Int. J. Anal. Appl. (2026), 24:49

International Journal of Analysis and Applications

Almost Near $\tau^\star(\sigma_1, \sigma_2)$-Continuity for Multifunctions

Jeeranunt Khampakdee[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1]*Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

[2]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

[*]*Corresponding author:* chawalit.b@msu.ac.th

Abstract. This paper presents a new concept of continuous multifunctions defined between an ideal topological space and a bitopological space, called almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are established.

1. Introduction

The notion of almost continuous functions was introduced by Singal and Singal [33]. Popa [27] defined almost quasi-continuous functions as a generalization of almost continuity [33] and quasi-continuity [21]. Munshi and Bassan [22] studied the notion of almost semi-continuous functions. Maheshwari et al. [19] introduced the concept of almost feebly continuous functions as a generalization of almost continuity [33]. Malghan and Hanchinamani [20] introduced the concept of N-continuous functions. Noiri and Ergun [25] investigated some characterizations of N-continuous functions. On the other hand, the present authors introduced and investigated the concepts of (τ_1, τ_2)-continuuos functions [1], almost (τ_1, τ_2)-continuuos functions [2], weakly (τ_1, τ_2)-continuuos functions [3], almost quasi (τ_1, τ_2)-continuuos functions [17], weakly quasi (τ_1, τ_2)-continuuos functions [7], nearly (τ_1, τ_2)-continuous functions [8], almost nearly (τ_1, τ_2)-continuous functions [15], $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [13], almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [37], weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [37], almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [16] and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [16]. Ekici [11] introduced and studied

Received: Dec. 12, 2025.

2020 *Mathematics Subject Classification.* 54C08; 54C60.

Key words and phrases. $\star$-open set; $\mathscr{N}(\tau_1, \tau_2)$-closed set; almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction.

https://doi.org/10.28924/2291-8639-24-2026-49
ISSN: 2291-8639

the concept of nearly continuous multifunctions as a generalization of semi-continuous multifunctions and N-continuous functions. Moreover, Ekici [10] introduced and investigated the notion of almost nearly continuous multifunctions as a generalization of nearly continuous multifunctions and almost continuous multifunctions [26]. Noiri and Popa [24] introduced and studied the notion of almost nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. Carpintero et al. [6] introduced and investigated the notion of nearly ω-continuous multifunctions as a weaker form of nearly continuous multifunctions. On the other hand, the present authors introduced and studied the notions of upper (τ_1, τ_2)-continuous multifunctions [32], lower (τ_1, τ_2)-continuous multifunctions [32], upper almost (τ_1, τ_2)-continuous multifunctions [14], lower almost (τ_1, τ_2)-continuous multifunctions [14], upper weakly (τ_1, τ_2)-continuous multifunctions [36], lower weakly (τ_1, τ_2)-continuous multifunctions [36], upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [13], lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [13], upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [38], lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [38], upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [29] and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions [29]. Pue-on et al. [31] introduced and investigated the concepts of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions. Furthermore, Pue-on et al. [30] introduced and investigated the concepts of upper weakly quasi (τ_1, τ_2)-continuous multifunctions and lower weakly quasi (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [34] introduced and studied the notions of upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, Thongmoon et al. [35] presented new classes of continuous multifunctions defined from a bitopological space into a bitopological space, namely upper nearly (τ_1, τ_2)-continuous multifunctions and lower nearly (τ_1, τ_2)-continuous multifunctions. Quite recently, Chutiman et al. [9] introduced and studied the notions of upper almost nearly (τ_1, τ_2)-continuous multifunctions and lower almost nearly (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the concept of almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A).

The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [5] of A and is denoted by $\tau_1\tau_2\text{-Int}(A)$.

Lemma 2.1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.*
(2) *If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.*
(3) *$\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.*
(5) *$\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [40] (resp. *$(\tau_1, \tau_2)s$-open* [4], *$(\tau_1, \tau_2)p$-open* [4], *$(\tau_1, \tau_2)\beta$-open* [4]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called *$(\tau_1, \tau_2)r$-closed* (resp. *$(\tau_1, \tau_2)s$-closed*, *$(\tau_1, \tau_2)p$-closed*, *$(\tau_1, \tau_2)\beta$-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\alpha(\tau_1, \tau_2)$-open* [39] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be *$\alpha(\tau_1, \tau_2)$-closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\mathscr{N}(\tau_1, \tau_2)$-closed* [36] if every cover of A by $(\tau_1, \tau_2)r$-open sets of X has a finite subcover.

Lemma 2.2. [9] *Let (X, τ_1, τ_2) be a bitopological space. If V is a $\tau_1\tau_2$-open set of X having $\mathscr{N}(\tau_1, \tau_2)$-closed complement, then $\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(V))$ is a $(\tau_1, \tau_2)r$-open set having $\mathscr{N}(\tau_1, \tau_2)$-closed complement.*

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [18], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [12] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\tau^\star$-semi-open* [23] (resp. *$\tau^\star$-pre-open* [23]) if $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$). The complement of a $\tau^\star$-semi-open (resp. $\tau^\star$-pre-open) set is called *$\tau^\star$-semi-closed* (resp. $\tau^\star$-pre-closed).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notion of almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, some characterizations of almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 3.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open sets V', V'' of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $x \in F^+(V') \cap F^-(V'')$, there exists a $\star$-open set U of X containing x such that*

$$U \subseteq F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V'))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V''))).$$

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point of X.

Theorem 3.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. *F is almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *for each $x \in X$ and for every $(\sigma_1, \sigma_2)r$-open sets V', V'' of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V'$ and $F(x) \cap V'' \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $F(z) \subseteq V'$ and $F(z) \cap V'' \neq \emptyset$ for every $z \in U$;*
3. *for each $x \in X$ and for every $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed sets K', K'' of Y such that*
$$x \in F^+(Y - K') \cap F^-(Y - K''),$$
there exists a $\star$-closed set $H \neq X$ such that $x \in X - H$ and
$$F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K'))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K''))) \subseteq H;$$
4. *$F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V'))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V'')))$ is $\star$-open in X for every $\sigma_1\sigma_2$-open sets V', V'' of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
5. *$F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K'))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K'')))$ is $\star$-closed in X for every $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed sets K', K'' of Y;*
6. *$F^+(V') \cup F^-(V'')$ is $\star$-open in X for every $(\sigma_1, \sigma_2)r$-open sets V', V'' of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
7. *$F^-(K') \cap F^+(K'')$ is $\star$-closed in X for every $(\sigma_1, \sigma_2)r$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed sets K', K'' of Y.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and V', V'' be any $(\sigma_1, \sigma_2)r$-open sets of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement such that $F(x) \subseteq V'$ and $F(x) \cap V'' \neq \emptyset$. By (1), there exists a $\star$-open set U of X containing x such that

$$U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V''))) = F^+(V') \cap F^-(V'').$$

(2) $\Rightarrow$ (1): It is sufficient to observe that for any $(\sigma_1,\sigma_2)r$-open sets V', V'' of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement the sets $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V'))$ and $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V''))$ are $(\sigma_1,\sigma_2)r$-open in Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement.

(1) $\Rightarrow$ (3): Let $x \in X$ and K', K'' be any $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1,\sigma_2)$-closed sets of Y such that

$$x \in F^+(Y - K') \cap F^-(Y - K'').$$

Then, $Y - K'$ and $Y - K''$ are $\sigma_1\sigma_2$-open sets having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Since F is almost nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that

$$\begin{aligned} U &\subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K''))) \\ &= X - [F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'))) \cup F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'')))]. \end{aligned}$$

It is clear that $H = X - U$ is $\star$-closed in X and the conclusion

$$F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'))) \cup F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K''))) \subseteq H$$

is satisfied.

(3) $\Rightarrow$ (1): The proof is similar to the proof (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (4): Let V', V'' be any $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Let

$$x \in F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V''))).$$

Then, we have $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V'))$ and $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V''))$ are $\sigma_1\sigma_2$-open sets of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. By the definition of almost nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous at a point x, there exists a $\star$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'')))$. Since U is $\star$-open, we have $x \in \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V''))))$ and hence

$$\begin{aligned} &F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V''))) \\ &\subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'')))). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'')))$ is $\star$-open in X.

(4) $\Rightarrow$ (1): The proof is clear.

(4) $\Rightarrow$ (5): Let K', K'' be any $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1,\sigma_2)$-closed sets of Y. Then, we have $Y - K'$ and $Y - K''$ are $\sigma_1\sigma_2$-open sets having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. By (4),

$$\begin{aligned} &F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K''))) \\ &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'))) \cap F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K''))) \\ &= [X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K')))] \cap [X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'')))] \\ &= X - [F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'')))] \end{aligned}$$

is $\star$-open in X. Thus, $F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K'')))$ is $\star$-closed in X.

(5) $\Rightarrow$ (4): It can be obtained similarly as (4) $\Rightarrow$ (5).

(4) ⇒ (6): It is easily seen that the set

$$F^+(V') \cap F^+(V'') = F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V'')))$$

for every $(\sigma_1, \sigma_2)r$-open sets V', V'' of Y.

(6) ⇒ (4): The proof is a consequence of Lemma 2.2.

(6) ⇒ (7): Let K', K'' be any $(\sigma_1, \sigma_2)r$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed sets of Y. Then, we have $Y - K'$ and $Y - K''$ are $(\sigma_1, \sigma_2)r$-open sets having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. By (6), we have

$$F^+(Y - K') \cup F^-(Y - K'') = X - (F^-(K') \cap F^+(K''))$$

is $\star$-open in X and so $F^-(K') \cap F^+(K'')$ is $\star$-closed in X.

(7) ⇒ (6): It can be obtained similarly as (6) ⇒ (7). □

Definition 3.2. [28] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point of X.*

Corollary 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
(2) *for each $x \in X$ and for every $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$;*
(3) *for each $x \in X$ and for every $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y such that $x \in f^{-1}(Y - K)$, there exists a $\star$-closed set $H \neq X$ such that $x \in X - H$ and $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq H$;*
(4) *$f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\star$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
(5) *$f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$ is $\star$-closed in X for every $\sigma_1\sigma_2$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y;*
(6) *$f^{-1}(V)$ is $\star$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
(7) *$f^{-1}(K)$ is $\star$-closed in X for every $(\sigma_1, \sigma_2)r$-closed $\mathscr{N}(\sigma_1, \sigma_2)$-closed set K of Y.*

Acknowledgements: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] C. Boonpok, N. Srisarakham, (τ_1, τ_2)-Continuity for Functions, Asia Pac. J. Math. 11 (2024), 21. https://doi.org/10.28924/APJM/11-21.

[2] C. Boonpok, P. Pue-On, Characterizations of Almost (τ_1, τ_2)-Continuous Functions, Int. J. Anal. Appl. 22 (2024), 33. https://doi.org/10.28924/2291-8639-22-2024-33.

[3] C. Boonpok, C. Klanarong, On Weakly (τ_1, τ_2)-Continuous Functions, Eur. J. Pure Appl. Math. 17 (2024), 416–425. https://doi.org/10.29020/nybg.ejpam.v17i1.4976.

[4] C. Boonpok, $(\tau_1, \tau_2)\delta$-Semicontinuous Multifunctions, Heliyon, 6 (2020), e05367. https://doi.org/10.1016/j.heliyon.2020.e05367.

[5] C. Boonpok, C. Viriyapong, M. Thongmoon, on Upper and Lower (τ_1, τ_2)-Precontinuous Multifunctions, J. Math. Computer Sci. 18 (2018), 282–293. https://doi.org/10.22436/jmcs.018.03.04.

[6] C. Carpintero, J. Pacheco, N. Rajesh, E. Rosas, S. Saranyasri, Properties of Nearly ω-Continuous Multifunctions, Acta Univ. Sapientiae, Math. 9 (2017), 13–25. https://doi.org/10.1515/ausm-2017-0002.

[7] M. Chiangpradit, S. Sompong, C. Boonpok, Weakly Quasi (τ_1, τ_2)-Continuous Functions, Int. J. Anal. Appl. 22 (2024), 125. https://doi.org/10.28924/2291-8639-22-2024-125.

[8] N. Chutiman, S. Sompong, C. Boonpok, Characterizations of Nearly (τ_1, τ_2)-Continuous Functions, Asia Pac. J. Math. 12 (2025), 27. https://doi.org/10.28924/APJM/12-27.

[9] N. Chutiman, A. Sama-Ae, C. Boonpok, Almost Near (τ_1, τ_2)-Continuity for Multifunctions, Eur. J. Pure Appl. Math. 18 (2025), 5650. https://doi.org/10.29020/nybg.ejpam.v18i1.5650.

[10] E. Ekici, Almost Nearly Continuous Multifunctions, Acta Math. Univ. Comen. New Ser. 73 (2004), 175–186. https://eudml.org/doc/126719.

[11] E. Ekici, Nearly Continuous Multifunctions, Acta Math. Comenianae, 72 (2003), 229–235.

[12] D. Jankovic, T.R. Hamlet, New Topologies from Old via Ideals, Am. Math. Mon. 97 (1990), 295–310. https://doi.org/10.2307/2324512.

[13] J. Khampakdee, A. Sama-Ae, C. Boonpok, Upper and Lower Continuous Multifunctions Defined Between an Ideal Topological Space and a Bitopological Space, Eur. J. Pure Appl. Math. 18 (2025), 6565. https://doi.org/10.29020/nybg.ejpam.v18i3.6565.

[14] C. Klanarong, S. Sompong, C. Boonpok, Upper and Lower Almost (τ_1, τ_2)-Continuous Multifunctions, Eur. J. Pure Appl. Math. 17 (2024), 1244–1253. https://doi.org/10.29020/nybg.ejpam.v17i2.5192.

[15] B. Kong-ied, A. Sama-Ae and C. Boonpok, Almost Nearly (τ_1, τ_2)-Continuous Functions, Int. J. Anal. Appl. 23 (2025), 14. https://doi.org/10.28924/2291-8639-23-2025-14.

[16] B. Kong-ied, A. Sama-Ae, C. Boonpok, Almost Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous and Weakly Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions, Eur. J. Pure Appl. Math. 18 (2025), 6572. https://doi.org/10.29020/nybg.ejpam.v18i3.6572.

[17] B. Kong-ied, S. Sompong, C. Boonpok, Almost Quasi (τ_1, τ_2)-Continuous Functions, Asia Pac. J. Math. 11 (2024), 64. https://doi.org/10.28924/APJM/11-64.

[18] K. Kuratowski, Topology, Academic Press, 1966.

[19] S.N. Maheshwari, G.I. Chae, P.C. Jain, Almost Feebly Continuous Functions, Ulsan Inst. Tech. Rep. 13 (1982), 195–197.

[20] S.R. Malghan, V.V. Hanchinamani, N-Continuous Functions, Ann. Soc. Sci. Bruxelles, 98 (1984), 69–79.

[21] S. Marcus, Sur les Fonctions Quasicontinues au Sense de S. Kempisty, Colloq. Math. 8 (1961), 47–53. http://eudml.org/doc/210887.

[22] B.M. Munshi, D.S. Bassan, Almost Semi-Continuous Mappings, Math. Student, 49 (1981), 239–248.

[23] T. Noiri, V. Popa, On (mI, nJ)-Continuous Multifunctions, Rom. J. Math. Comput. Sci. 15 (2025), 1–8.

[24] T. Noiri, V. Popa, A Unified Theory of Upper and Lower Almost Nearly Continuous Multifunctions, Math. Balkanica, 23 (2009), 51–72.

[25] T. Noiri, N. Ergun, Notes on N-Continuous Functions, Res. Rep. Yatsushiro Coll. Tech. 11 (1989), 65–68.

[26] V. Popa, Almost Continuous Multifunctions, Mat. Vesnik, 6 (1982), 75–84.

[27] V. Popa, on a Decomposition of Quasicontinuity in Topological Spaces, Stud. Cerc. Mat. 30 (1978), 31–35.

[28] P. Pue-on, A. Sama-Ae, C. Boonpok, Characterizations of Nearly $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions, (Submitted).

[29] P. Pue-on, A. Sama-Ae, C. Boonpok, On Weak Forms of Upper and Lower Continuous Multifunctions Between an Ideal Topological Space and a Bitopological Space, Eur. J. Pure Appl. Math. 18 (2025), 6567. https://doi.org/10.29020/nybg.ejpam.v18i3.6567.

[30] P. Pue-on, S. Sompong, C. Boonpok, Weakly Quasi (τ_1, τ_2)-Continuous Multifunctions, Eur. J. Pure Appl. Math. 17 (2024), 1553–1564. https://doi.org/10.29020/nybg.ejpam.v17i3.5191.

[31] P. Pue-on, S. Sompong, C. Boonpok, Almost Quasi (τ_1, τ_2)-Continuity for Multifunctions, Int. J. Anal. Appl. 22 (2024), 97. https://doi.org/10.28924/2291-8639-22-2024-97.

[32] P. Pue-on, S. Sompong, C. Boonpok, Upper and Lower (τ_1, τ_2)-Continuous Mulfunctions, Int. J. Math. Comput. Sci. 19 (2024), 1305–1310.

[33] M.K. Singal, A.R. Singal, Almost Continuous Mappings, Yokohama J. Math. 16 (1968), 63–73.

[34] M. Thongmoon, A. Sama-Ae, C. Boonpok, Almost Quasi Continuity and Weak Quasi Continuity for Multifunctions Between an Ideal Topological Space and a Bitopological Space, Eur. J. Pure Appl. Math. 18 (2025), 6571. https://doi.org/10.29020/nybg.ejpam.v18i3.6571.

[35] M. Thongmoon, A. Sama-Ae, C. Boonpok, Upper and Lower Near (τ_1, τ_2)-Continuity, Eur. J. Pure Appl. Math. 18 (2025), 5633. https://doi.org/10.29020/nybg.ejpam.v18i1.5633.

[36] M. Thongmoon, S. Sompong, C. Boonpok, Upper and Lower Weak (τ_1, τ_2)-Continuity, Eur. J. Pure Appl. Math. 17 (2024), 1705–1716. https://doi.org/10.29020/nybg.ejpam.v17i3.5238.

[37] N. Viriyapong, A. Sama-Ae, C. Boonpok, On Almost $\tau^{\star}(\sigma_1, \sigma_2)$-Continuity and Weak $\tau^{\star}(\sigma_1, \sigma_2)$-Continuity, Eur. J. Pure Appl. Math. 18 (2025), 6568. https://doi.org/10.29020/nybg.ejpam.v18i3.6568.

[38] C. Viriyapong, A. Sama-Ae, C. Boonpok, Almost Continuity for Multifunctions Defined from an Ideal Topological Space into a Bitopological Space, Eur. J. Pure Appl. Math. 18 (2025), 6566. https://doi.org/10.29020/nybg.ejpam.v18i3.6566.

[39] N. Viriyapong, S. Sompong, C. Boonpok, (τ_1, τ_2)-Extremal Disconnectedness in Bitopological Spaces, Int. J. Math. Comput. Sci. 19 (2024), 855–860.

[40] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-Continuity for Multifunctions, J. Math. 2020 (2020), 6285763. https://doi.org/10.1155/2020/6285763.