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On Two Broad Subfamilies of Liouville–Caputo-Type Fractional Derivatives Governed by Gregory Numbers

Feras Yousef[1], Tariq Al-Hawary[2], Jamal Salah[3,*], Basem Aref Frasin[4], Mohamed Illafe[5]

[1]*Department of Mathematics, The University of Jordan, Amman 11942, Jordan*
[2]*Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan*
[3]*College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box 42, Ibra 400, Oman*
[4]*Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan*
[5]*School of Engineering, Math, & Technology, Navajo Technical University, Crownpoint, NM 87313, USA*

[*]*Corresponding author:* damous73@yahoo.com

Abstract. In this paper, by employing Liouville–Caputo-type fractional derivatives and subordination to the generating function of the Gregory coefficients, we introduce two comprehensive subfamilies, denoted by $E_F(\Psi_u, \rho, \daleth, \gimel)$ and $C_F(\Psi_u, \phi)$, within the family of bi-univalent functions, and demonstrate that these subfamilies are non-empty. We establish estimates for the initial Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as for the Fekete–Szegö functional associated with functions belonging to these classes. The originality of the proofs and the resulting characterizations are expected to inspire further investigation into these analytic bi-univalent function subfamilies.

1. Introduction and preliminaries

Let $\mathcal{H}$ be the family of analytic functions A on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $A(0) = A'(0) - 1 = 0$. Consequently, every function $A \in \mathcal{H}$ has the form:

$$A(z) = z + \sum_{u=2}^{\infty} a_u z^u, \quad (z \in \Delta). \tag{1.1}$$

Furthermore, let $\mathcal{D}$ be the subfamily of $\mathcal{H}$ comprising all univalent functions (for details on the family $\mathcal{D}$, see [1–3]).

The analytic function A is said to be subordinate to B, denoted by $A \prec B$, if there exists a Schwarz function w, analytic in Δ, with $|w(z)| < 1$ and $w(0) = 0$, such that

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$$A(z) = B(w(z)), \text{ for all } z \in \Delta.$$

Moreover, if B is univalent in Δ, then the subordination of A to B holds if and only if $A(0) = B(0)$ and $A(\Delta) \subset B(\Delta)$ (see [4,5]).

Let $A(z)$ be an analytic and univalent function. The inverse function $Q(z) = A^{-1}(z)$ is defined by (see [6]):

$$A^{-1}(A(z)) = z \quad (z \in \Delta)$$

and

$$A\left(A^{-1}(w)\right) = w \quad \left(|w| < r_0(A); r_0(A) \geq \frac{1}{4}\right).$$

The inverse function is actually

$$A^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (a_4 - 5a_2a_3 + 5a_2^3)w^4 + \cdots. \tag{1.2}$$

If $A \in \mathcal{H}$ and both A and A^{-1} are univalent in Δ, then the function A given by the equation (1.1) belongs to the family F, where F denotes the family of bi-univalent functions in Δ. For more details on the family F, see [7–10].

By examining the family F, Lewin [11] established that $a_2 < 1.51$. Subsequently, Brannan and Clunie [12] conjectured that $a_2 < \sqrt{2}$, while Tan [13] obtained the estimate $|a_2| < 1.485$. Later, Netanyahu [14] provided a sharp result, proving that $\max a_2 = \frac{4}{3}$. Despite these significant contributions, determining accurate bounds for the higher-order coefficients a_u (for $u \geq 3, u \in \mathbb{N}$) remains an open and challenging problem in the theory of bi-univalent functions.

It is well known that the Fekete-Szegö problems relate to the function coefficients in $\mathcal{D}$. Fekete and Szegö [15] were the first to do so; if $A \in F$, who said that

$$|a_3 - \xi a_2^2| \leq 1 + 2e^{-2\xi/(1-m)}, \quad \xi \in \mathbb{R}.$$

Gregory coefficients Π_u, are the numbers $\frac{1}{2}, \frac{-1}{12}, \frac{1}{24}, \frac{-19}{720}, \frac{3}{160}, \frac{-863}{60.480}, \cdots$. They can be found in the reciprocal logarithm Maclaurin series expansion (see [16])

$$\frac{z}{\log(z+1)} = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \frac{1}{24}z^3 - \frac{19}{720}z^4 + \frac{3}{160}z^5 - \frac{863}{60.480}z^6 + \cdots.$$

These numbers were first introduced by James Gregory in 1670, and many mathematicians have since revived them in the writings of modern authors.

The generating function of the Gregory coefficients Π_u given by (see [17])

$$\Xi(z) = \frac{z}{\log(z+1)} = \sum_{u=0}^{\infty} \Pi_u z^u = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \frac{1}{24}z^3 - \frac{19}{720}z^4 + \frac{3}{160}z^5 - \frac{863}{60.480}z^6 + \cdots, \quad z \in \Delta. \tag{1.3}$$

Clearly, for some values of $u \in \mathbb{N}$

$$\Pi_0 = 1,\ \Pi_1 = \frac{1}{2},\ \Pi_2 = \frac{-1}{12},\ \Pi_3 = \frac{1}{24},\ \Pi_4 = \frac{-19}{720},\ \Pi_5 = \frac{3}{160},\ \Pi_6 = \frac{-863}{60.480}.$$

The following operator $R^{\epsilon}: \mathcal{D} \rightarrow \mathcal{D}$ was dfined by Srivastava and Owa [18] as

$$R^{\epsilon}D(z) = \urcorner(2-\epsilon)z^{\epsilon}\mathcal{I}_z^{\epsilon}A(z) = z + \sum_{u=2}^{\infty} \frac{\urcorner(u+1)\urcorner(2-\epsilon)}{\urcorner(u+1-\epsilon)} a_u z^u = z + \sum_{u=2}^{\infty} \Lambda(u,\epsilon) a_u z^u,$$

where $\epsilon \in \mathbb{R}$; $\epsilon \neq 2, 3, 4, \cdots$.

Definition 1.1. *Let $A \in \mathcal{H}$ in a simply connected region contains the origin of the z-plane, and the fractional integral (FI) of A of order η is as follows:*

$$\mathcal{I}_z^{-\eta}A(z) = \frac{1}{\urcorner(\eta)} \int_0^z \frac{A(\chi)}{(z-\chi)^{1-\eta}} d\chi, \quad \eta > 0,$$

and the fractional derivatives (FD) of A of order η is given as

$$\mathcal{I}_z^{\eta}A(z) = \frac{1}{\urcorner(1-\eta)} \frac{d}{dz} \int_0^z \frac{A(\chi)}{(z-\chi)^{\eta}} d\chi, \quad 0 \leq \eta < 1$$

where $(z-\chi)^{\eta-1}$ and $(z-\chi)^{-\eta}$ is removed to be $log(z-\chi)$ real, when $z > \chi$.

Definition 1.2. *The FD of A of order $n+\eta$ is*

$$\mathcal{I}_z^{-n+\eta}A(z) = \frac{d^n}{dz^n}\mathcal{I}_z^{\eta}A(z), \quad 0 \leq \eta < 1, n \in \mathbb{N}_0.$$

The article analyzes the fractional-order derivative according to Liouville–Caputo's definition [19] with the presumption that:

$$\mathcal{I}^{\eta}A(z) = \frac{1}{\urcorner(u-\eta)} \int_a^z \frac{A^{(u)}(\chi)}{(u-\chi)^{\eta+1-u}} d\chi$$

where $u-1 < \mathrm{Re}(\eta) \leq u,\ u \in \mathbb{N}$, and $\eta \in \mathbb{C}$, η is the initial value of A.

The generalization operator of Libera integral operator [20] and Salagean derivative operator [21], was given by Owa [22]

$$J^{\varsigma}A(z) = \urcorner(2-\varsigma)z^{\varsigma}\mathcal{I}^{\eta}A(z) = z + \sum_{u=2}^{\infty} a_u\, z^u\,, \quad \varsigma \in \mathbb{R}.$$

Lately, Salah et al. in [23], gave

$$K_{\eta}^{\varsigma}A(z) = \frac{\urcorner(2+\varsigma-\eta)}{\urcorner(\varsigma-\eta)} z^{\eta-\varsigma} \int_0^z \frac{J^{\varsigma}A(\chi)}{(z-\chi)^{\eta+1-\varsigma}} d\chi$$

where $\varsigma \in \mathbb{R}$ and $(\varsigma - 1 < \eta < \varsigma < 2)$. Simple direct calculations for $A \in \mathcal{H}$ yield

$$K_{\eta}^{\varsigma}A(z) = z + \sum_{u=2}^{\infty} \Psi_u\, a_u\, z^u\,, \ z \in \Delta$$

where

$$\Psi_u = \frac{\urcorner(2+\varsigma-\eta)\urcorner(2-\varsigma)(\urcorner(u+1))^2}{\urcorner(u-\varsigma+1)\urcorner(u+\varsigma-\eta+1)}. \tag{1.4}$$

Further, note that $K_0^0A(z) = A(z)$ and $K_1^1A(z) = zA'(z)$, see [24].

$$K_\eta^\varsigma A(z) = z + \Psi_2 a_2 z^2 + \Psi_3 a_3 z^3 + \cdots \quad z \in \Delta \tag{1.5}$$

$$K_\eta^\varsigma Q(w) = w - \Psi_2 a_2 w^2 + \Psi_3 (2a_2^2 - a_3) w^3 + \cdots \quad w \in \Delta. \tag{1.6}$$

The purpose of this work is to improve the coefficients $|a_2|$, $|a_3|$ and $\left|a_3 - \xi a_2^2\right|$ for certain subfamilies of family F which are defined by Liouville–Caputo-type fractional derivatives subordinate to the generating function of the Gregory coefficients.

2. Coefficient bounds for the subfamily $E_F(\Psi_u, \rho, \daleth, \beth)$

In this section, we examine a subfamily $E_F(\Psi_u, \rho, \daleth, \beth)$ using the Liouville–Caputo-type fractional derivatives and the generating functions of the Gregory coefficients, and obtain the initial coefficients $|a_2|$, $|a_3|$ and Fekete–Szegö inequality $\left|a_3 - \xi a_2^2\right|$.

Lemma 2.1. *([25]) If $\varepsilon \in \mathcal{V}$, then $|a_u| \le 2$, $u \ge 1$ for each u, where ε analytic in Δ and has the form $\varepsilon(z) = 1 + a_1 z + a_2 z^2 + \cdots$, and sharp for all $u \in \mathbb{N}$.*

Lemma 2.2. *([26]) Let $X, Y \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1|, |z_2| < \mu$, then*

$$\left|(X+Y)z_1 + (X-Y)z_2\right| \le \begin{cases} 2|X|\mu & \text{for } |X| \ge |Y|, \\ 2|Y|\mu & \text{for } |X| \le |Y|. \end{cases}$$

Definition 2.1. *Let $\rho \ge 0$, $\beth \ge 1$, $\daleth \in \mathbb{C}$ and $\mathrm{Re}(\daleth) \ge 0$. A function $A \in F$ given by (1.1) belongs to the subfamily $E_F(\Psi_u, \rho, \daleth, \beth)$ if satisfied*

$$(1-\rho)\left(K_\eta^\varsigma A(z)\right)' + \rho\left(\left(K_\eta^\varsigma A(z)\right)'\right)^\beth \left(\frac{K_\eta^\varsigma A(z)}{z}\right)^{\daleth - 1} \prec \Xi(z) \tag{2.1}$$

and

$$(1-\rho)\left(K_\eta^\varsigma Q(w)\right)' + \rho\left(\left(K_\eta^\varsigma Q(w)\right)'\right)^\beth \left(\frac{K_\eta^\varsigma Q(w)}{w}\right)^{\daleth - 1} \prec \Xi(w), \tag{2.2}$$

where $\Xi(z)$ is given by (1.3) and $Q(w) = A^{-1}(w)$.

Remark 2.1. *1) Using specific values for the parameters ρ, $\beth$, $\daleth$, ς and η in Definition 2.1, we get a lot of subfamilies in $\mathcal{H}$ examined by a number of authors.*

2) If we take into $K_\eta^\varsigma A(z)_* = \frac{z}{1-sz}$, $|s| \le 1$, then we can verify that $K_\eta^\varsigma A(z)_* \in E_F(\Psi_u, \rho, \daleth, \beth)$, so the family $E_F(\Psi_u, \rho, \daleth, \beth)$ not empty (Remark 1 in [26]).

Theorem 2.1. *Let $\rho \ge 0$, $\beth \ge 1$, $\daleth \in \mathbb{C}$ and $\mathrm{Re}(\daleth) \ge 0$, if $A \in E_F(\Psi_u, \rho, \daleth, \beth)$. Then*

$$|a_2| \le \min\left\{\frac{1}{2\Psi_2\left|\rho\,(2\beth + \daleth - 3) + 2\right|},\right.$$

$$\left.\frac{1}{\Psi_2\sqrt{\left|\rho\left((\daleth+2)(\daleth-3) + 4\beth(\daleth-1) + 2\beth(2\beth+1)\right) + \frac{14}{3}\left(\rho\,(2\beth + \daleth - 3) + 2\right)^2 + 6\right|}}\right\},$$

$$|a_3| \le \min\left\{\frac{1}{4\Psi_3\left|\rho\left(2\gimel+\daleth-3\right)+2\right|^2}+\frac{1}{2\Psi_3\left|\rho\left(3\gimel+\daleth-4\right)+3\right|},\right.$$
$$\frac{1}{\Psi_3\left|\rho\left(\left(\daleth+2\right)\left(\daleth-3\right)+4\gimel\left(\daleth-1\right)+2\gimel(2\gimel+1)\right)+\frac{14}{3}\left(\rho\left(2\gimel+\daleth-3\right)+2\right)^2+6\right|}$$
$$\left.+\frac{1}{2\Psi_3\left|\rho\left(3\gimel+\daleth-4\right)+3\right|}\right\}$$

and

$$\left|a_3-\xi a_2^2\right| \le \begin{cases} \frac{1}{2\Psi_3(\rho(3\gimel+\daleth-4)+3)} & \text{if } \left|M(\xi)\right| \le \frac{1}{8\Psi_3(\rho(3\gimel+\daleth-4)+3)}, \\ 4\left|M(\xi)\right| & \text{if } \left|M(\xi)\right| \ge \frac{1}{8\Psi_3(\rho(3\gimel+\daleth-4)+3)}. \end{cases}$$

where

$$M(\xi)=\frac{\frac{\Psi_2^2}{\Psi_3}-\xi}{4\Psi_2^2\left[\rho\left(\left(\daleth+2\right)\left(\daleth-3\right)+4\gimel\left(\daleth-1\right)+2\gimel(2\gimel+1)\right)+\frac{14}{3}\left(\rho\left(2\gimel+\daleth-3\right)+2\right)^2+6\right]}.$$

Proof. Let $A \in E_F(\Psi_u, \rho, \daleth, \gimel)$. From (2.1) and (2.2), there are analytic functions C and r such that $C(0)=r(0)=0$, $|C(z)|<1$ and $|r(w)|<1$, while

$$(1-\rho)\left(K_\eta^\varsigma A(z)\right)'+\rho\left(\left(K_\eta^\varsigma A(z)\right)'\right)^{\gimel}\left(\frac{K_\eta^\varsigma A(z)}{z}\right)^{\daleth-1}=\Xi(C(z)),\ \ z\in\Delta \tag{2.3}$$

and

$$(1-\rho)\left(K_\eta^\varsigma Q(w)\right)'+\rho\left(\left(K_\eta^\varsigma Q(w)\right)'\right)^{\gimel}\left(\frac{K_\eta^\varsigma Q(w)}{w}\right)^{\daleth-1}=\Xi(r(w)),\ \ \ w\in\Delta. \tag{2.4}$$

So

$$t(z)=\frac{1+C(z)}{1-C(z)}=1+s_1z+s_2z^2+\cdots\in\mathcal{V},$$

hence

$$C(z)=\frac{s_1}{2}z+\frac{1}{2}\left(s_2-\frac{s_1^2}{2}\right)z^2+\frac{1}{2}\left(s_3-s_1s_2+\frac{s_1^3}{4}\right)z^3+\cdots$$

and

$$\Xi(C(z))=1+\frac{s_1}{4}z+\frac{1}{48}\left(12s_2-7s_1^2\right)z^2+\frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right)z^3+\cdots,\ z\in\Delta.$$

Also, the function

$$u(w)=\frac{1+r(w)}{1-r(w)}=1+b_1w+b_2w^2+\cdots\in\mathcal{V},$$

$$r(w)=\frac{b_1}{2}w+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)w^2+\frac{1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)w^3+\cdots$$

and

$$\Xi(r(w))=1+\frac{b_1}{4}w+\frac{1}{48}\left(12b_2-7b_1^2\right)w^2+\frac{1}{192}\left(17b_1^3-56b_1b_2+48b_3\right)w^3+\cdots,\ w\in\Delta.$$

Consequently, we get

$$\begin{aligned}&(1-\rho)\left(K_{\eta}^{\varsigma}A(z)\right)'+\rho\left(\left(K_{\eta}^{\varsigma}A(z)\right)'\right)^{\gimel}\left(\frac{K_{\eta}^{\varsigma}A(z)}{z}\right)^{\daleth-1}\\&=1+\frac{s_1}{4}z+\frac{1}{48}\left(12s_2-7s_1^2\right)z^2+\frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right)z^3+\cdots,\ z\in\Delta.\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&(1-\rho)\left(K_{\eta}^{\varsigma}Q(w)\right)'+\rho\left(\left(K_{\eta}^{\varsigma}Q(w)\right)'\right)^{\gimel}\left(\frac{K_{\eta}^{\varsigma}Q(w)}{w}\right)^{\daleth-1}\\&=1+\frac{b_1}{4}w+\frac{1}{48}\left(12b_2-7b_1^2\right)w^2+\frac{1}{192}\left(17b_1^3-56b_1b_2+48b_3\right)w^3+\cdots,\ w\in\Delta.\end{aligned}\tag{2.6}$$

Comparing the coefficients in the equations (2.5) and (2.6), we get

$$\left(\rho\left(2\gimel+\daleth-3\right)+2\right)\Psi_2a_2=\frac{s_1}{4},\tag{2.7}$$

$$\begin{aligned}&\left[\rho\left(\frac{\left(\daleth-1\right)\left(\daleth-2\right)}{2}+2\gimel\left(\daleth-1\right)+2\gimel(\gimel-1)\right)\right]\Psi_2^2a_2^2+\left[\rho\left(3\gimel+\daleth-4\right)+3\right]\Psi_3a_3\\&=\frac{1}{48}\left(12s_2-7s_1^2\right),\end{aligned}\tag{2.8}$$

$$-\left(\rho\left(2\gimel+\daleth-3\right)+2\right)\Psi_2a_2=\frac{b_1}{4},\tag{2.9}$$

and

$$\begin{aligned}&\left[\rho\left(\frac{\left(\daleth-2\right)\left(\daleth+3\right)}{2}+2\gimel\left(\daleth-1\right)+2\gimel(\gimel+2)-4\right)+6\right]\Psi_2^2a_2^2-\left[\rho\left(3\gimel+\daleth-4\right)+3\right]\Psi_3a_3\\&=\frac{1}{48}\left(12b_2-7b_1^2\right).\end{aligned}\tag{2.10}$$

From (2.7) and (2.9) it follows that

$$s_1=-b_1\tag{2.11}$$

and

$$s_1^2+b_1^2=32\left(\rho\left(2\gimel+\daleth-3\right)+2\right)^2\Psi_2^2a_2^2.\tag{2.12}$$

If we add the equations (2.8) and (2.10), we get

$$\begin{aligned}&\left[\rho\left(\left(\daleth+2\right)\left(\daleth-3\right)+4\gimel\left(\daleth-1\right)+2\gimel(2\gimel+1)\right)+6\right]\Psi_2^2a_2^2\\&=\frac{1}{4}\left(s_2+b_2\right)-\frac{7}{48}\left(s_1^2+b_1^2\right).\end{aligned}\tag{2.13}$$

Substituting the value of $s_1^2+b_1^2$ from (2.12) in (2.13), we get

$$\begin{aligned}&\left\{\left[\rho\left(\left(\daleth+2\right)\left(\daleth-3\right)+4\gimel\left(\daleth-1\right)+2\gimel(2\gimel+1)\right)+6\right]+\frac{14}{3}\left(\rho\left(2\gimel+\daleth-3\right)+2\right)^2\right\}\Psi_2^2a_2^2\\&=\frac{1}{4}\left(s_2+b_2\right).\end{aligned}\tag{2.14}$$

Using the triangle inequality and Lemma 2.1 for the equations (2.7) and (2.14), we get, respectively

$$|a_2| \leq \frac{1}{2\Psi_2 \left|\rho\left(2\gimel + \daleth - 3\right) + 2\right|}$$

and

$$|a_2| \leq \frac{1}{\Psi_2 \sqrt{\left|\rho\left(\left(\daleth + 2\right)\left(\daleth - 3\right) + 4\gimel\left(\daleth - 1\right) + 2\gimel(2\gimel + 1)\right) + \frac{14}{3}\left(\rho\left(2\gimel + \daleth - 3\right) + 2\right)^2 + 6\right|}}.$$

Moreover, if we subtract (2.10) from (2.8), we get

$$2\left[\rho\left(3\gimel + \daleth - 4\right) + 3\right]\left(\Psi_3 a_3 - \Psi_2^2 a_2^2\right) = \frac{1}{4}\left(s_2 - b_2\right) - \frac{7}{48}\left(s_1^2 - b_1^2\right). \tag{2.15}$$

Then, in view of (2.11), last equation becomes

$$\Psi_3 a_3 = \Psi_2^2 a_2^2 + \frac{s_2 - b_2}{8\rho\left(3\gimel + \daleth - 4\right) + 24}. \tag{2.16}$$

In view of (2.7), the above equation becomes

$$\Psi_3 a_3 = \frac{s_1^2}{16\left(\rho\left(2\gimel + \daleth - 3\right) + 2\right)^2} + \frac{s_2 - b_2}{8\rho\left(3\gimel + \daleth - 4\right) + 24}. \tag{2.17}$$

By applying Lemma 2.1 and the triangle inequality for (2.17), we get

$$|a_3| \leq \frac{1}{4\Psi_3 \left|\rho\left(2\gimel + \daleth - 3\right) + 2\right|^2} + \frac{1}{2\Psi_3 \left|\rho\left(3\gimel + \daleth - 4\right) + 3\right|}.$$

And using the assertion (2.14) with (2.16), it follows that

$$|a_3| \leq \frac{1}{\Psi_3 \left|\rho\left(\left(\daleth + 2\right)\left(\daleth - 3\right) + 4\gimel\left(\daleth - 1\right) + 2\gimel(2\gimel + 1)\right) + \frac{14}{3}\left(\rho\left(2\gimel + \daleth - 3\right) + 2\right)^2 + 6\right|}$$
$$+ \frac{1}{\Psi_3 \left|2\rho\left(3\gimel + \daleth - 4\right) + 6\right|}.$$

Also, from (2.16) we get

$$\begin{aligned}
a_3 - \xi a_2^2 &= \frac{s_2 - b_2}{8\Psi_3\left(\rho\left(3\gimel + \daleth - 4\right) + 3\right)} + \left(\frac{\Psi_2^2}{\Psi_3} - \xi\right)a_2^2 \\
&= \frac{s_2 - b_2}{8\Psi_3\left(\rho\left(3\gimel + \daleth - 4\right) + 3\right)} + \\
&\frac{\left(\frac{\Psi_2^2}{\Psi_3} - \xi\right)\left(s_2 + b_2\right)}{4\Psi_2^2\left[\rho\left(\left(\daleth + 2\right)\left(\daleth - 3\right) + 4\gimel\left(\daleth - 1\right) + 2\gimel(2\gimel + 1)\right) + \frac{14}{3}\left(\rho\left(2\gimel + \daleth - 3\right) + 2\right)^2 + 6\right]} \\
&= \left(M(\xi) + \frac{1}{8\Psi_3\left(\rho\left(3\gimel + \daleth - 4\right) + 3\right)}\right)s_2 + \left(M(\xi) - \frac{1}{8\Psi_3\left(\rho\left(3\gimel + \daleth - 4\right) + 3\right)}\right)b_2,
\end{aligned}$$

where

$$M(\xi) = \frac{\frac{\Psi_2^2}{\Psi_3} - \xi}{4\Psi_2^2\left[\rho\left(\left(\daleth + 2\right)\left(\daleth - 3\right) + 4\gimel\left(\daleth - 1\right) + 2\gimel(2\gimel + 1)\right) + \frac{14}{3}\left(\rho\left(2\gimel + \daleth - 3\right) + 2\right)^2 + 6\right]}.$$

Then, using Lemma 2.1 for s_2 and b_2, and Lemma 2.2, we get

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{1}{2\Psi_3(\rho(3ℷ+ℸ-4)+3)} & \text{if } \left|M(\xi)\right| \leq \frac{1}{8\Psi_3(\rho(3ℷ+ℸ-4)+3)}, \\ \\ 4\left|M(\xi)\right| & \text{if } \left|M(\xi)\right| \geq \frac{1}{8\Psi_3(\rho(3ℷ+ℸ-4)+3)}. \end{cases}$$

Which completes the proof. □

Fixing $\rho = \varsigma = \eta = 0$ or $\varsigma = \eta = 0$, $\rho = ℷ = ℸ = 1$ in Theorem 2.1, we get the following Corollary, which is due to [28].

Corollary 2.1. *If* $A \in E_F(\Psi_u, \rho, ℸ, ℷ)$. *Then*

$$|a_2| \leq \sqrt{\frac{3}{74}}, \quad |a_3| \leq \frac{23}{111}$$

and

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{1}{6} & \text{if } \left|M(\xi)\right| \leq \frac{1}{24}, \\ \\ 4\left|M(\xi)\right| & \text{if } \left|M(\xi)\right| \geq \frac{1}{24}. \end{cases}$$

3. Coefficient bounds of the subfamily $C_F(\Psi_u, \phi)$

In this section, we examine a subfamily $C_F(\Psi_u, \phi)$ of Liouville–Caputo-type fractional derivatives and the generating functions of the Gregory coefficients, and obtain initial coefficients $|a_2|$, $|a_3|$ and Fekete–Szegö inequality $\left|a_3 - \varrho a_2^2\right|$.

We need the following lemma to demonstrate the proof of results regarding the subfamily $C_F(\Psi_u, \phi)$.

Lemma 3.1. *([27]) If* $\varepsilon(z) = 1 + a_1 z + a_2 z^2 + \cdots \in \mathcal{V}$, $z \in \Delta$, *then there exist some* α, δ *with* $|\alpha|, |\delta| \leq 1$, *such that*

$$2s_2 = s_1^2 + \alpha(4 - s_1^2) \text{ and } 4s_3 = s_1^3 + 2s_1\alpha(4 - s_1^2) - (4 - s_1^2)s_1\alpha^2 + 2(4 - s_1^2)(1 - |\alpha|^2)\delta.$$

Definition 3.1. *A function* $A \in F$ *given by (1.1) is be in the subfamily* $C_F(\Psi_u, \phi)$ *where* $\phi \in (-\pi, \pi]$, *if satisfied*

$$\left(K_\eta^\varsigma A(z)\right)' + \left(\frac{e^{i\phi} + 1}{2}\right) z \left(K_\eta^\varsigma A(z)\right)'' \prec \Xi(z) \tag{3.1}$$

and

$$\left(K_\eta^\varsigma Q(w)\right)' + \left(\frac{e^{i\phi} + 1}{2}\right) w \left(K_\eta^\varsigma Q(w)\right)'' \prec \Xi(w), \tag{3.2}$$

where $\Xi(z)$ *is given by (1.3) and* $Q(w) = A^{-1}(w)$.

Remark 3.1. *1) We obtain several subfamilies of* $\mathcal{H}$ *by utilizing particular values for* $\phi \in (-\pi, \pi]$ *in Definition 3.1.*

2) If we take $K_\eta^\varsigma A(z)_* = \frac{z}{1-sz}$, $|s| \leq 1$, *then we can verify that* $K_\eta^\varsigma A(z)_* \in C_F(\Psi_u, \phi)$, *so the subfamily* $C_F(\Psi_u, \phi)$ *not empty (Remark 5 in [26]).*

Theorem 3.1. *Let $\phi \in (-\pi, \pi]$, if $A \in C_F(\Psi_u, \phi)$. Then*

$$|a_2| \leq \min\left\{ \frac{1}{\sqrt{2}\Psi_2 \left|e^{i\phi}+3\right|}, \frac{1}{\Psi_2\sqrt{6\left|e^{i\phi}+2\right| + \frac{7}{3}\left|e^{i\phi}+3\right|^2}} \right\},$$

$$|a_3| \leq \left\{ \frac{1}{2\Psi_3\left|e^{i\phi}+3\right|^2} + \frac{1}{6\Psi_3\left|e^{i\phi}+2\right|}, \frac{1}{\Psi_3\left|33+20e^{i\phi}+\frac{7}{3}e^{i2\phi}\right|} + \frac{1}{6\Psi_3\left|e^{i\phi}+2\right|} \right\}$$

and

$$\left|a_3 - \varrho a_2^2\right| \leq \begin{cases} \frac{1}{6\Psi_3\left|e^{i\phi}+2\right|} & \text{if } \left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| \leq \frac{\Psi_2^2\left|e^{i\phi}+3\right|^2}{3\Psi_3\left|e^{i\phi}+2\right|}, \\ \frac{\left|\frac{\Psi_2^2}{\Psi_3}-\varrho\right|}{2\Psi_2^2\left|e^{i\phi}+3\right|^2} & \text{if } \left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| \geq \frac{\Psi_2^2\left|e^{i\phi}+3\right|^2}{3\Psi_3\left|e^{i\phi}+2\right|}. \end{cases} \tag{3.3}$$

Proof. Let $A \in C_F(\Psi_u, \phi)$. From (3.1) and (3.2), there are C and r analytic functions, so that $C(0) = r(0) = 0$, $|C(z)|$ and $|r(w)| < 1$, such that

$$\left(K_\eta^\varsigma A(z)\right)' + \left(\frac{e^{i\phi}+1}{2}\right) z \left(K_\eta^\varsigma A(z)\right)'' = \Xi(C(z)), \quad z \in \Delta \tag{3.4}$$

and

$$\left(K_\eta^\varsigma Q(w)\right)' + \left(\frac{e^{i\phi}+1}{2}\right) w \left(K_\eta^\varsigma Q(w)\right)'' = \Xi(r(w)), \quad w \in \Delta.$$

Thus we have

$$\begin{aligned} &\left(K_\eta^\varsigma A(z)\right)' + \left(\frac{e^{i\phi}+1}{2}\right) z \left(K_\eta^\varsigma A(z)\right)'' \\ &= 1 + \frac{s_1}{4}z + \frac{1}{48}\left(12s_2 - 7s_1^2\right)z^2 + \frac{1}{192}\left(17s_1^3 - 56s_1s_2 + 48s_3\right)z^3 + \cdots, \; z \in \Delta. \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\left(K_\eta^\varsigma Q(w)\right)' + \left(\frac{e^{i\phi}+1}{2}\right) w \left(K_\eta^\varsigma Q(w)\right)'' \\ &= 1 + \frac{b_1}{4}w + \frac{1}{48}\left(12b_2 - 7b_1^2\right)w^2 + \frac{1}{192}\left(17b_1^3 - 56b_1b_2 + 48b_3\right)w^3 + \cdots, \; w \in \Delta. \end{aligned} \tag{3.6}$$

Comparing the coefficients in equations (3.5) and (3.6), we get

$$\left(e^{i\phi}+3\right)\Psi_2 a_2 = \frac{s_1}{4}, \tag{3.7}$$

$$3\left(e^{i\phi}+2\right)\Psi_3 a_3 = \frac{1}{48}\left(12s_2 - 7s_1^2\right), \tag{3.8}$$

$$-\left(e^{i\phi}+3\right)\Psi_2 a_2 = \frac{b_1}{4}, \tag{3.9}$$

and

$$3\left(e^{i\phi}+2\right)\left(2\Psi_2^2 a_2^2 - \Psi_3 a_3\right) = \frac{1}{48}\left(12b_2 - 7b_1^2\right). \tag{3.10}$$

From (2.7) and (2.9) it follows that

$$s_1 = -b_1 \tag{3.11}$$

and

$$16\left(e^{i\phi}+3\right)^2 \Psi_2^2 a_2^2 = s_1^2 + b_1^2. \tag{3.12}$$

If we add (3.8) to (3.10), we get

$$6\left(e^{i\phi}+2\right)\Psi_2^2 a_2^2 = \frac{1}{4}\left(s_2+b_2\right) - \frac{7}{48}\left(s_1^2+b_1^2\right). \tag{3.13}$$

Substituting the value of $s_1^2 + b_1^2$ from (3.12) in (3.13), we get

$$\left\{6\left(e^{i\phi}+2\right)+\frac{7}{3}\left(e^{i\phi}+3\right)^2\right\}\Psi_2^2 a_2^2 = \frac{1}{4}\left(s_2+b_2\right) \tag{3.14}$$

Using the triangle inequality and Lemma 2.1 for (3.12) and (3.14), we get, respectively

$$|a_2| \leq \frac{1}{\sqrt{2}\Psi_2\left|e^{i\phi}+3\right|} \quad \text{and} \quad |a_2| \leq \frac{1}{\Psi_2\sqrt{6\left|e^{i\phi}+2\right|+\frac{7}{3}\left|e^{i\phi}+3\right|^2}}.$$

Also, if we subtract (3.10) from (3.8), we get

$$6\left(e^{i\phi}+2\right)\left(\Psi_3 a_3 - \Psi_2^2 a_2^2\right) = \frac{1}{4}\left(s_2-b_2\right) - \frac{7}{48}\left(s_1^2-b_1^2\right). \tag{3.15}$$

Then, in view of (3.11), the equation (3.15) becomes

$$\Psi_3 a_3 = \Psi_2^2 a_2^2 + \frac{s_2-b_2}{24\left(e^{i\phi}+2\right)}. \tag{3.16}$$

The equation (3.16) with equation (3.12) becomes

$$a_3 = \frac{s_1^2+b_1^2}{16\Psi_3\left(e^{i\phi}+3\right)^2} + \frac{s_2-b_2}{24\Psi_3\left(e^{i\phi}+2\right)}.$$

Using the triangle inequality and Lemma 2.1 for the last relation, we get

$$|a_3| \leq \frac{1}{2\Psi_3\left|e^{i\phi}+3\right|^2} + \frac{1}{6\Psi_3\left|e^{i\phi}+2\right|}.$$

Similarly, using of (3.14) in (3.16) follows that

$$|a_3| \leq \frac{1}{\Psi_3\left|33+20e^{i\phi}+\frac{7}{3}e^{i2\phi}\right|} + \frac{1}{6\Psi_3\left|e^{i\phi}+2\right|}.$$

Also, using (3.11) and (3.12), we get $a_2^2 = \frac{s_1^2}{8\Psi_2^2\left(e^{i\phi}+3\right)^2}$. Thus, from (3.16), we have

$$\begin{aligned} a_3 - \varrho a_2^2 &= \frac{s_2-b_2}{24\Psi_3\left(e^{i\phi}+2\right)} + \left(\frac{\Psi_2^2}{\Psi_3}-\varrho\right)a_2^2 \\ &= \frac{s_2-b_2}{24\Psi_3\left(e^{i\phi}+2\right)} + \left(\frac{\Psi_2^2}{\Psi_3}-\varrho\right)\frac{s_1^2}{8\Psi_2^2\left(e^{i\phi}+3\right)^2}. \end{aligned}$$

From Lemma 3.1, we have $2s_2 = s_1^2 + \alpha(4 - s_1^2)$ and $2b_2 = b_1^2 + \delta(4 - b_1^2)$, $|\alpha|, |\delta| \le 1$, and using (3.11), we get

$$s_2 - b_2 = \frac{4 - s_1^2}{2}(\alpha - \delta),$$

and thus

$$a_3 - \varrho a_2^2 = \frac{(4 - s_1^2)(\alpha - \delta)}{48\Psi_3\,(e^{i\phi} + 2)} + \frac{(\frac{\Psi_2^2}{\Psi_3} - \varrho)s_1^2}{8\Psi_2^2\,(e^{i\phi} + 3)^2}.$$

Using the triangle inequality, taking $|\alpha| = \mu$, $|\delta| = y$, μ, $y \in [0,1]$, and assuming that $s_1 \in \mathbb{R}$, $s_1 = d \in [0,2]$; thus, we have

$$\left|a_3 - \varrho a_2^2\right| \le \frac{(4 - d^2)(\mu + y)}{48\Psi_3\left|e^{i\phi} + 2\right|} + \frac{\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| d^2}{8\Psi_2^2\left|e^{i\phi} + 3\right|^2}. \tag{3.17}$$

Assume that: $O(d) = \frac{\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| d^2}{8\Psi_2^2\left|e^{i\phi}+3\right|^2} \ge 0$ and $B(d) = \frac{4-d^2}{48\Psi_3\left|e^{i\phi}+2\right|} \ge 0$, the relation (3.17) can be written as

$$\left|a_3 - \varrho a_2^2\right| \le O(d) + B(d)(\mu + y) =: \mathcal{L}(\mu, y),\ \mu, y \in [0,1].$$

Therefore

$$\max\{\mathcal{L}(\mu, y):\ \mu, y \in [0,1]\} = \mathcal{L}(1,1) = O(d) + 2B(d) =: P(d),\ d \in [0,2]$$

where

$$P(d) = \frac{1}{8\Psi_2^2\left|e^{i\phi} + 3\right|^2}\left(\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| - \frac{\Psi_2^2\left|e^{i\phi} + 3\right|^2}{3\Psi_3\left|e^{i\phi} + 2\right|}\right)d^2 + \frac{1}{6\Psi_3\left|e^{i\phi} + 2\right|}.$$

Since

$$P'(d) = \frac{1}{4\Psi_2^2\left|e^{i\phi} + 3\right|^2}\left(\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| - \frac{\Psi_2^2\left|e^{i\phi} + 3\right|^2}{3\Psi_3\left|e^{i\phi} + 2\right|}\right)d,$$

it is clear that $P'(d) \le 0$ iff $\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| \le \frac{\Psi_2^2\left|e^{i\phi}+3\right|^2}{3\Psi_3\left|e^{i\phi}+2\right|}$; hence, the function P is a decreasing on $[0,2]$; therefore

$$\max\{P(d):\ d \in [0,2]\} = P(0) = \frac{1}{6\Psi_3\left|e^{i\phi} + 2\right|}.$$

Also, $P'(d) \ge 0$ iff $\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right| \ge \frac{\Psi_2^2\left|e^{i\phi}+3\right|^2}{3\Psi_3\left|e^{i\phi}+2\right|}$; So, P is an increasing function over $[0,2]$, therefore

$$\max\{P(d):\ d \in [0,2]\} = P(2) = \frac{\left|\frac{\Psi_2^2}{\Psi_3} - \varrho\right|}{2\Psi_2^2\left|e^{i\phi} + 3\right|^2}$$

and the estimation (3.3) has been validated. □

Remark 3.2. *For specific values for ρ, $\daleth$, $\gimel$, ς, and η in Theorem 2.1, and for ϕ, ς and η in Theorem 3.1, we have a lot of subfamilies of F. Also, There are a lot of functions $\Xi(z)$ that may produce a lot of subfamily of function family F. For example, if*

$$\Xi(z) = \frac{1+(1-2\kappa)z}{1-z} = 1 + 2(1-\kappa)z + 2(1-\kappa)z^2 + \cdots \ (0 \le \kappa < 1)$$

or

$$\Xi(z) = \left(\frac{1+z}{1-z}\right)^{\kappa} = 1 + 2\kappa z + 2\kappa^2 z^2 + \cdots \ (0 < \kappa \le 1).$$

Which provides specific instances for our previous results.

Remark 3.3. *Fixing $\phi = \pi$ in Theorem 3.1, we get the Corollary 2.1, which is due to [28].*

Remark 3.4. *Fixing $\varsigma = \eta = 0$ (which is equivalent to $\Psi_u = 1$) in Theorem 2.1 and Theorem 3.1, we get the result due to [29] in Theorem 2.5 and Theorem 3.4, respectively.*

4. Conclusions

In this work, utilizing the Liouville–Caputo-type fractional derivatives and subordinate to the generating function of the Gregory coefficients, we presented two comprehensive subfamilies, $E_F(\Psi_u, \rho, \daleth, \gimel)$ and $C_F(\Psi_u, \phi)$ for the family of bi-univalent functions. We estimate the initial coefficients $|a_2|$ and $|a_3|$ and the Fekete–Szegö issues for the functions in these subfamilies, and provide justification that these subfamilies are non-empty. Additionally, additional corollaries can be made regarding the parameter selection for Liouville–Caputo-type fractional derivatives, which include a large number of fractional derivatives and integral operators. The novelty of the characterizations and the proofs in this study might stimulate further study of such similarly specified subfamilies of analytic bi-univalent functions.

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