z).$$

Proof. From (2.5) and Theorem 3.3 (ii), we have

$$T_{k,\ell,m}F^*_{\lambda,g}(y,a) = \mathcal{F}_k^{-1}\left(\frac{m(az)}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right]\right)(y).$$

The function $z \to \frac{m(az)}{1+\lambda e^{s|z|^2}}\left[\int_0^\infty \overline{m(bz)}\mathcal{F}_k(g(.,b))(z)\frac{\mathrm{d}b}{b}\right]$ belongs to $L^1(\mu_k)$. Then by Theorem 2.1 (ii), we obtain the result. □

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