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A Unified Theory of Near Continuity for Functions Defined Between an Ideal Topological Space and a Bitopological Space

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Abstract. This paper introduces a new class of continuous functions defined from an ideal topological space into a bitopological space, namely nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Furthermore, several characterizations and some properties concerning nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are investigated.

1. Introduction

In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of continuity for functions. Weaker and stronger forms of open sets play an important role in the generalization of different forms of continuity. Using different forms of open sets, several authors have introduced and investigated various types of continuity. Carnahan [11] introduced the concept of N-closed sets in topological spaces. Noiri [26] studied several properties of N-closed sets and some separation axioms. The notion of N-continuous functions was introduced by Malghan and Hanchinamani [23]. Noiri and Ergun [25] investigated some characterizations of N-continuous functions. Janković and Hamlett [18] introduced the concept of $\mathscr{I}$-open sets in topological spaces via ideals. Abd El-Monsef et al. [1] studied some properties of $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. As generalizations of open sets in an ideal topological space, many authors introduced the notions of semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets. Hatir and Noiri [17] introduced and studied

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the concepts of semi-$\mathscr{I}$-continuous functions and α-$\mathscr{I}$-conttinuous functions by utilizing the notions of semi-$\mathscr{I}$-open sets and α-$\mathscr{I}$-open sets, respectively. Furthermore, Hatir and Noiri [16] introduced and investigated the notions of β-$\mathscr{I}$-continuous functions and β-$\mathscr{I}$-irresolute. On the other hand, the present authors introduced and investigated the concepts of (τ_1, τ_2)-continuous functions [5], almost (τ_1, τ_2)-continuous functions [3], weakly (τ_1, τ_2)-continuous functions [4], almost quasi (τ_1, τ_2)-continuuos functions [21], weakly quasi (τ_1, τ_2)-continuuos functions [14], $\star$-continuous functions [6], weakly $\star$-continuous functions [8], $\theta(\star)$-continuous functions [8], $\theta(\star)$-precontinuous functions [7], almost $\star$-precontinuous functions [7], weakly $\star$-precontinuous functions [7] and *pι*-continuous functions [2]. Khampakdee et al. [19] introduced and investigated notion of $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Viriyapong et al [28] introduced and studied the concepts of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Chutiman et al. [14] introduced and investigated the notion of nearly (τ_1, τ_2)-continuous functions. Quite recently, Kong-ied et al. [20] presented a new class of continuous functions defined between bitopological spaces, namely almost nearly (τ_1, τ_2)-continuous functions. In this paper, we introduce the notion of nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. We also investigate some characterizations of nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [10] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [10] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [10] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [10] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.
(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*
(4) *A is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.
(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*r-open* [30] (resp. (τ_1, τ_2)*s-open* [9], (τ_1, τ_2)*p-open* [9], $(\tau_1, \tau_2)\beta$*-open* [9]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq$

$\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called *$(\tau_1, \tau_2)r$-closed* (resp. *$(\tau_1, \tau_2)s$-closed*, *$(\tau_1, \tau_2)p$-closed*, *$(\tau_1, \tau_2)\beta$-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\alpha(\tau_1, \tau_2)$-open* [29] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be *$\alpha(\tau_1, \tau_2)$-closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\mathscr{N}(\tau_1, \tau_2)$-closed* [27] if every cover of A by $(\tau_1, \tau_2)r$-open sets of X has a finite subcover. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a *$(\tau_1, \tau_2)\theta$-cluster point* [30] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the *$(\tau_1, \tau_2)\theta$-closure* [30] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)\theta$-closed* [30] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be *$(\tau_1, \tau_2)\theta$-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the *$(\tau_1, \tau_2)\theta$-interior* [30] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2.2. [30] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.*
(2) *$(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.*

Lemma 2.3. [12] *Let (X, τ_1, τ_2) be a bitopological space. If V is a $\tau_1\tau_2$-open set of X having $\mathscr{N}(\tau_1, \tau_2)$-closed complement, then $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(V))$ is a $(\tau_1, \tau_2)r$-open set having $\mathscr{N}(\tau_1, \tau_2)$-closed complement.*

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [22], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [18] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\tau^\star$-semi-open* [24] (resp. *$\tau^\star$-pre-open* [24]) if $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$). The complement of a $\tau^\star$-semi-open (resp. $\tau^\star$-pre-open) set is called *$\tau^\star$-semi-closed* (resp. $\tau^\star$-pre-closed).

3. Nearly $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions

In this section, we introduce the notion of nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Furthermore, several characterizations of nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are investigated.

Definition 3.1. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement,* *there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;*
(2) *$x \in Int^\star(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
(3) *$x \in f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y having the $\mathscr{N}(\sigma_1, \sigma_2)$-closed $\sigma_1\sigma_2$-closure such that $x \in Cl^\star(f^{-1}(B))$;*
(4) *$x \in Int^\star(f^{-1}(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2\text{-}Int(B)$ is $\mathscr{N}(\sigma_1, \sigma_2)$-closed and $x \in f^{-1}(\sigma_1\sigma_2\text{-}Int(B))$.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement and $x \in f^{-1}(V)$. By (1), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$. Since U is $\star$-open, we have $x \in \text{Int}^\star(f^{-1}(V))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y having the $\mathscr{N}(\sigma_1, \sigma_2)$-closed $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed and $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-open set having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. Suppose that

$$x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

Then, $x \in X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)) = f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(B))$ and so $f(x) \in Y - \sigma_1\sigma_2\text{-Cl}(B)$. Since $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-open set having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement, by (2) we have

$$\begin{aligned} x \in \text{Int}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(B))) &= \text{Int}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))) \\ &= X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))) \\ &\subseteq X - \text{Cl}^\star(f^{-1}(B)). \end{aligned}$$

Therefore, $x \notin \text{Cl}^\star(f^{-1}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\mathscr{N}(\sigma_1, \sigma_2)$-closed. Suppose that $x \notin \text{Int}^\star(f^{-1}(B))$. Then, we have $x \in X - \text{Int}^\star(f^{-1}(B)) = \text{Cl}^\star(X - f^{-1}(B)) = \text{Cl}^\star(f^{-1}(Y - B))$ and by (3), $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) = f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$. Thus, $x \notin f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement. Then, $Y - \sigma_1\sigma_2\text{-Int}(V) = Y - V$ which is $\mathscr{N}(\sigma_1, \sigma_2)$-closed and $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(V))$. By (4), we have $x \in \text{Int}^\star(f^{-1}(V))$. Therefore, there exists a $\star$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$. This shows that f is nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x. □

Theorem 3.2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
(2) *$f^{-1}(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\mathscr{N}(\sigma_1, \sigma_2)$-closed complement;*
(3) *$f^{-1}(K)$ is $\star$-closed in X for every $\mathscr{N}(\sigma_1, \sigma_2)$-closed $\sigma_1\sigma_2$-closed set K of Y;*

(4) *$Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y having the $\mathscr{N}(\sigma_1,\sigma_2)$-closed $\sigma_1\sigma_2$-closure;*

(5) *$f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2\text{-}Int(B)$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement and $x \in f^{-1}(V)$. Since f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous, by Theorem 3.1 we have $x \in \text{Int}^\star(f^{-1}(V))$. Thus, $\text{Int}^\star(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $f^{-1}(V)$ is $\star$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let B be any subset of Y having the $\mathscr{N}(\sigma_1,\sigma_2)$-closed $\sigma_1\sigma_2$-closure. Then, we have $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ is $\star$-closed in X. Thus,

$$f^{-1}(B) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)) = \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)))$$

and so $\text{Cl}^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed. Then by (4), we have

$$\begin{aligned} X - \text{Int}^\star(f^{-1}(B)) &= \text{Cl}^\star(X - f^{-1}(B)) \\ &= \text{Cl}^\star(f^{-1}(Y - B)) \\ &\subseteq \text{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B))) \\ &\subseteq f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(f^{-1}(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$ and having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Thus by (5), $x \in f^{-1}(V) = f^{-1}(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \text{Int}^\star(f^{-1}(V))$. By Theorem 3.1, f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous at x. This shows that f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous. □

Corollary 3.1. *A function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous if $f^{-1}(K)$ is $\star$-closed in X for every $\mathscr{N}(\sigma_1,\sigma_2)$-closed set K of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Then, $Y - V$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed. By the hypothesis, $X - f^{-1}(V) = f^{-1}(Y - V) = \text{Cl}^\star(X - f^{-1}(V)) = X - \text{Int}^\star(f^{-1}(V))$ and hence $f^{-1}(V) = \text{Int}^\star(f^{-1}(V))$. It follows from Theorem 3.2 that f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous. □

Recall that a bitopological space (X,τ_1,τ_2) is said to be (τ_1,τ_2)-*regular* [15] if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 3.3. *For a function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, the following properties are equivalent:*

(1) *f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous;*

(2) $f^{-1}((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y *such that* $(\sigma_1,\sigma_2)\theta\text{-}Cl(B)$ *is* $\mathscr{N}(\sigma_1,\sigma_2)$*-closed;*

(3) $f^{-1}(K)$ *is* $\star$*-closed in* X *for every* $\mathscr{N}(\sigma_1,\sigma_2)$*-closed* $(\sigma_1,\sigma_2)\theta$*-closed set* K *of* Y*;*

(4) $f^{-1}(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1,\sigma_2)\theta$*-open set* V *of* Y *having* $\mathscr{N}(\sigma_1,\sigma_2)$*-closed complement.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y such that $(\sigma_1,\sigma_2)\theta$-Cl$(B)$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed. Therefore, $(\sigma_1,\sigma_2)\theta$-Cl$(B)$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed and $\sigma_1\sigma_2$-closed. Thus by Theorem 3.2, $f^{-1}((\sigma_1,\sigma_2)\theta\text{-Cl}(B))$ is $\star$-closed in X .

(2) $\Rightarrow$ (3): Let K be any $\mathscr{N}(\sigma_1,\sigma_2)$-closed $(\sigma_1,\sigma_2)\theta$-closed set of Y. Then, we have $K = (\sigma_1,\sigma_2)\theta\text{-Cl}(K)$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed and by (2), $f^{-1}(K)$ is $\star$-closed in X.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1,\sigma_2)\theta$-open set of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Then, $Y - V$ is $\mathscr{N}(\sigma_1,\sigma_2)$-closed and $(\sigma_1,\sigma_2)\theta$-closed. By (3), we have

$$X - f^{-1}(V) = f^{-1}(Y - V) = \text{Cl}^\star(f^{-1}(Y - V)) = \text{Cl}^\star(X - f^{-1}(V)) = X - \text{Int}^\star(f^{-1}(V))$$

and hence $f^{-1}(V)$ is $\star$-open in X.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Since (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, V is $(\sigma_1,\sigma_2)\theta$-open in Y and having $\mathscr{N}(\sigma_1,\sigma_2)$-closed complement. Then by (4), we have $f^{-1}(V)$ is $\star$-open in X. By Theorem 3.2, f is nearly $\tau^\star(\sigma_1,\sigma_2)$-continuous. □

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