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Analysis of Coupled $\mathbb{G}$-Caputo Variable-Order Fractional Differential Equations Subject to Nonlocal Integral Boundary Conditions

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Abstract. This paper investigates a system of two coupled $\mathbb{G}$-Caputo fractional differential equations. The derivatives are of piecewise-constant variable order, and the system is complemented by nonlocal integral boundary conditions. The $\mathbb{G}$-Caputo framework, constructed from a strictly increasing differentiable kernel function $\mathbb{G}$, encompasses the classical Caputo, Hadamard–Caputo and Katugampola–Caputo operators as particular cases. To the best of our knowledge, coupled $\mathbb{G}$-Caputo systems with piecewise-constant variable orders under nonlocal integral boundary conditions have not been treated previously. Furthermore, the proposed framework accommodates nonlinearities that may exhibit singular behavior at the initial point via a $\mathbb{G}^{\delta_i}$-weighted Lipschitz condition, thereby significantly extending the scope of existing bounded-growth theories. By reducing the problem to a system of coupled constant-order integral equations on a suitable partition, we establish existence via Darbo's fixed-point theorem and the Kuratowski measure of noncompactness, uniqueness through Banach's contraction principle, and Ulam–Hyers stability with an explicit stability constant. Three illustrative examples are provided: a Hadamard fractional system with constant orders, a

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variable-order thermo-diffusion model in a stratified medium, and a numerical experiment that confirms the predicted geometric convergence rate.

1. Introduction

Fractional differential equations provide a powerful mathematical framework for modeling physical systems with memory effects. Their ability to capture hereditary properties makes them particularly attractive in many applications involving memory and anomalous diffusion, including viscoelastic deformations and electrochemical processes [1–3]. Among the broad family of fractional operators, the $\mathbb{G}$-Caputo derivative introduced by Almeida [4] is particularly notable for its unifying nature. By selecting different strictly increasing auxiliary functions $\mathbb{G}$, one recovers several classical fractional derivatives as special cases, including the Caputo derivative ($\mathbb{G}(t) = t$), the Hadamard–Caputo derivative ($\mathbb{G}(t) = \ln t$), and the Katugampola–Caputo derivative ($\mathbb{G}(t) = t^\rho/\rho$, $\rho > 0$).

Variable-order fractional operators, in which the differentiation order varies with time, provide an effective framework for modeling processes whose memory characteristics evolve dynamically. Such models arise, for example, in heat transfer through media with spatially varying thermal properties and in anomalous diffusion within heterogeneous materials [5, 6]. In contrast to their constant-order counterparts, variable-order derivatives generally fail to satisfy the classical semigroup property [7], leading to significant analytical challenges. Consequently, piecewise-constant order approximations have become an important tool for both theoretical investigations and practical applications.

Recent studies on scalar $\mathbb{G}$-Caputo variable-order equations [8–11] have successfully used piecewise-constant order approximations. Together with Darbo's fixed-point theorem and measures of noncompactness, this approach has proved effective in establishing existence results. Recent developments on ψ-Caputo operators with applications to controllability, boundary value problems, and integrodifferential systems can be found in [12–15], while stochastic and pantograph dynamics under generalized Caputo settings have also been investigated [16], and generalized proportional variable-order settings are treated in [17]. However, several important challenges remain open when moving toward more realistic models. In this work, we address three such issues within the framework of coupled $\mathbb{G}$-Caputo variable-order systems; see also [18, 19] for related coupled fractional frameworks.

First, we treat a genuinely coupled system rather than a scalar equation, so that the two unknowns interact through the nonlinearities and the whole analysis must be carried out in the product space $C(J, \mathbb{R})^2$.

Second, we consider nonlocal integral boundary conditions, which are more realistic than classical Dirichlet conditions. In particular, boundary constraints of the form

$$x(a_2) = \int_{a_1}^{a_2} g_1(s)\, y(s)\, ds$$

introduce a boundary-level coupling between the unknowns, significantly increasing the analytical complexity.

Third, we introduce a weighted singularity framework that allows for nonlinearities with integrable singular behavior near the initial point. Unlike classical assumptions requiring bounded growth with continuous weights, our approach incorporates $\mathbb{G}^{\delta_i}$-weighted Lipschitz conditions, enabling the treatment of models exhibiting singular kernels such as $f(t,x,y) \sim (\ln t)^{-\delta}$ in the Hadamard setting.

Main contributions. The existing $\mathbb{G}$-Caputo variable-order literature [8–11] is confined to scalar equations with bounded-growth nonlinearities. To the best of our knowledge, the combination of the following features has not been addressed before, and each of them lies beyond the scope of that theory:

(1) *Coupled, kernel-unified, variable-order formulation.* We move from the scalar setting of [8–11] to a *coupled* system of $\mathbb{G}$-Caputo equations of piecewise-constant variable order, posed in the product space $C(J,\mathbb{R})^2$. Through the kernel $\mathbb{G}$ this single framework covers the Caputo ($\mathbb{G}(t) = t$), Hadamard–Caputo ($\mathbb{G}(t) = \ln t$) and Katugampola–Caputo ($\mathbb{G}(t) = t^\rho/\rho$) variable-order systems at once.

(2) *Boundary-level coupling.* The two unknowns interact not only through the nonlinearities but also through the nonlocal integral boundary conditions $x(a_2) = \int_{a_1}^{a_2} g_1\, y$, $y(a_2) = \int_{a_1}^{a_2} g_2\, x$, which are more realistic than Dirichlet conditions and require control of the cross terms in the fixed-point analysis.

(3) *Weighted-singularity framework.* We replace the usual bounded-growth hypothesis by the $\mathbb{G}^{\delta_i}$-weighted continuity and Lipschitz conditions (H1)–(H2), which admit nonlinearities with integrable singularities of order $\mathbb{G}^{-\delta_i}$ at the left endpoint (e.g. $f_i \sim (\ln t)^{-\delta_i}$ in the Hadamard case). The framework reduces to the classical Lipschitz setting when $\mathbb{G}(a_1) > 0$, and is therefore a genuine extension.

(4) *Complete analysis with explicit constants.* We establish existence via Darbo's fixed-point theorem and the Kuratowski measure of noncompactness (condensing constant κ^*), uniqueness via the Banach principle with the *same* κ^*, and Ulam–Hyers stability with the explicit constant $C_{\mathrm{UH}} = 2\max\{\hat{\Phi}, \hat{\Psi}\}/(1-\kappa^*)$, all given in closed form in terms of $\mathbb{G}$, the orders and the data ((3.10), (3.15)).

(5) *Constructive reduction and validation.* The variable-order problem is reduced to a system of constant-order coupled integral equations on a partition, including the correction term enforcing the nonlocal condition at a_2; the theory is illustrated on a constant-order Hadamard system (Example 4.1), a variable-order thermo-diffusion model (Example 4.2), and a numerical experiment (Example 4.3) in which the successive-approximation scheme inherits the contraction rate κ^*.

Specifically, we study the coupled system

$$\begin{cases} {}^{c}D_{a_1^+}^{u(t);\mathbb{G}} x(t) = f_1(t, x(t), y(t)), \quad t \in J = [a_1, a_2], \\ {}^{c}D_{a_1^+}^{v(t);\mathbb{G}} y(t) = f_2(t, x(t), y(t)), \quad t \in J, \\ \qquad x(a_1) = 0, \quad x(a_2) = \int_{a_1}^{a_2} g_1(s)\, y(s)\, ds, \\ \qquad y(a_1) = 0, \quad y(a_2) = \int_{a_1}^{a_2} g_2(s)\, x(s)\, ds, \end{cases} \tag{1.1}$$

where ${}^{c}D_{a_1^+}^{\alpha;\mathbb{G}}$ denotes the $\mathbb{G}$-Caputo derivative with $u, v : J \to (1, 2]$ piecewise constant, and g_1, g_2 are given boundary kernels.

The analysis rests on the Kuratowski measure of noncompactness [20,21] and Darbo's fixed-point theorem for existence, Banach's contraction principle for uniqueness, and a direct perturbation argument for Ulam–Hyers stability [22].

The paper is organized as follows. Section 2 collects the prerequisites on $\mathbb{G}$-fractional calculus, measures of noncompactness, and the definition of Ulam–Hyers stability. Section 3 contains the main results: integral formulation, existence, uniqueness, and stability. Section 4 presents illustrative examples. Section 5 concludes with a discussion.

2. Preliminaries

Throughout the paper, $J = [a_1, a_2]$ with $a_1 < a_2$, and the kernel $\mathbb{G} \in C^2(J, \mathbb{R})$ is assumed to be strictly increasing with $\mathbb{G}'(t) > 0$ for all $t \in J$ and $\mathbb{G}(a_1) \geq 0$; consequently $\mathbb{G}(t) > 0$ on $(a_1, a_2]$, so that the weights $\mathbb{G}^{\pm\delta}$ used in Section 3 are well defined. The three model kernels satisfy these requirements on the natural intervals: $\mathbb{G}(t) = t$ on $J \subset [0, \infty)$, $\mathbb{G}(t) = \ln t$ on $J \subset [1, \infty)$, and $\mathbb{G}(t) = t^\rho / \rho$ ($\rho > 0$) on $J \subset (0, \infty)$.

We recall the $\mathbb{G}$-fractional calculus framework introduced in Almeida [4].

Definition 2.1. *For $\alpha > 0$ and a measurable function $h : J \to \mathbb{R}$, the left-sided $\mathbb{G}$-fractional integral of order α is*

$$I_{a_1^+}^{\alpha;\mathbb{G}} h(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{t} \mathbb{G}'(s) \Big(\mathbb{G}(t) - \mathbb{G}(s) \Big)^{\alpha - 1} h(s)\, ds, \quad t \in J, \tag{2.1}$$

provided the integral exists.

Definition 2.2. *Let $n \in \mathbb{N}$ and $\alpha \in (n-1, n]$. The left-sided $\mathbb{G}$-Caputo fractional derivative of order α is*

$${}^{c}D_{a_1^+}^{\alpha;\mathbb{G}} h(t) = I_{a_1^+}^{n-\alpha;\mathbb{G}} h^{[n]}(t), \tag{2.2}$$

where $h^{[k]}(t) = \left(\frac{1}{\mathbb{G}'(t)} \frac{d}{dt} \right)^k h(t)$.

We recall the following result concerning the interaction between the $\mathbb{G}$-fractional integral and the $\mathbb{G}$-Caputo derivative [4]; recall that $h^{[1]} = h' / \mathbb{G}'$.

Lemma 2.1. *Let $\alpha \in (1, 2]$ and $h \in C^2(J, \mathbb{R})$. Then*

$$I_{a_1^+}^{\alpha;\mathbb{G}c} D_{a_1^+}^{\alpha;\mathbb{G}} h(t) = h(t) - h(a_1) - \Big(\mathbb{G}(t) - \mathbb{G}(a_1)\Big) h^{[1]}(a_1). \tag{2.3}$$

To establish our existence result, we recall the Kuratowski measure of noncompactness and a version of Darbo's fixed-point theorem. Let $(X, \|\cdot\|)$ be a Banach space. The Kuratowski measure of noncompactness μ, defined on the bounded subsets of X, is given by

$$\mu(\Omega) = \inf\Big\{\varepsilon > 0 : \ \Omega \text{ admits a finite cover by sets of diameter } < \varepsilon\Big\}. \tag{2.4}$$

We recall some basic properties of the Kuratowski measure of noncompactness μ, which will be used.

Proposition 2.1 ([20]). *For bounded sets $\Omega, \Omega_1, \Omega_2 \subset X$:*

(i) $\mu(\Omega) = 0$ *if and only if* $\overline{\Omega}$ *is compact,*
(ii) $\mu(\Omega_1) \leq \mu(\Omega_2)$ *whenever* $\Omega_1 \subset \Omega_2$,
(iii) $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$,
(iv) $\mu(c \cdot \Omega) = |c| \, \mu(\Omega)$ *for any scalar* c,
(v) $\mu(\Omega_1 + \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$.

We recall a useful property of the Kuratowski measure of noncompactness in product spaces.

Lemma 2.2. *Endow $C(J, \mathbb{R})^2$ with the norm $\big\|(x, y)\big\|_\infty = \max\{\|x\|_\infty, \big\|y\big\|_\infty\}$. For bounded sets $\Omega_1, \Omega_2 \subset C(J, \mathbb{R})$, the product $\Omega = \Omega_1 \times \Omega_2$ satisfies*

$$\mu(\Omega) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}. \tag{2.5}$$

Proof. Let π_1, π_2 denote the canonical projections from $C(J, \mathbb{R})^2$ onto the first and second component. Since each π_j is 1-Lipschitz and $\pi_j(\Omega) = \Omega_j$, the monotonicity and Lipschitz property of the Kuratowski measure yield $\mu(\Omega_j) \leq \mu(\Omega)$ for $j = 1, 2$, hence $\max\{\mu(\Omega_1), \mu(\Omega_2)\} \leq \mu(\Omega)$.

Conversely, fix $\varepsilon > \max\{\mu(\Omega_1), \mu(\Omega_2)\}$. There exist finite covers $\{A_i\}_{i=1}^N$ of Ω_1 and $\{B_j\}_{j=1}^M$ of Ω_2 with $\operatorname{diam}(A_i) < \varepsilon$ and $\operatorname{diam}(B_j) < \varepsilon$. Then $\{A_i \times B_j\}_{i,j}$ is a finite cover of Ω and, for the max-norm, $\operatorname{diam}(A_i \times B_j) = \max\{\operatorname{diam}(A_i), \operatorname{diam}(B_j)\} < \varepsilon$. Thus $\mu(\Omega) \leq \varepsilon$. Letting $\varepsilon \downarrow \max\{\mu(\Omega_1), \mu(\Omega_2)\}$ gives the reverse inequality. □

We now recall Darbo's fixed-point theorem, which is a key tool in our existence analysis.

Theorem 2.1 ([21]). *Let C be a nonempty, closed, convex and bounded subset of a Banach space X, and $T : C \to C$ continuous. If $\mu(T(\Omega)) \leq k \, \mu(\Omega)$ for all bounded $\Omega \subset C$, with $k \in [0, 1)$, then T has at least one fixed point in C.*

We conclude the preliminaries by recalling the notion of Ulam–Hyers stability that will be used throughout the paper.

Definition 2.3. *System* (1.1) *is Ulam–Hyers stable if there exists* $C_{\mathrm{UH}} > 0$ *such that for every* $\varepsilon > 0$ *and every* $(\tilde{x}, \tilde{y}) \in C(J, \mathbb{R})^2$ *satisfying the boundary conditions in* (1.1) *and*

$$\left| {}^{c}D^{u(t);\mathbb{G}}_{a_1^+}\tilde{x}(t) - f_1(t, \tilde{x}, \tilde{y}) \right| \le \varepsilon, \quad \left| {}^{c}D^{v(t);\mathbb{G}}_{a_1^+}\tilde{y}(t) - f_2(t, \tilde{x}, \tilde{y}) \right| \le \varepsilon, \quad t \in J, \tag{2.6}$$

there exists a solution (x^*, y^*) *of* (1.1) *with* $\max\{\|\tilde{x} - x^*\|_\infty, \left\|\tilde{y} - y^*\right\|_\infty\} \le C_{\mathrm{UH}}\,\varepsilon$.

3. Main Results

To derive the integral representation in the following lemma, we use the fact that $u(\cdot)$ and $v(\cdot)$ are piecewise constant. Hence, there exists a partition $a_1 = \tau_0 < \tau_1 < \cdots < \tau_m = a_2$ such that $u(t) \equiv u_\ell$ and $v(t) \equiv v_\ell$ on each interval $J_\ell = (\tau_{\ell-1}, \tau_\ell]$, $\ell = 1, \dots, m$.

To keep the notation compact, we denote by

$$\mathcal{I}^{\alpha}_{k}[h](t) = \frac{1}{\Gamma(\alpha)} \int_{\tau_{k-1}}^{\min\{t,\tau_k\}} \mathbb{G}'(s)\Big(\mathbb{G}(t) - \mathbb{G}(s)\Big)^{\alpha-1} h(s)\, ds, \qquad \alpha > 1, \tag{3.1}$$

the contribution of the k-th subinterval, with the convention $\mathcal{I}^{\alpha}_{k}[h](t) = 0$ for $t \le \tau_{k-1}$. Thus $\mathcal{I}^{\alpha}_{k}[h]$ coincides with $I^{\alpha;\mathbb{G}}_{\tau_{k-1}^+} h$ on J_k, and the integration is truncated at τ_k for $t > \tau_k$.

Because variable-order operators do not obey the semigroup property [7], we work with a piecewise-constant convention in which the memory created on each subinterval carries the order prevailing on that subinterval. Specifically, for $h \in C(J, \mathbb{R})$ we set

$$I^{u(\cdot);\mathbb{G}}_{a_1^+} h(t) := \sum_{k=1}^{m} \mathcal{I}^{u_k}_{k}[h](t), \qquad t \in J, \tag{3.2}$$

and we *define* the left-sided $\mathbb{G}$-Caputo derivative of piecewise-constant variable order $u(\cdot)$ as the left inverse of (3.2): for $h \in C(J, \mathbb{R})$ and $x \in C(J, \mathbb{R})$,

$${}^{c}D^{u(\cdot);\mathbb{G}}_{a_1^+} x = h \quad \Longleftrightarrow \quad x(t) = c_0 + c_1\Big(\mathbb{G}(t) - \mathbb{G}(a_1)\Big) + I^{u(\cdot);\mathbb{G}}_{a_1^+} h(t) \ \text{ for some } c_0, c_1 \in \mathbb{R}. \tag{3.3}$$

When u is constant ($m = 1$) both operators reduce to Definitions 2.1 and 2.2, and (3.3) coincides with Lemma 2.1 with $c_0 = x(a_1)$ and $c_1 = x^{[1]}(a_1)$. The same convention is applied to $v(\cdot)$ with the orders v_k.

Lemma 3.1. *A pair* $(x, y) \in C(J, \mathbb{R})^2$ *solves* (1.1) *if and only if, for every* $t \in J$,

$$x(t) = \Lambda_1[y](t) + \mathcal{K}_1[f_1(\cdot, x, y)](t), \tag{3.4}$$

$$y(t) = \Lambda_2[x](t) + \mathcal{K}_2[f_2(\cdot, x, y)](t), \tag{3.5}$$

where the boundary functionals and the integral operators are

$$\Lambda_1[y](t) = \frac{\mathbb{G}(t) - \mathbb{G}(a_1)}{\mathbb{G}(a_2) - \mathbb{G}(a_1)} \int_{a_1}^{a_2} g_1(s)\, y(s)\, ds, \tag{3.6}$$

$$\mathcal{K}_1[h](t) = \sum_{k=1}^{m} \mathcal{I}^{u_k}_{k}[h](t) - \frac{\mathbb{G}(t) - \mathbb{G}(a_1)}{\mathbb{G}(a_2) - \mathbb{G}(a_1)} \sum_{k=1}^{m} \mathcal{I}^{u_k}_{k}[h](a_2), \tag{3.7}$$

and Λ_2, $\mathcal{K}_2$ *are defined analogously, with* g_2 *in place of* g_1 *and the orders* v_k *in place of* u_k.

Proof. By (3.3), the first equation of (1.1) holds if and only if

$$x(t) = c_0 + c_1\Big(\mathbb{G}(t) - \mathbb{G}(a_1)\Big) + \sum_{k=1}^{m} \mathcal{I}_k^{u_k}[f_1(\cdot, x, y)](t), \qquad t \in J,$$

for some $c_0, c_1 \in \mathbb{R}$. The condition $x(a_1) = 0$ forces $c_0 = 0$, since $\mathcal{I}_k^{u_k}[h](a_1) = 0$ for every k. Evaluating at $t = a_2$ and imposing $x(a_2) = \int_{a_1}^{a_2} g_1(s)y(s)\,ds$ gives

$$c_1 = \frac{1}{\mathbb{G}(a_2) - \mathbb{G}(a_1)}\Big[\int_{a_1}^{a_2} g_1(s)y(s)\,ds - \sum_{k=1}^{m} \mathcal{I}_k^{u_k}[f_1(\cdot, x, y)](a_2)\Big],$$

which is the value of $x^{[1]}(a_1)$ in the constant-order case. Substituting yields (3.4). The derivation for y is identical. □

Remark 3.1. *Each $\mathcal{I}_k^{u_k}[h]$ is continuous on J and vanishes on $[a_1, \tau_{k-1}]$, so the right-hand sides of* (3.4)–(3.5) *are automatically continuous across the interior nodes $\tau_1, \ldots, \tau_{m-1}$ and no matching condition has to be imposed there. By contrast $x^{[1]}$ is in general discontinuous at the nodes, because the order jumps: the slope change observed at $t = e$ in Figure 3 is exactly this effect.*

We next state the hypotheses that will be assumed throughout the paper:

(H1) For each $i = 1, 2$, there exists $\delta_i \in (0, 1)$ such that the map $(t, x, y) \mapsto \mathbb{G}^{\delta_i}(t)\, f_i(t, x, y)$ extends to a continuous function on $J \times \mathbb{R}^2$, with $f_i^\star = \sup_{t \in J} \left|\mathbb{G}^{\delta_i}(t)\, f_i(t, 0, 0)\right| < \infty$.

(H2) With the same δ_i as in (H1), there exist $K_i > 0$ with

$$\mathbb{G}^{\delta_i}(t)\left|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)\right| \le K_i\Big(|x_1 - x_2| + \big|y_1 - y_2\big|\Big) \tag{3.8}$$

for all $t \in J$ and $x_j, y_j \in \mathbb{R}$.

(H3) $g_i \in C(J, [0, \infty))$ for $i = 1, 2$; we write $G_i = \int_{a_1}^{a_2} g_i(s)\,ds$.

(H4) Define the weighted integral constants

$$\Phi_k = \frac{(\mathbb{G}(a_2) - \mathbb{G}(\tau_{k-1}))^{u_k - 1}\left(\mathbb{G}^{1-\delta_1}(\tau_k) - \mathbb{G}^{1-\delta_1}(\tau_{k-1})\right)}{(1 - \delta_1)\,\Gamma(u_k)}, \tag{3.9}$$

and Ψ_k analogously with v_k, δ_2, and set $\Phi = \sum_{k=1}^{m} \Phi_k$, $\Psi = \sum_{k=1}^{m} \Psi_k$. Require

$$\kappa^* = \max\Big\{G_1 + 4K_1\Phi,\ G_2 + 4K_2\Psi\Big\} < 1. \tag{3.10}$$

Remark 3.2. *The exponents δ_i quantify the admissible singularity of f_i near $t = a_1$. When $\mathbb{G}(a_1) > 0$, the weight $\mathbb{G}^{\delta_i}$ is bounded and (H1)–(H2) reduce to the standard Lipschitz condition; the framework is therefore a genuine extension.*

We now establish the existence of solutions via Darbo's fixed-point theorem.

Theorem 3.1. *Assume (H1)–(H4). Then system* (1.1) *admits at least one solution* $(x, y) \in C(J, \mathbb{R})^2$.

Proof. Define $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : C(J, \mathbb{R})^2 \to C(J, \mathbb{R})^2$ by the right-hand sides of (3.4)–(3.5). We work in the closed ball $\mathcal{B}_R = \{(x, y) \in C(J, \mathbb{R})^2 : \|(x, y)\|_\infty \le R\}$, where $R > 0$ is specified below.

Key estimate on $\mathcal{K}_i$. Let $h : J \to \mathbb{R}$ satisfy $|h(s)| \le \mathbb{G}^{-\delta_i}(s)\, M$ for some $M \ge 0$. For each subinterval $J_k = (\tau_{k-1}, \tau_k]$, since $u_k > 1$ implies $(\mathbb{G}(t) - \mathbb{G}(s))^{u_k-1} \le (\mathbb{G}(a_2) - \mathbb{G}(\tau_{k-1}))^{u_k-1}$, we obtain

$$\left|\mathcal{I}_k^{u_k}[h](t)\right| \le \frac{M}{\Gamma(u_k)}(\mathbb{G}(a_2) - \mathbb{G}(\tau_{k-1}))^{u_k-1} \int_{\tau_{k-1}}^{\tau_k} \mathbb{G}'(s)\mathbb{G}^{-\delta_i}(s)\, ds = M\,\Phi_k,$$

where Φ_k is as in (3.9); for $i = 2$ the same computation with v_k and δ_2 gives $M\,\Psi_k$. The structure of $\mathcal{K}_1$ (direct sum plus correction, each bounded by $\Phi = \sum_k \Phi_k$, since $0 \le \frac{\mathbb{G}(t)-\mathbb{G}(a_1)}{\mathbb{G}(a_2)-\mathbb{G}(a_1)} \le 1$) gives

$$\left\|\mathcal{K}_1[h]\right\|_\infty \le 2\Phi\, M, \qquad \left\|\mathcal{K}_2[h]\right\|_\infty \le 2\Psi\, M, \tag{3.11}$$

where the factor 2 accounts for the correction term in (3.7).

We split the proof into the following three steps:

Step 1: $\mathcal{T}$ maps $\mathcal{B}_R$ into itself. By (H1)–(H2), $|f_1(s, x, y)| \le \mathbb{G}^{-\delta_1}(s)(f_1^\star + K_1(|x| + |y|))$. For $(x, y) \in \mathcal{B}_R$,

$$\left|\mathcal{T}_1(x, y)(t)\right| \le G_1 R + 2\Phi\Big(f_1^\star + 2K_1 R\Big) = 2\Phi f_1^\star + (G_1 + 4K_1\Phi)R.$$

Thus $\|\mathcal{T}_1(x, y)\|_\infty \le R$ whenever $R \ge \frac{2\Phi f_1^\star}{1-G_1-4K_1\Phi}$. An analogous bound holds for $\mathcal{T}_2$ with $\Psi, f_2^\star$. Since $G_1 + 4K_1\Phi \le \kappa^* < 1$ and $G_2 + 4K_2\Psi \le \kappa^* < 1$ by (H4), such an R exists; if $f_1^\star = f_2^\star = 0$, any $R > 0$ suffices. Hence $\mathcal{T}(\mathcal{B}_R) \subset \mathcal{B}_R$.

Step 2: $\mathcal{T}$ is continuous. Let $(x_n, y_n) \to (x, y)$ in $C(J, \mathbb{R})^2$. By (H1), the function $\mathbb{G}^{\delta_i} f_i$ is uniformly continuous on the compact set $J \times [-R, R]^2$, so $\mathbb{G}^{\delta_i}(t) f_i(t, x_n, y_n) \to \mathbb{G}^{\delta_i}(t) f_i(t, x, y)$ uniformly on J. Combined with the integrable dominating function $\mathbb{G}^{-\delta_i}(s)(f_i^\star + 2K_i R)$ against the kernel $\mathbb{G}'(s)(\mathbb{G}(t) - \mathbb{G}(s))^{u_k-1}$, the dominated convergence theorem yields $\mathcal{K}_i[f_i(\cdot, x_n, y_n)] \to \mathcal{K}_i[f_i(\cdot, x, y)]$ uniformly. Combined with the continuity of Λ_i, the operator $\mathcal{T}$ is continuous.

Step 3: $\mathcal{T}$ is κ^*-condensing.

Step 3: $\mathcal{T}$ is κ^*-condensing. Let $\Omega \subset \mathcal{B}_R$ be bounded and set $\Omega_1 = \pi_1(\Omega)$, $\Omega_2 = \pi_2(\Omega)$, where π_j are the canonical projections. Then $\Omega \subset \Omega_1 \times \Omega_2$ and, by monotonicity and Lemma 2.2,

$$\mu(\Omega) \le \mu(\Omega_1 \times \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}.$$

The boundary functional $\Lambda_1[y](t) = c_y\, w(t)$, with $c_y = \int_{a_1}^{a_2} g_1 y$ and $w(t) = \frac{\mathbb{G}(t)-\mathbb{G}(a_1)}{\mathbb{G}(a_2)-\mathbb{G}(a_1)} \in [0, 1]$, gives $|c_y - c_{y'}| \le G_1 \|y - y'\|_\infty$, whence $\mu(\Lambda_1[\Omega_2]) \le G_1\mu(\Omega_2)$. For the integral operator, the weighted Lipschitz (H2) and the bound (3.11) yield, for any two pairs in Ω,

$$\left\|\mathcal{K}_1[f_1(\cdot, x, y) - f_1(\cdot, x', y')]\right\|_\infty \le 2\Phi K_1\Big(\|x - x'\|_\infty + \|y - y'\|_\infty\Big).$$

By the Lipschitz property of the Kuratowski measure, $\mu(\mathcal{K}_1[f_1(\cdot, \Omega)]) \le 2K_1\Phi\Big(\mu(\Omega_1) + \mu(\Omega_2)\Big) \le 4K_1\Phi\, \mu(\Omega_1 \times \Omega_2)$. Therefore

$$\mu(\mathcal{T}_1(\Omega)) \le (G_1 + 4K_1\Phi)\, \mu(\Omega_1 \times \Omega_2) \le \kappa^* \mu(\Omega_1 \times \Omega_2).$$

The same estimate holds for $\mathcal{T}_2$ with $(G_2 + 4K_2\Psi)$. By Lemma 2.2,

$$\mu(\mathcal{T}(\Omega)) = \max\Big\{\mu(\mathcal{T}_1(\Omega)), \mu(\mathcal{T}_2(\Omega))\Big\} \leq \kappa^* \mu(\Omega_1 \times \Omega_2) = \kappa^* \mu(\Omega).$$

Since $\kappa^* < 1$, Theorem 2.1 provides a fixed point $(x, y) \in \mathcal{B}_R$, which solves (1.1) by Lemma 3.1. □

We next state the uniqueness result for system (1.1).

Theorem 3.2. *Under (H2)–(H4), system* (1.1) *has at most one solution in* $C(J, \mathbb{R})^2$.

Proof. First observe that the same estimates as in Step 3 of the proof of Theorem 3.1 show that $\mathcal{T}$ is a contraction on $\mathcal{B}_R$ with respect to the norm $\left\|(x, y)\right\|_\infty = \max\{\|x\|_\infty, \left\|y\right\|_\infty\}$: for any $(x, y), (x', y') \in \mathcal{B}_R$,

$$\left\|\mathcal{T}(x, y) - \mathcal{T}(x', y')\right\|_\infty \leq \kappa^* \left\|(x, y) - (x', y')\right\|_\infty.$$

Hence system (1.1) can have at most one solution in $\mathcal{B}_R$. For completeness, we also give a direct argument.

Let (x_1, y_1) and (x_2, y_2) be two solutions. Setting $e_x = x_1 - x_2$ and $e_y = y_1 - y_2$, the integral equations (3.4)–(3.5) and the weighted Lipschitz (H2) give

$$\|e_x\|_\infty \leq G_1 \left\|e_y\right\|_\infty + 2\Phi K_1\Big(\|e_x\|_\infty + \left\|e_y\right\|_\infty\Big) \leq (G_1 + 4K_1\Phi)\,\mathcal{E}, \tag{3.12}$$

$$\left\|e_y\right\|_\infty \leq (G_2 + 4K_2\Psi)\,\mathcal{E}, \tag{3.13}$$

where $\mathcal{E} = \max\{\|e_x\|_\infty, \left\|e_y\right\|_\infty\}$. Thus $\mathcal{E} \leq \kappa^*\mathcal{E}$, and since $\kappa^* < 1$, we conclude $\mathcal{E} = 0$. □

For the Ulam–Hyers stability analysis we introduce the *unweighted* residual constants (they bound $\mathcal{K}_i$ acting on the constant function 1, and are therefore not of the form (3.9))

$$\hat{\Phi} = \sum_{k=1}^{m} \frac{(\mathbb{G}(a_2) - \mathbb{G}(\tau_{k-1}))^{u_k} - (\mathbb{G}(a_2) - \mathbb{G}(\tau_k))^{u_k}}{\Gamma(u_k + 1)}, \tag{3.14}$$

and define $\hat{\Psi}$ analogously with respect to v_k.

Theorem 3.3. *Assume (H1)–(H4). Then system* (1.1) *is Ulam–Hyers stable with constant*

$$C_{\mathrm{UH}} = \frac{2\max\{\hat{\Phi}, \hat{\Psi}\}}{1 - \kappa^*}. \tag{3.15}$$

Proof. Let $(\tilde{x}, \tilde{y})$ satisfy the boundary conditions and the ε-inequalities in Definition 2.3. Denote the unique solution by (x^*, y^*) (Theorem 3.2), and set $e_x = \tilde{x} - x^*$, $e_y = \tilde{y} - y^*$, $\mathcal{E} = \max\{\|e_x\|_\infty, \left\|e_y\right\|_\infty\}$.

Write $r_1(t) = {}^cD_{a_1^+}^{u(t);\mathbb{G}}\tilde{x}(t) - f_1(t, \tilde{x}, \tilde{y})$ and $r_2(t) = {}^cD_{a_1^+}^{v(t);\mathbb{G}}\tilde{y}(t) - f_2(t, \tilde{x}, \tilde{y})$ for the residuals, so that $\left|r_i(t)\right| \leq \varepsilon$ on J (the symbols r_i are used here to avoid any confusion with the singularity exponents δ_i). Applying $\mathcal{K}_1$ to the *unweighted* bound $|r_1| \leq \varepsilon$ yields $\left\|\mathcal{K}_1[r_1]\right\|_\infty \leq 2\hat{\Phi}\,\varepsilon$, since $\sup_{t\in J} \sum_{k=1}^{m} \mathcal{I}_k^{u_k}[1](t) = \hat{\Phi}$ with $\hat{\Phi}$ as in (3.14).

Since $\tilde{x}$ and x^* satisfy the same boundary conditions, Lemma 3.1 applied to both gives

$$e_x = \Lambda_1[\tilde{y} - y^*] + \mathcal{K}_1[f_1(\cdot, \tilde{x}, \tilde{y}) - f_1(\cdot, x^*, y^*)] + \mathcal{K}_1[r_1].$$

The weighted Lipschitz condition and the operator bound give $\|e_x\|_\infty \le (G_1 + 4K_1\Phi)\mathcal{E} + 2\hat{\Phi}\,\varepsilon \le \kappa^*\mathcal{E} + 2\hat{\Phi}\,\varepsilon$. Similarly, $\left\|e_y\right\|_\infty \le \kappa^*\mathcal{E} + 2\hat{\Psi}\,\varepsilon$ (with the residual r_2). Taking the maximum: $\mathcal{E} \le \kappa^*\mathcal{E} + 2\max\{\hat{\Phi},\hat{\Psi}\}\,\varepsilon$, whence $\mathcal{E} \le \frac{2\max\{\hat{\Phi},\hat{\Psi}\}}{1-\kappa^*}\,\varepsilon = C_{\mathrm{UH}}\,\varepsilon$. □

4. Examples

We now present illustrative examples. The first concerns a Hadamard-type system with constant fractional orders.

Example 4.1. *Consider the coupled system on $J = [1, e]$ with $\mathbb{G}(t) = \ln t$, constant orders $u \equiv 1.5$, $v \equiv 1.7$:*

$$\begin{cases} {}^cD_{1^+}^{1.5;\mathbb{G}}x(t) = \dfrac{\sin(x+y)}{20\,(\ln t)^{1/3}}, & t \in [1,e], \\ {}^cD_{1^+}^{1.7;\mathbb{G}}y(t) = \dfrac{|x-y|}{30\,(\ln t)^{1/4}}, & t \in [1,e], \\ x(1) = 0, \quad x(e) = \displaystyle\int_1^e \frac{y(s)}{20s}\,ds, & \\ y(1) = 0, \quad y(e) = \displaystyle\int_1^e \frac{x(s)}{25s}\,ds. & \end{cases} \tag{4.1}$$

We now verify the four hypotheses. Here, $\mathbb{G}(t) = \ln t$, $\mathbb{G}(1) = 0$, $\mathbb{G}(e) = 1$, and $m = 1$.

- ***(H1)–(H2):*** *We have $f_1(t,x,y) = \frac{\sin(x+y)}{20\,(\ln t)^{1/3}}$, which has a singularity $(\ln t)^{-1/3}$ at $t = 1$. Choosing $\delta_1 = \frac{1}{3}$: $(\ln t)^{1/3}\left|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)\right| \le \frac{|(x_1+y_1)-(x_2+y_2)|}{20} \le \frac{1}{20}(|e_x| + \left|e_y\right|)$, so $K_1 = \frac{1}{20} = 0.05$, and $f_1^\star = 0$ (since $\sin(0) = 0$).*

  For f_2, the singularity is $(\ln t)^{-1/4}$, so $\delta_2 = \frac{1}{4}$: using $\big||x_1 - y_1| - |x_2 - y_2|\big| \le |x_1 - x_2| + |y_1 - y_2|$, we obtain $(\ln t)^{1/4}\left|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)\right| \le \frac{1}{30}(|e_x| + \left|e_y\right|)$, so $K_2 = \frac{1}{30} \approx 0.0333$, and $f_2^\star = 0$.
- ***(H3):*** $G_1 = \int_1^e \frac{ds}{20s} = \frac{1}{20} = 0.05$, $G_2 = \int_1^e \frac{ds}{25s} = \frac{1}{25} = 0.04$.
- ***(H4):*** *With $\tau_0 = 1, \tau_1 = e$ ($\mathbb{G}$-values $0, 1$):*

$$\Phi = \frac{(1-0)^{0.5}(1^{2/3} - 0^{2/3})}{(2/3)\,\Gamma(1.5)} = \frac{1}{(2/3)\cdot 0.8862} \approx 1.693,$$
$$\Psi = \frac{(1-0)^{0.7}(1^{3/4} - 0^{3/4})}{(3/4)\,\Gamma(1.7)} = \frac{1}{(3/4)\cdot 0.9086} \approx 1.467.$$

  The condensing constant: $\kappa^* = \max\{0.05 + 4\cdot 0.05\cdot 1.693,\ \ 0.04 + 4\cdot 0.0333\cdot 1.467\} = \max\{0.389,\ 0.236\} \approx 0.389 < 1$.

By Theorems 3.1–3.3, system (4.1) *has a unique solution in $C([1,e],\mathbb{R})^2$ that is Ulam–Hyers stable. The singular weight functions $(\ln t)^{-1/3}$ and $(\ln t)^{-1/4}$ are depicted in Figure 1. Here $\hat{\Phi} = \frac{1}{\Gamma(2.5)} \approx 0.752$ and $\hat{\Psi} = \frac{1}{\Gamma(2.7)} \approx 0.647$, so $\max\{\hat{\Phi},\hat{\Psi}\} = \hat{\Phi}$ and the stability constant* (3.15) *evaluates to $C_{\mathrm{UH}} = \frac{2\hat{\Phi}}{1-\kappa^*} = \frac{2\cdot 0.752}{1-0.389} \approx 2.46$.*

The second example deals with a variable-order thermo-diffusion model in a stratified medium.

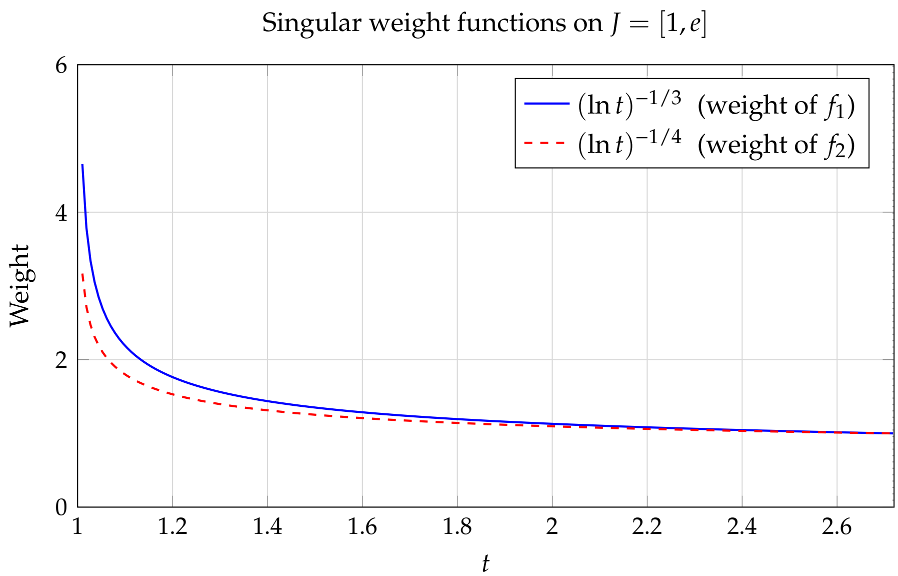

FIGURE 1. Singular weight functions $(\ln t)^{-\delta_i}$ for Example 4.1. Both weights blow up as $t \to 1^+$ ($\mathbb{G}(t) = \ln t \to 0$), illustrating the integrable singularity at the left endpoint. The $\mathbb{G}^{\delta_i}$-weighted Lipschitz framework (H1)–(H2) absorbs these singularities, yielding a well-posed coupled system.

Example 4.2. *Consider a bi-layer medium on $[1, e^2]$ with interface at $t = e$. The temperature $x(t)$ and concentration $y(t)$ satisfy*

$$\begin{cases} {}^cD_{1^+}^{u(t);\mathbb{G}} x(t) = \dfrac{x + 2y}{80\,(\ln t)^{1/3}}, & t \in [1, e^2], \\ {}^cD_{1^+}^{v(t);\mathbb{G}} y(t) = \dfrac{y - x}{80\,(\ln t)^{1/4}}, & t \in [1, e^2], \\ x(1) = 0, \quad x(e^2) = \displaystyle\int_1^{e^2} \frac{y(s)}{30\sqrt{s}}\, ds, \\ y(1) = 0, \quad y(e^2) = \displaystyle\int_1^{e^2} \frac{e^{-s}}{8}\, x(s)\, ds, \end{cases} \tag{4.2}$$

where

$$u(t) = \begin{cases} 1.2, & t \in [1, e], \\ 1.5, & t \in (e, e^2], \end{cases} \qquad v(t) = \begin{cases} 1.25, & t \in [1, e], \\ 1.6, & t \in (e, e^2]. \end{cases}$$

We now verify the hypotheses for this example. Let $\mathbb{G}(t) = \ln t$, $a_1 = 1$, $a_2 = e^2$, $m = 2$, $\tau_0 = 1, \tau_1 = e, \tau_2 = e^2$ ($\mathbb{G}$-values: $0, 1, 2$).

- ***(H1)–(H2):*** *The singularity exponents are the same as Example 4.1: $\delta_1 = \frac{1}{3}$ for f_1 and $\delta_2 = \frac{1}{4}$ for f_2.*

  For f_1: $(\ln t)^{1/3} \left| f_1(t, x_1, y_1) - f_1(t, x_2, y_2) \right| \le \frac{|e_x| + 2|e_y|}{80} \le \frac{2}{80}(|e_x| + |e_y|)$, so $K_1 = \frac{1}{40} = 0.025$; $f_1^\star = 0$.

  For f_2: $K_2 = \frac{1}{80} = 0.0125$; $f_2^\star = 0$.

- ***(H3):*** $G_1 = \int_1^{e^2} \frac{ds}{30\sqrt{s}} = \frac{2(e-1)}{30} \approx 0.115$, $G_2 = \int_1^{e^2} \frac{e^{-s}}{8}\, ds = \frac{e^{-1}-e^{-e^2}}{8} \approx 0.046$.
- ***(H4):*** *For* $k = 1$ *(*$\mathbb{G}$*-values* $0 \to 1$*, orders* $u_1 = 1.2$*,* $v_1 = 1.25$*):*

$$\Phi_1 = \frac{(2-0)^{0.2}(1^{2/3}-0)}{(2/3)\,\Gamma(1.2)} = \frac{2^{0.2}}{(2/3)\cdot 0.918} \approx 1.877,$$
$$\Psi_1 = \frac{(2)^{0.25}(1^{3/4}-0)}{(3/4)\,\Gamma(1.25)} \approx 1.749.$$

For $k = 2$ *(*$\mathbb{G}$*-values* $1 \to 2$*, orders* $u_2 = 1.5$*,* $v_2 = 1.6$*):*

$$\Phi_2 = \frac{(2-1)^{0.5}(2^{2/3}-1^{2/3})}{(2/3)\,\Gamma(1.5)} \approx 0.994,$$
$$\Psi_2 = \frac{(2-1)^{0.6}(2^{3/4}-1^{3/4})}{(3/4)\,\Gamma(1.6)} \approx 1.017.$$

Totals: $\Phi = \Phi_1 + \Phi_2 \approx 2.871$, $\Psi = \Psi_1 + \Psi_2 \approx 2.767$.

Condensing constant: $\kappa^* = \max\{0.115 + 4\cdot 0.025\cdot 2.871,\ \ 0.046 + 4\cdot 0.0125\cdot 2.767\} = \max\{0.402,\ 0.184\} \approx 0.402 < 1$.

By Theorems 3.1–3.3, system (4.2) *possesses a unique solution in* $C([1,e^2],\mathbb{R})^2$ *that is Ulam–Hyers stable. Here* $\hat{\Phi} \approx 1.930$ *and* $\hat{\Psi} \approx 1.916$*, so* $\max\{\hat{\Phi},\hat{\Psi}\} = \hat{\Phi}$ *and* $C_{\mathrm{UH}} = \frac{2\hat{\Phi}}{1-\kappa^*} = \frac{2\cdot 1.930}{1-0.402} \approx 6.45$.

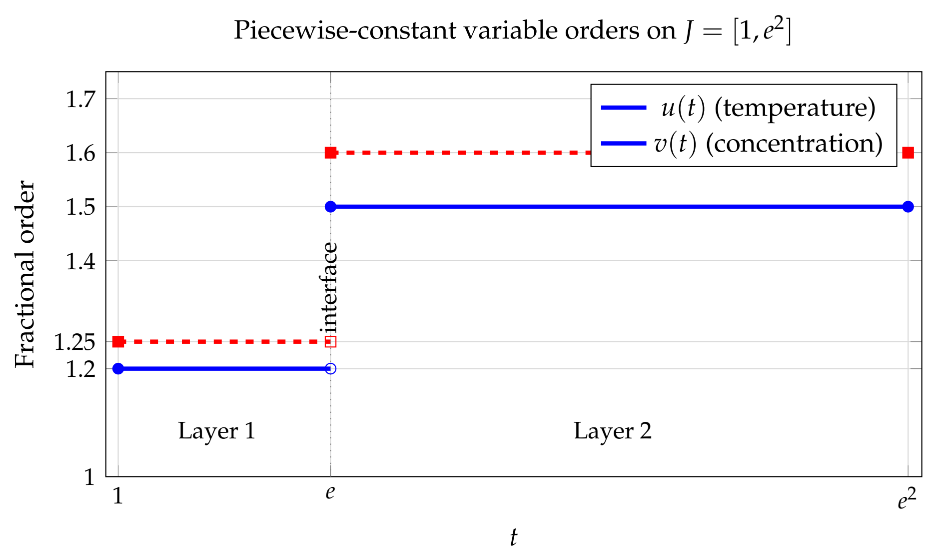

FIGURE 2. Piecewise-constant fractional orders $u(t)$ and $v(t)$ for the variable-order thermo-diffusion model of Example 4.2. The interface at $t = e$ separates two layers of a stratified medium; within each layer the memory parameters are constant but differ across the interface, reflecting distinct material properties. The jump discontinuity in the orders is handled by the partition-based integral reformulation of Lemma 3.1.

Example 4.3. *To complement the qualitative results with a quantitative illustration, we add bounded source terms (modeling heat and mass injection) to the thermo-diffusion model of Example 4.2, on $J = [1, e^2]$ with $\mathbb{G}(t) = \ln t$ and the same orders $u(t), v(t)$:*

$$\begin{cases} {}^{c}D_{1^+}^{u(t);\mathbb{G}} x(t) = \dfrac{x+2y}{80\,(\ln t)^{1/3}} + \dfrac{1}{2\,(\ln t)^{1/3}}, & t \in [1, e^2], \\ {}^{c}D_{1^+}^{v(t);\mathbb{G}} y(t) = \dfrac{y-x}{80\,(\ln t)^{1/4}} + \dfrac{1}{4\,(\ln t)^{1/4}}, & t \in [1, e^2], \\ x(1) = 0, \quad x(e^2) = \displaystyle\int_1^{e^2} \frac{y(s)}{30\sqrt{s}}\, ds, \\ y(1) = 0, \quad y(e^2) = \displaystyle\int_1^{e^2} \frac{e^{-s}}{8}\, x(s)\, ds. \end{cases} \tag{4.3}$$

Here $f_1(t,x,y) = (x+2y+40)/\big(80(\ln t)^{1/3}\big)$ and $f_2(t,x,y) = (y-x+20)/\big(80(\ln t)^{1/4}\big)$, so that the $\mathbb{G}^{\delta_i}$-weighted source is constant; the Lipschitz constants are unchanged, $K_1 = \frac{1}{40}$, $K_2 = \frac{1}{80}$, while now $f_1^\star = \frac{1}{2}$, $f_2^\star = \frac{1}{4}$. Consequently the constants of Example 4.2 are preserved, namely $\Phi \approx 2.871$, $\Psi \approx 2.767$, $\kappa^* \approx 0.402 < 1$ and $C_{\mathrm{UH}} \approx 6.45$, and Theorems 3.1–3.3 again guarantee a unique, Ulam–Hyers stable solution—now non-trivial because of the forcing.

We compute it by successive approximations $(x_{n+1}, y_{n+1}) = \mathcal{T}(x_n, y_n)$, where $\mathcal{T}$ is the integral operator of Lemma 3.1 and $(x_0, y_0) = (0,0)$. By the estimate in the proof of Theorem 3.2, $\mathcal{T}$ is a contraction with ratio κ^* on $\big(C(J,\mathbb{R})^2, \|\cdot\|_\infty\big)$, so the iterates converge geometrically to the unique solution. The $\mathbb{G}$-fractional integrals are evaluated by product integration in the variable $\xi = \mathbb{G}(t) = \ln t$, which resolves both the kernel singularity $(\mathbb{G}(t) - \mathbb{G}(s))^{u_k - 1}$ and the weighted endpoint singularity $\mathbb{G}^{-\delta_i}(s)$ of f_i. The scheme reaches an increment below 10^{-10} in 8 iterations; the computed components and the geometric decay of the iteration error are shown in Figure 3. The error stays below the $(\kappa^*)^n$ envelope predicted by the contraction, the solution satisfies $x(1) = y(1) = 0$ and the nonlocal conditions at $t = e^2$ to machine precision, and a slope change is visible at the interface $t = e$ where the fractional orders jump.

5. Conclusion

We have established a rigorous analytical framework for coupled $\mathbb{G}$-Caputo variable-order fractional boundary value problems (1.1) under nonlocal integral boundary constraints.

Our approach rests on three main ingredients. First, the variable-order problem is transformed into a system of coupled integral equations with piecewise-constant orders, including the correction term required to satisfy the nonlocal boundary conditions at a_2. Second, the existence of solutions is established by means of Darbo's fixed-point theorem, using the explicit condensing constant κ^*. Finally, Ulam–Hyers stability is proved, and an explicit stability constant is derived.

A key methodological contribution is the *weighted-singularity framework* (hypotheses (H1)–(H2)), which allows the nonlinearities f_i to develop integrable singularities of order $\mathbb{G}^{-\delta_i}$ at the left endpoint. This substantially extends the scope of the standard bounded-growth hypotheses used in the existing scalar $\mathbb{G}$-Caputo variable-order literature [8–11].

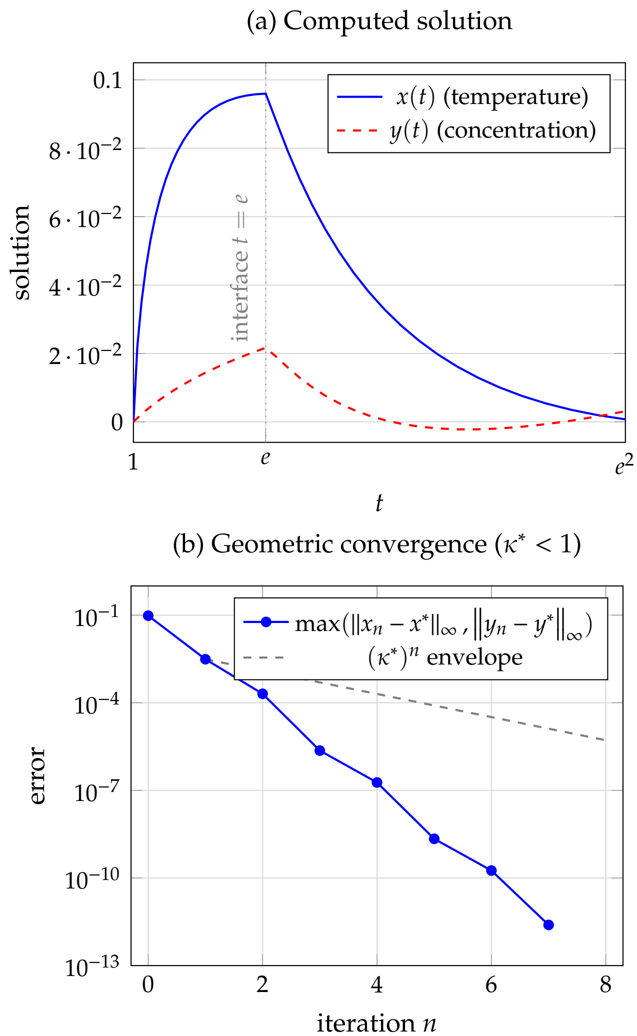

FIGURE 3. Numerical experiment of Example 4.3 for the coupled $\mathbb{G}$-Caputo variable-order thermo-diffusion system (4.3). (a) The temperature $x(t)$ and concentration $y(t)$ obtained by successive approximations; both vanish at $t = 1$, satisfy the nonlocal integral conditions at $t = e^2$, and exhibit a slope change at the interface $t = e$ where the orders jump from $(1.2, 1.25)$ to $(1.5, 1.6)$. (b) The iteration error decays geometrically, remaining below the $(\kappa^*)^n$ envelope with $\kappa^* \approx 0.402$ predicted by the contraction estimate, in agreement with Theorems 3.1–3.2.

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