

## Interval-Valued Intuitionistic $(T, S)$ -Fuzzy Subsemirings in Semirings

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**Abstract.** In this paper, we introduce the notion of an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemirings and investigate some of the properties. Finally, we study under homomorphism, anti-homomorphism and prove some results on these.

### 1. INTRODUCTION

Semirings are algebraic structures with two binary operations, introduced by Vandiver [7]. In more recent times semirings have been deeply studied, especially in relation with applications. Semirings have also been used for studying optimization, graph theory, theory of discrete event, dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes.

In 1965, Zadeh introduced the concept of fuzzy sets. Since then, fuzzy set has been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld and he introduced the notion of fuzzy subgroups. Ahsan initiated the study of fuzzy semirings. The fuzzy algebraic structures play an important role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces.

In 1975, the concept of interval-valued fuzzy sets was introduced by Zadeh [3], as a generalization of the notion of fuzzy sets. Narayanan and Manikanran initiated the notion of interval valued fuzzy ideal in semigroup. In 2008, Shabir and Khan introduced the concept of the interval valued

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fuzzy ideals generated by interval valued fuzzy subset in ordered semigroup. Recently, H. Hedayati initiated interval valued  $(\alpha, \beta)$ -fuzzy bi-ideal of semigroup and obtained some fundamental results. The concept of interval valued fuzzy sets in algebra was initiated by Biswas and further this concept was investigated in [5]. Xueling and Zhan defined interval valued fuzzy h-ideals of hemirings and discussed some results associated with interval valued fuzzy h-ideals of hemirings.

The idea of intuitionistic fuzzy sets (IFS) was first published in 1986 by K.T. Atanassov [4] which is characterized by a membership function and a non-membership function. Later, W.A. Dudek introduced the notion of intuitionistic fuzzy h-ideal and intuitionistic fuzzy left h-ideal in hemiring. The notion of intuitionistic fuzzy sets introduced by Atanassov [2].

In 1989 Atanassov and Gargov initiated the notion of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval valued fuzzy sets. Their concept is characterized by a membership function and a non-membership function whose values are intervals rather than real number. Recently, Gujin and Xiapping, introduced the concept of interval-valued fuzzy subgroups induced by T-triangular norms. Akram and Dar introduced the idea of fuzzy left h-ideal in hemirings with respect to an s-norm. Later, Zhan studied the fuzzy left h-ideals in hemirings with t-norms. There are several authors who applied the theory of intuitionistic  $(S, T)$ -fuzzy sets to different algebraic structures for instance. N. Yaqoob, M. Kaha, M. Akram and A. Khan introduced the interval-valued intuitionistic  $(\bar{S}, \bar{T})$ -fuzzy ideals of ternary semigroups. In later H. Hedayati studied the interval-valued intuitionistic  $(\bar{S}, \bar{T})$ -fuzzy substructures in semirings.

In this paper, we studied the concept of interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of a hemiring and investigated some of the properties. The homomorphism, anti-homomorphism, image and inverse image are investigated properties.

## 2. PRELIMINARIES

Throughout this paper,  $S$  will denote a semiring.

**Definition 2.1.** [2, p.16] A semiring is algebraic system  $(S, +, \cdot)$  consisting of non-empty set  $S$  together with two binary operations on  $S$ , called addition and multiplication, defined on  $S$ , such that the following axioms are satisfied:

- (1)  $(S, +)$  is a semigroup,
- (2)  $(S, \cdot)$  is a semigroup,
- (3)  $(a + b) \cdot c = a \cdot c + b \cdot c$  and  $a \cdot (b + c) = a \cdot b + a \cdot c$ , for all  $a, b, c \in S$ .

**Definition 2.2.** [2, p.16] A non-empty subset  $A$  of a semiring  $(S, +, \cdot)$  is called a subsemiring if  $A$  itself is a semiring under the same operation as in  $S$ .

**Theorem 2.1.** [6, p.20] A non-empty subset  $A$  of semiring  $S$  is a subsemiring of  $S$  if  $a + b \in A$  and  $ab \in A$ , for all  $a, b \in A$ .

**Definition 2.3.** [8, p.1850] Let  $X$  be a non-empty set. A fuzzy subset  $A$  of  $X$  is a function  $A : X \rightarrow [0, 1]$

**Definition 2.4.** [8, p.1850] Let  $S$  be a semiring. A fuzzy subset  $A$  of  $S$  is said to be a fuzzy subsemiring of  $S$  if it satisfies the following conditions:

- (1)  $A(x + y) \geq \min(A(x), A(y))$ ,
- (2)  $A(xy) \geq \min(A(x), A(y))$ , for all  $x, y \in S$ .

**Definition 2.5.** [8, p.1850] Let  $S$  be a semiring. A fuzzy subset  $A$  of  $S$  is said to be an anti-fuzzy subsemiring of  $S$  if it satisfies the following conditions:

- (1)  $A(x + y) \leq \max(A(x), A(y))$ ,
- (2)  $A(xy) \leq \max(A(x), A(y))$ , for all  $x, y \in S$ .

**Definition 2.6.** [8, p.1850] An intuitionistic fuzzy subset  $A$  in  $X$  is defined as an object of the form  $A = \{\langle x, \mu_A, \nu_A \rangle \mid x \in X\}$  where  $\mu_A : X \rightarrow [0, 1]$  and  $\nu_A : X \rightarrow [0, 1]$  define the degree of membership and the degree of non-membership for the element  $x \in X$  respectively and for every  $x \in X$  satisfying  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ .

**Definition 2.7.** [8, p.1850] Let  $S$  be a semiring. An intuitionistic fuzzy subset  $A$  of  $S$  is said to be an intuitionistic fuzzy subsemiring of  $S$  if it satisfies the following conditions:

- (1)  $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$ ,
- (2)  $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ ,
- (3)  $\nu_A(x + y) \leq \max\{\nu_A(x), \nu_A(y)\}$ ,
- (4)  $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\}$ , for all  $x, y \in S$ .

Now we will introduce a new relation of an interval. By an interval  $[0, 1]$ , say  $\bar{a}$  is a closed subinterval of  $[0, 1]$ , that is  $\bar{a} = [a^-, a^+]$ , where  $0 \leq a^- \leq a^+ \leq 1$ . Let  $D[0, 1]$  denoted the family of all closed subinterval of  $[0, 1]$ , i.e.,

$$D[0, 1] = \{[a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

The interval  $[a, a]$  is identified with the number  $a \in [0, 1]$ .

For interval numbers  $\bar{a}_i = [a_i^-, a_i^+] \in D[0, 1]$  for all  $i \in I$  where  $I$  is an index set. We define  $\bigwedge_{i \in I} \bar{a}_i^- = \inf\{a_i^- : i \in I\}$  and  $\bigvee_{i \in I} \bar{a}_i^- = \sup\{a_i^- : i \in I\}$ .

Consider two elements  $\bar{a} = [a^-, a^+]$  and  $\bar{b} = [b^-, b^+] \in D[0, 1]$ , we define

- (1)  $\bar{a} \leq \bar{b}$  if and only if  $a^- \leq b^-$  and  $a^+ \leq b^+$ ,
- (2)  $\bar{a} = \bar{b}$  if and only if  $a^- = b^-$  and  $a^+ = b^+$ ,
- (3)  $\bar{a} < \bar{b}$  if and only if  $\bar{a} \leq \bar{b}$  and  $\bar{a} \neq \bar{b}$ ,
- (4)  $k\bar{a} = [ka^-, ka^+]$ , whenever  $0 \leq k \leq 1$ .

Obviously,  $(D[0, 1], \leq, \vee, \wedge)$  is a complete lattice with  $0 = [0, 0]$  as the least element and  $1 = [1, 1]$  as the greatest element [9].

Let  $X$  be a set. An interval valued fuzzy set  $A$  on  $X$  is defined as

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) \mid x \in X\},$$

where  $\mu_A^-$  and  $\mu_A^+$  are two fuzzy sets of  $X$  such that  $\mu_A^-(x) \leq \mu_A^+(x)$  for all  $x \in X$ . Putting  $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ , we see that  $A = \{x, \mu_A(x) \mid x \in X\}$ , where  $\mu_A(x) : X \rightarrow D[0, 1]$ .

**Definition 2.8.** [9, p.155] A mapping  $\delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called an interval  $T$ -norm (resp.  $S$ -norm) if it is satisfying the following condition;

- (1) boundary conditions,  $\delta(1, x) = x$  (resp.  $\delta(0, x) = 0$ ),
- (2) commutative,  $\delta(x, y) = \delta(y, x)$ ,
- (3) associative,  $\delta(x, \delta(y, z)) = \delta(\delta(x, y), z)$ ,
- (4) monotony, if  $w \leq z$  then  $\delta(x, w) \leq \delta(x, z)$ , for all  $x, y, w, z \in [0, 1]$ .

If  $\delta(a, a) = a$  for all  $a \in [0, 1]$ , then  $\delta$  is called an idempotent interval  $t$ -norm (resp.  $s$ -norm).

According to Atanassov [9], an interval-valued intuitionistic fuzzy set on  $X$  is defined as an object of the form

$$A = \{(x, M_A(x), N_A(x)) \mid x \in X\}$$

where  $M_A(x)$  and  $N_A(x)$  are interval valued fuzzy sets on  $X$  such that

$$0 \leq \sup M_A(x) + \sup N_A(x) \leq 1$$

for all  $x \in X$ . For the sake of simplicity, in the following, such interval-valued intuitionistic fuzzy set will be denoted by  $A = (M_A, N_A)$ .

**Definition 2.9.** [11, p.5420] Let  $A = (M_A, N_A)$  and  $B = (M_B, N_B)$  be interval-valued intuitionistic fuzzy sets. Then

- (1)  $A \subseteq B$  if and only if  $M_A \leq M_B$  and  $N_A \geq N_B$ ,
- (2)  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ ,
- (3)  $A^c = (N_A, M_A)$ ,
- (4)  $A \cap B = (M_A \wedge M_B, N_A \vee N_B)$ ,
- (5)  $A \cup B = (M_A \vee M_B, N_A \wedge N_B)$ ,
- (6) if  $\{A_i : i \in I\}$  is a family of interval-valued intuitionistic fuzzy sets, then

$$\begin{aligned} \bigcup_{i \in I} A_i &= (\bigcup_{i \in I} M_{A_i}, \bigcap_{i \in I} N_{A_i}) \\ &= (\bigvee_{i \in I} M_{A_i}, \bigwedge_{i \in I} N_{A_i}) \end{aligned}$$

and

$$\begin{aligned} \bigcap_{i \in I} A_i &= (\bigcap_{i \in I} M_{A_i}, \bigcup_{i \in I} N_{A_i}) \\ &= (\bigwedge_{i \in I} M_{A_i}, \bigvee_{i \in I} N_{A_i}), \end{aligned}$$

- (7)  $\square A = (M_A, M_A^*)$  where  $M_A^* = 1 - M_A$ ,
- (8)  $\diamond A = (N_A^*, N_A)$  where  $N_A^* = 1 - N_A$ .

3. INTERVAL-VALUED INTUITIONISTIC  $(T, S)$ -FUZZY SUBSEMIRING IN SEMIRINGS

In this section, we define interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring and study basic properties of subsemiring.

**Definition 3.1.** An interval-valued intuitionistic fuzzy set  $A = (M_A, N_A)$  of a semiring  $S$  is called an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring if for all  $x, y \in S$  :

- (1)  $M_A(x + y) \geq T(M_A(x), M_A(y))$ ,
- (2)  $M_A(xy) \geq T(M_A(x), M_A(y))$ ,
- (3)  $N_A(x + y) \leq S(N_A(x), N_A(y))$ ,
- (4)  $N_A(xy) \leq S(N_A(x), N_A(y))$ .

**Example 3.1.** On four element semiring  $(S, +, \cdot)$  defined by the following two tables:

$S : +$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

$S : \cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Consider the interval valued fuzzy set  $A = (M_A, N_A)$  as follows

$$M_A(x) = \begin{cases} [0.4, 0.5], & \text{if } x = 0 \\ [0.2, 0.25], & \text{if } x \neq 0 \end{cases}$$

and

$$N_A(x) = \begin{cases} [0.2, 0.3], & \text{if } x = 0 \\ [0.7, 0.75], & \text{if } x \neq 0. \end{cases}$$

Note that

- (1)  $M_A(b + c) = M_A(c) = [0.2, 0.25]$  and  
 $T(M_A(b), M_A(c)) = [0.2, 0.25]$   
 Then  $M_A(b + c) \geq T(M_A(b), M_A(c))$ ,
- (2)  $M_A(b \cdot c) = M_A(a) = [0.2, 0.25]$  and  
 $T(M_A(b), M_A(c)) = [0.2, 0.25]$   
 Then  $M_A(b \cdot c) \geq T(M_A(b), M_A(c))$ ,

- (3)  $N_A(b + c) = N_A(c) = [0.7, 0.75]$  and  
 $S(N_A(b) = [0.7, 0.75], N_A(c) = [0.7, 0.75]) = [0.7, 0.75]$   
 Then  $N_A(b + c) \leq S(N_A(b), N_A(c))$ ,
- (4)  $N_A(b \cdot c) = M_A(a) = [0.7, 0.75]$  and  
 $S(N_A(b) = [0.7, 0.75], M_A(c) = [0.7, 0.75]) = [0.7, 0.75]$   
 Then  $N_A(b \cdot c) \leq S(N_A(b), N_A(c))$ .

Thus  $A = (M_A, N_A)$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring.

**Theorem 3.1.** Let  $A = (M_A, N_A)$  and  $B = (M_B, N_B)$  be two interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S$ . Then  $(T(M_A, M_B), S(N_A, N_B))$  is also an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .

*Proof.* Let  $x, y \in S$ . Then

$$\begin{aligned} T(M_A, M_B)(x + y) &= T(M_A(x + y), M_B(x + y)) \\ &\geq T(T(M_A(x), M_A(y)), T(M_B(x), M_B(y))), \text{ by Definition 3.1} \\ &= T(T(M_A(x), M_B(x)), T(M_A(y), M_B(y))) \\ &= T(T(M_A, M_B)(x), T(M_A, M_B)(y)). \end{aligned}$$

Thus  $T(M_A, M_B)(x + y) \geq T(T(M_A, M_B)(x), T(M_A, M_B)(y))$  and

$$\begin{aligned} T(M_A, M_B)(xy) &= T(M_A(xy), M_B(xy)) \\ &\geq T(T(M_A(x), M_A(y)), T(M_B(x), M_B(y))), \text{ by Definition 3.1} \\ &= T(T(M_A(x), M_B(x)), T(M_A(y), M_B(y))) \\ &= T(T(M_A, M_B)(x), T(M_A, M_B)(y)). \end{aligned}$$

Thus  $T(M_A, M_B)(xy) \geq T(T(M_A, M_B)(x), T(M_A, M_B)(y))$ . Now,

$$\begin{aligned} S(N_A, N_B)(x + y) &= S(N_A(x + y), N_B(x + y)) \\ &\leq S(S(N_A(x), N_A(y)), S(N_B(x), N_B(y))), \text{ by Definition 3.1} \\ &= S(S(N_A(x), N_B(x)), S(N_A(y), N_B(y))) \\ &= S(S(N_A, N_B)(x), S(N_A, N_B)(y)). \end{aligned}$$

Thus  $S(N_A, N_B)(x + y) \leq S(S(N_A, N_B)(x), S(N_A, N_B)(y))$  and

$$\begin{aligned} S(N_A, N_B)(xy) &= S(N_A(xy), N_B(xy)) \\ &\leq S(S(N_A(x), N_A(y)), S(N_B(x), N_B(y))), \text{ by Definition 3.1} \\ &= S(S(N_A(x), N_B(x)), S(N_A(y), N_B(y))) \\ &= S(S(N_A, N_B)(x), S(N_A, N_B)(y)) \end{aligned}$$

Thus  $S(N_A, N_B)(xy) \leq S(S(N_A, N_B)(x), S(N_A, N_B)(y))$ .

Therefore  $(T(M_A, M_B), S(N_A, N_B))$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .  $\square$

**Theorem 3.2.** Intersection of any two interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring  $S$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .

*Proof.* Let  $A = (M_A, N_A)$  and  $B = (M_B, N_B)$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring  $S$  and let  $C = A \cap B = \{\langle x, [M_A(x) \wedge M_B(x)], [N_A(x) \vee N_B(x)] \rangle \mid x \in S\}$  where  $M_C(x) = (M_A(x) \wedge M_B(x))$  and  $N_C(x) = (N_A(x) \vee N_B(x))$ . Let  $x, y \in S$ . Then

$$\begin{aligned} M_C(x+y) &= M_A(x+y) \wedge M_B(x+y) \\ &\geq T(M_A(x), M_A(y)) \wedge T(M_B(x), M_B(y)), \quad \text{by Definition 3.1} \\ &= T(M_A(x) \wedge M_B(x), T(M_A(y) \wedge M_B(y))) \\ &= T(M_C(x), M_C(y)). \end{aligned}$$

Thus  $M_C(x+y) \geq T(M_C(x), M_C(y))$  and

$$\begin{aligned} M_C(xy) &= M_A(xy) \wedge M_B(xy) \\ &\geq T(M_A(x), M_A(y)) \wedge T(M_B(x), M_B(y)), \quad \text{by Definition 3.1} \\ &= T(M_A(x) \wedge M_B(x), T(M_A(y) \wedge M_B(y))) \\ &= T(M_C(x), M_C(y)). \end{aligned}$$

Thus  $M_C(xy) \geq T(M_C(x), M_C(y))$ . Now,

$$\begin{aligned} N_C(x+y) &= N_A(x+y) \vee N_B(x+y) \\ &\leq S(N_A(x), N_A(y)) \vee S(N_B(x), N_B(y)), \quad \text{by Definition 3.1} \\ &= S(N_A(x) \vee N_B(x), S(N_A(y) \vee N_B(y))) \\ &= S(N_C(x), N_C(y)). \end{aligned}$$

Thus  $N_C(x+y) \leq S(N_C(x), N_C(y))$  and

$$\begin{aligned} N_C(xy) &= N_A(xy) \vee N_B(xy) \\ &\leq S(N_A(x), N_A(y)) \vee S(N_B(x), N_B(y)), \quad \text{by Definition 3.1} \\ &= S(N_A(x) \vee N_B(x), S(N_A(y) \vee N_B(y))) \\ &= S(N_C(x), N_C(y)). \end{aligned}$$

Thus  $N_C(xy) \leq S(N_C(x), N_C(y))$ .

Therefore  $C = A \cap B$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ . □

**Corollary 3.1.** *Union of any two interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring  $S$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .*

**Theorem 3.3.** *Let  $\{A_i\}_{i \in I}$  be a family of interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S$ . Then  $\bigcap_{i \in I} A_i$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ , where  $\bigcap_{i \in I} A_i = (\bigwedge_{i \in I} M_{A_i}, \bigvee_{i \in I} N_{A_i})$ .*

*Proof.* Let  $V = \bigcap_{i \in I} A_i = (\bigwedge_{i \in I} M_{A_i}, \bigvee_{i \in I} N_{A_i})$ . To show that  $\bigcap_{i \in I} A_i$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ , let  $x, y \in S$ . Now

$$\begin{aligned} M_V(x+y) &= \bigwedge_{i \in I} M_{A_i}(x+y) \\ &\geq \bigwedge_{i \in I} T(M_{A_i}(x), M_{A_i}(y)), \quad \text{by Definition 3.1} \\ &= T(\bigwedge_{i \in I} M_{A_i}(x), \bigwedge_{i \in I} M_{A_i}(y)) \\ &= T(M_V(x), M_V(y)) \end{aligned}$$

Thus  $M_V(x + y) \geq T(M_V(x), M_V(y))$  and

$$\begin{aligned} M_V(xy) &= \wedge_{i \in I} M_{A_i}(xy) \\ &\geq \wedge_{i \in I} T(M_{A_i}(x), M_{A_i}(y)), && \text{by Definition 3.1} \\ &= T(\wedge_{i \in I} M_{A_i}(x), \wedge_{i \in I} M_{A_i}(y)) \\ &= T(M_V(x), M_V(y)). \end{aligned}$$

Thus  $M_V(xy) \geq T(M_V(x), M_V(y))$ . Now,

$$\begin{aligned} N_V(x + y) &= \vee_{i \in I} N_{A_i}(x + y) \\ &\leq \vee_{i \in I} S(N_{A_i}(x), N_{A_i}(y)), && \text{by Definition 3.1} \\ &= S(\vee_{i \in I} N_{A_i}(x), \vee_{i \in I} N_{A_i}(y)) \\ &= S(N_V(x), N_V(y)). \end{aligned}$$

Thus  $N_V(x + y) \leq S(N_V(x), N_V(y))$  and

$$\begin{aligned} N_V(xy) &= \vee_{i \in I} N_{A_i}(xy) \\ &\leq \vee_{i \in I} S(N_{A_i}(x), N_{A_i}(y)), && \text{by Definition 3.1} \\ &= S(\vee_{i \in I} N_{A_i}(x), \vee_{i \in I} N_{A_i}(y)) \\ &= S(N_V(x), N_V(y)). \end{aligned}$$

Thus  $M_V(xy) \leq S(N_V(x), N_V(y))$ .

Therefore,  $V = \bigcap_{i \in I} A_i$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .  $\square$

**Corollary 3.2.** Let  $\{A_i\}_{i \in I}$  be a family of interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S$ . Then  $\bigcup_{i \in I} A_i$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ , where  $\bigcup_{i \in I} A_i = (\vee_{i \in I} M_{A_i}, \wedge_{i \in I} N_{A_i})$ .

**Definition 3.2.** Let  $A$  and  $B$  be an interval-valued intuitionistic fuzzy subsets of the set  $I$  and  $J$ , respectively. The product of  $A$  and  $B$  denoted by  $A \times B$  is defined as  $A \times B = \{(x, y), M_{A \times B}(x, y), N_{A \times B}(x, y) \mid \text{for all } x \in I \text{ and } y \in J\}$

**Theorem 3.4.** Let  $A = (M_A, N_A)$  and  $B = (M_B, N_B)$  be two interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S_1$  and  $S_2$  respectively. Then  $A \times B$  is also an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1 \times S_2$ .

*Proof.* Let  $x_1, x_2 \in S_1$  and  $y_1, y_2 \in S_2$ . Then  $(x_1, y_1)$  and  $(x_2, y_2) \in S_1 \times S_2$ . Thus

$$\begin{aligned} M_{A \times B}[(x_1, y_1) + (x_2, y_2)] &= M_{A \times B}(x_1 + x_2, y_1 + y_2) \\ &= M_A(x_1 + x_2), M_B(y_1 + y_2) \\ &\geq T(M_A(x_1), M_A(x_2), T(M_B(y_1), M_B(y_2))), && \text{by Definition 3.1} \\ &= T(M_A(x_1), M_B(y_1), T(M_A(x_2), M_B(y_2))) \\ &= T(M_{A \times B}(x_1, y_1), T(M_{A \times B}(x_2, y_2))), && \text{by Definition 3.2} \\ &= T(M_{A \times B}(x_1, y_1), M_{A \times B}(x_2, y_2)) \end{aligned}$$

Hence  $M_{A \times B}[(x_1, y_1) + (x_2, y_2)] \geq T(M_{A \times B}(x_1, y_1), M_{A \times B}(x_2, y_2))$ . Also

$$\begin{aligned} M_{A \times B}[(x_1, y_1)(x_2, y_2)] &= M_{A \times B}(x_1 x_2, y_1 y_2) \\ &= M_A(x_1 x_2), M_B(y_1 y_2) \\ &\geq T(M_A(x_1), M_A(x_2), T(M_B(y_1), M_B(y_2))), \text{ by Definition 3.1} \\ &= T(M_A(x_1), M_B(y_1), T(M_A(x_2), M_B(y_2))) \\ &= T(M_{A \times B}(x_1, y_1), T(M_{A \times B}(x_2, y_2))), \text{ by Definition 3.2} \\ &= T(M_{A \times B}(x_1, y_1), M_{A \times B}(x_2, y_2)). \end{aligned}$$

Hence  $M_{A \times B}[(x_1, y_1)(x_2, y_2)] \geq T(M_{A \times B}(x_1, y_1), M_{A \times B}(x_2, y_2))$ . Now

$$\begin{aligned} N_{A \times B}[(x_1, y_1) + (x_2, y_2)] &= N_{A \times B}(x_1 + x_2, y_1 + y_2) \\ &= N_A(x_1 + x_2), N_B(y_1 + y_2) \\ &\leq S(N_A(x_1), N_A(x_2), S(N_B(y_1), N_B(y_2))), \text{ by Definition 3.1} \\ &= S(N_A(x_1), N_B(y_1), S(N_A(x_2), N_B(y_2))) \\ &= S(N_{A \times B}(x_1, y_1), S(N_{A \times B}(x_2, y_2))), \text{ by Definition 3.2} \\ &= S(N_{A \times B}(x_1, y_1), N_{A \times B}(x_2, y_2)). \end{aligned}$$

Hence  $N_{A \times B}(x_1, y_1) + (x_2, y_2) \geq S(N_{A \times B}(x_1, y_1), N_{A \times B}(x_2, y_2))$  and

$$\begin{aligned} N_{A \times B}[(x_1, y_1)(x_2, y_2)] &= N_{A \times B}(x_1 x_2, y_1 y_2) \\ &= N_A(x_1 x_2), N_B(y_1 y_2) \\ &\leq S(N_A(x_1), N_A(x_2), S(N_B(y_1), N_B(y_2))), \text{ by Definition 3.1} \\ &= S(N_A(x_1), N_B(y_1), S(N_A(x_2), N_B(y_2))) \\ &= S(N_{A \times B}(x_1, y_1), S(N_{A \times B}(x_2, y_2))), \text{ by Definition 3.2} \\ &= S(N_{A \times B}(x_1, y_1), N_{A \times B}(x_2, y_2)). \end{aligned}$$

Hence  $N_{A \times B}[(x_1, y_1)(x_2, y_2)] \leq S(N_{A \times B}(x_1, y_1), N_{A \times B}(x_2, y_2))$ .

Therefore  $A \times B$  is also interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1 \times S_2$ . □

**Definition 3.3.** Let  $S$  be a semiring and let  $\emptyset \neq A \subseteq S$ . Then interval-value intuitionistic fuzzy characteristic function  $\mathcal{X}_A = (M_{\mathcal{X}_A}, N_{\mathcal{X}_A})$  of is defined as

$$M_{\mathcal{X}_A} = \begin{cases} [1, 1], & \text{if } x \in A, \\ [0, 0], & \text{if } x \notin A. \end{cases}$$

and

$$N_{\mathcal{X}_A} = \begin{cases} [0, 0], & \text{if } x \in A, \\ [1, 1], & \text{if } x \notin A. \end{cases}$$

**Theorem 3.5.** Let  $S$  be a semiring and let  $\emptyset \neq A \subseteq S$ . Then  $A$  is a subsemiring of  $S$  if and only if  $\mathcal{X}_A$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .

*Proof.* Suppose that  $A$  is a subsemiring of  $S$ . To show that  $\mathcal{X}_A$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ , let  $x, y \in S$ . Then the following cases:

Case(1): If  $x, y \in A$  then  $x + y \in A$  and  $xy \in A$ , since  $A$  is a subsemiring of  $S$ . Thus  $M_{\mathcal{X}_A}(x) = M_{\mathcal{X}_A}(y) = M_{\mathcal{X}_A}(x + y) = M_{\mathcal{X}_A}(xy) = [1, 1]$ . Hence

$$M_{\mathcal{X}_A}(x + y) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) \quad \text{and} \quad M_{\mathcal{X}_A}(xy) \geq T(M_{\mathcal{X}_A}(x)M_{\mathcal{X}_A}(y)).$$

And  $N_{\mathcal{X}_A}(x) = N_{\mathcal{X}_A}(y) = N_{\mathcal{X}_A}(x + y) = N_{\mathcal{X}_A}(xy) = [0, 0]$ . Then

$$N_{\mathcal{X}_A}(x + y) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) \quad \text{and} \quad N_{\mathcal{X}_A}(xy) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)).$$

Case(2): If  $x \in A$  and  $y \notin A$  then  $x + y$  and  $xy$  may or may not be in  $A$ . Thus  $M_{\mathcal{X}_A}(x) = [1, 1]$ ,  $M_{\mathcal{X}_A}(y) = [0, 0]$  and  $M_{\mathcal{X}_A}(x + y) = M_{\mathcal{X}_A}(xy) = [1, 1]$ . Hence

$$M_{\mathcal{X}_A}(x + y) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) \quad \text{and} \quad M_{\mathcal{X}_A}(xy) \geq T(M_{\mathcal{X}_A}(x)M_{\mathcal{X}_A}(y)).$$

And  $N_{\mathcal{X}_A}(x) = [0, 0]$ ,  $N_{\mathcal{X}_A}(y) = [1, 1]$  and  $N_{\mathcal{X}_A}(x + y) = N_{\mathcal{X}_A}(xy) = [0, 0]$ . Then

$$N_{\mathcal{X}_A}(x + y) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) \quad \text{and} \quad N_{\mathcal{X}_A}(xy) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)).$$

Case(3): If  $x \notin A$  and  $y \in A$  then  $x + y$  and  $xy$  may or may not be in  $A$ . Thus  $M_{\mathcal{X}_A}(x) = [0, 0]$ ,  $M_{\mathcal{X}_A}(y) = [1, 1]$  and  $M_{\mathcal{X}_A}(x + y) = M_{\mathcal{X}_A}(xy) = [0, 0]$ . Hence

$$M_{\mathcal{X}_A}(x + y) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) \quad \text{and} \quad M_{\mathcal{X}_A}(xy) \geq T(M_{\mathcal{X}_A}(x)M_{\mathcal{X}_A}(y))$$

And  $N_{\mathcal{X}_A}(x) = [1, 1]$ ,  $N_{\mathcal{X}_A}(y) = [0, 0]$  and  $N_{\mathcal{X}_A}(x + y) = N_{\mathcal{X}_A}(xy) = [1, 1]$ . Then

$$N_{\mathcal{X}_A}(x + y) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) \quad \text{and} \quad N_{\mathcal{X}_A}(xy) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)).$$

Case(4): If  $x, y \notin A$  then  $x + y$  and  $xy$  may or may not be in  $A$ . Thus  $M_{\mathcal{X}_A}(x) = M_{\mathcal{X}_A}(y) = [0, 0]$  and  $M_{\mathcal{X}_A}(x + y) = M_{\mathcal{X}_A}(xy) = [1, 1]$  or  $[0, 0]$ . Hence

$$M_{\mathcal{X}_A}(x + y) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) \quad \text{and} \quad M_{\mathcal{X}_A}(xy) \geq T(M_{\mathcal{X}_A}(x)M_{\mathcal{X}_A}(y)).$$

And  $N_{\mathcal{X}_A}(x) = N_{\mathcal{X}_A}(y) = [1, 1]$  and  $N_{\mathcal{X}_A}(x + y) = N_{\mathcal{X}_A}(xy) = [0, 0]$  or  $[1, 1]$ . Then

$$N_{\mathcal{X}_A}(x + y) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) \quad \text{and} \quad N_{\mathcal{X}_A}(xy) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)).$$

From case(1-4) we have  $\mathcal{X}_A$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .

Conversely, suppose that  $\mathcal{X}_A$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ . To show that  $A$  is a subsemiring of  $S$ , let  $x, y \in A$ . Then  $M_{\mathcal{X}_A}(x) = M_{\mathcal{X}_A}(y) = [1, 1]$  and  $N_{\mathcal{X}_A}(x) = N_{\mathcal{X}_A}(y) = [0, 0]$ .

Since  $\mathcal{X}_A$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  we have  $M_{\mathcal{X}_A}(x + y) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) = [1, 1]$  and  $M_{\mathcal{X}_A}(xy) \geq T(M_{\mathcal{X}_A}(x), M_{\mathcal{X}_A}(y)) = [1, 1]$ . Thus,  $M_{\mathcal{X}_A}(x + y) = M_{\mathcal{X}_A}(xy) \geq [1, 1]$ . And  $N_{\mathcal{X}_A}(x + y) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) = [0, 0]$  and  $N_{\mathcal{X}_A}(xy) \leq S(N_{\mathcal{X}_A}(x), N_{\mathcal{X}_A}(y)) = [0, 0]$ . Thus  $N_{\mathcal{X}_A}(x + y) = N_{\mathcal{X}_A}(xy) \leq [0, 0]$ . Hence  $x + y \in A$  and  $xy \in A$  so  $A$  is a subsemiring of  $S$ .  $\square$

**Definition 3.4.** Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  and  $a \in S$ . Then the pseudo interval-valued intuitionistic  $(T, S)$ -fuzzy coset  $(aA)^p$  is defined by  $((aM_A)^p)(x) = p(a)M_A(x)$  and  $((aN_A)^p)(x) = p(a)N_A(x)$ , for all  $x \in S$  and for some  $p \in P$ .

**Theorem 3.6.** *Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ . Then the pseudo interval-valued intuitionistic  $(T, S)$ -fuzzy coset is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .*

*Proof.* Let  $x, y \in S$ . Then

$$\begin{aligned} ((aM_A)^p)(x + y) &= p(a)M_A(x + y), && \text{by Definition 3.4} \\ &\geq p(a)(T(M_A(x), M_A(y))), && \text{by Definition 3.1} \\ &= T(p(a)M_A(x), p(a)M_A(y)) \\ &= T((aM_A)^p)(x), (aM_A)^p)(y), && \text{by Definition 3.4.} \end{aligned}$$

Thus  $((aM_A)^p)(x + y) \geq T((aM_A)^p)(x), (aM_A)^p)(y)$  and

$$\begin{aligned} ((aM_A)^p)(xy) &= p(a)M_A(xy), && \text{by Definition 3.4} \\ &\geq p(a)(T(M_A(x), M_A(y))), && \text{by Definition 3.1} \\ &= T(p(a)M_A(x), p(a)M_A(y)) \\ &= T((aM_A)^p)(x), (aM_A)^p)(y), && \text{by Definition 3.4.} \end{aligned}$$

Thus  $((aM_A)^p)(xy) \geq T((aM_A)^p)(x), (aM_A)^p)(y)$ . Similarly

$$\begin{aligned} ((aN_A)^p)(x + y) &= p(a)N_A(x + y), && \text{by Definition 3.4} \\ &\leq p(a)(S(N_A(x), N_A(y))), && \text{by Definition 3.1} \\ &= S(p(a)N_A(x), p(a)N_A(y)) \\ &= S((aN_A)^p)(x), (aN_A)^p)(y), && \text{by Definition 3.4} \end{aligned}$$

Thus  $((aN_A)^p)(x + y) \leq S((aN_A)^p)(x), (aN_A)^p)(y)$  and

$$\begin{aligned} ((aN_A)^p)(xy) &= p(a)N_A(xy), && \text{by Definition 3.4} \\ &\leq p(a)(S(N_A(x), N_A(y))), && \text{by Definition 3.1} \\ &= S(p(a)N_A(x), p(a)N_A(y)) \\ &= S((aN_A)^p)(x), (aN_A)^p)(y), && \text{by Definition 3.4} \end{aligned}$$

Thus  $((aN_A)^p)(xy) \leq S((aN_A)^p)(x), (aN_A)^p)(y)$ . Hence the pseudo interval-valued intuitionistic  $(T, S)$ -fuzzy coset is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .  $\square$

**Definition 3.5.** [11, p.5425] *A mapping  $\eta : D[0, 1] \rightarrow D[0, 1]$  is called a negation if it satisfies*

- (1)  $\eta([0, 0]) = [1, 1], \eta([1, 1]) = [0, 0]$ ,
- (2)  $\eta$  is non-increasing,
- (3)  $\eta(\eta(x)) = x$ .

**Remark 3.1.** *The interval t-norm and interval s-conorm are said to be dual with respect to the negation  $\eta(x) = [1, 1] - x$  if  $T(x, y) = T(\eta(\eta(x), \eta(y)))$ . This holds with respect to  $\eta$  if and only if  $S(x, y) = S(\eta(\eta(x), \eta(y)))$ .*

**Theorem 3.7.** *Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  then  $\square A = (M_A, M_A^*)$  an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  where  $M_A^* = [1, 1] - M_A$ .*

*Proof.* Since  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  we have  $M_A(x + y) \geq T(M_A(x), M_A(y))$  and  $M_A(xy) \geq T(M_A(x), M_A(y))$ , for all  $x, y \in S$ . It is sufficient to prove for all  $x, y \in S$  with  $M_A^*$  satisfies

$$M_A^*(x + y) \leq S(M_A^*(x), M_A^*(y)) \quad \text{and} \quad M_A^*(xy) \leq S(M_A^*(x), M_A^*(y)).$$

Let  $x, y \in S$ . Then

$$\begin{aligned} M_A^*(x + y) &= [1, 1] - M_A(x + y) \\ &\leq [1, 1] - \{T(M_A(x), M_A(y))\}, && \text{by Definition 3.1} \\ &= \eta\{T(M_A(x), M_A(y))\}, && \text{by Remark 3.1} \\ &= T(\eta N_A(x), \eta N_A(y)) \\ &= S(\eta N_A(x), \eta N_A(y)) \\ &= S([1, 1] - M_A(x), [1, 1] - M_A(y)), && \text{by Remark 3.1} \\ &= S(M_A^*(x), M_A^*(y)), && \text{by } M_A^* = [1, 1] - M_A. \end{aligned}$$

Thus  $M_A^*(x + y) \leq S(M_A^*(x), M_A^*(y))$ . And

$$\begin{aligned} M_A^*(xy) &= [1, 1] - M_A(xy) \\ &\leq [1, 1] - \{T(M_A(x), M_A(y))\}, && \text{by Definition 3.1} \\ &= \eta\{T(M_A(x), M_A(y))\}, && \text{by Remark 3.1} \\ &= T(\eta N_A(x), \eta N_A(y)) \\ &= S(\eta N_A(x), \eta N_A(y)) \\ &= S([1, 1] - M_A(x), [1, 1] - M_A(y)), && \text{by Remark 3.1} \\ &= S(M_A^*(x), M_A^*(y)), && \text{by } M_A^* = [1, 1] - M_A. \end{aligned}$$

Thus  $M_A^*(xy) \leq S(M_A^*(x), M_A^*(y))$ . Therefore  $\square A = (M_A, M_A^*)$  an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ .  $\square$

**Theorem 3.8.** Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  then  $\diamond A = (N_A^*, N_A)$  an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  where  $N_A^* = [1, 1] - N_A$ .

*Proof.* Since  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$  we have  $N_A(x + y) \leq S(N_A(x), N_A(y))$  and  $N_A(xy) \leq S(N_A(x), N_A(y))$ , for all  $x, y \in S$ . It is sufficient to prove for all  $x, y \in S$  with  $N_A^*$  satisfies

$$N_A^*(x + y) \geq T(N_A^*(x), N_A^*(y)) \quad \text{and} \quad N_A^*(xy) \geq T(N_A^*(x), N_A^*(y)).$$

Let  $x, y \in S$ . Then

$$\begin{aligned} N_A^*(x + y) &= [1, 1] - N_A(x + y) \\ &\geq [1, 1] - \{S(N_A(x), N_A(y))\}, && \text{by Definition 3.1} \\ &= \eta\{S(N_A(x), N_A(y))\}, && \text{by Remark 3.1} \\ &= S(\eta N_A(x), \eta N_A(y)) \\ &= T(\eta N_A(x), \eta N_A(y)) \end{aligned}$$

$$\begin{aligned} &= T([1, 1] - N_A(x), [1, 1] - N_A(y)), && \text{by Remark 3.1} \\ &= T(N_A^*(x), N_A^*(y)), && \text{by } N_A^* = [1, 1] - N_A. \end{aligned}$$

Thus  $N_A^*(x + y) \geq T(N_A^*(x), N_A^*(y))$ . And

$$\begin{aligned} N_A^*(xy) &= [1, 1] - N_A(xy) \\ &\geq [1, 1] - \{S(N_A(x), N_A(y))\}, && \text{by Definition 3.1} \\ &= \eta\{S(N_A(x), N_A(y))\}, && \text{by Remark 3.1} \\ &= S(\eta N_A(x), \eta N_A(y)) \\ &= T(\eta N_A(x), \eta N_A(y)) \\ &= T([1, 1] - N_A(x), [1, 1] - N_A(y)), && \text{by Remark 3.1} \\ &= T(N_A^*(x), N_A^*(y)), && \text{by } N_A^* = [1, 1] - N_A. \end{aligned}$$

Thus,  $N_A^*(xy) \geq T(N_A^*(x), N_A^*(y))$ . Therefore,  $\diamond A = (N_A^*, N_A)$  an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S$ . □

#### 4. HOMOMORPHIC INVERSE IMAGE AND COMPOSITION OPERATION TO GET INTERVAL-VALUED INTUITIONISTIC $(T, S)$ -FUZZY SUBSEMIRING

In this section, we study some properties of homomorphic and anti-homomorphic inverse image of interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring.

**Definition 4.1.** [12, p.5] Let  $f : X \rightarrow Y$  be a mapping and  $A = (M_A, N_A)$  and  $B = (M_B, N_B)$  be an interval-valued intuitionistic sets of  $X$  and  $Y$  respectively. Then the image  $f[A] = (f(M_A), f(N_A))$  of  $A$  is an interval-valued intuitionistic fuzzy set of  $Y$  defined by

$$f(M_A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} M_A(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ [0, 0], & \text{otherwise} \end{cases}$$

$$f(N_A)(y) = \begin{cases} \inf_{z \in f^{-1}(y)} N_A(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ [0, 0], & \text{otherwise} \end{cases}$$

for all  $y \in Y$ . The inverse image  $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$  of  $B$  is an interval-valued intuitionistic fuzzy set of  $X$  defined by

$$\begin{aligned} f^{-1}(M_B)(x) &= M_{f^{-1}(B)}(x) = M_B(f(x)), \\ f^{-1}(N_B)(x) &= N_{f^{-1}(B)}(x) = N_B(f(x)) \end{aligned}$$

for all  $x \in X$ .

**Definition 4.2.** [2, p.17] Let  $S_1$  and  $S_2$  be any two semirings. A function  $f : S_1 \rightarrow S_2$  is called semiring homomorphism if it satisfies the following:

- (1)  $f(x + y) = f(x) + f(y)$ ,
- (2)  $f(xy) = f(x)f(y)$ , for all  $x, y \in S$ .

**Definition 4.3.** [2, p.18] Let  $S_1$  and  $S_2$  be any two semirings. A function  $f : S_1 \rightarrow S_2$  is called semiring anti-homomorphism if it satisfies the following:

- (1)  $f(x + y) = f(y) + f(x)$ ,
- (2)  $f(xy) = f(y)f(x)$ , for all  $x, y \in S$ .

**Theorem 4.1.** Let  $f : S_1 \rightarrow S_2$  be a homomorphism from a semiring  $S_1$  to a semiring  $S_2$ . If  $B = (M_B, N_B)$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_2$ , then the inverse image  $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$  of  $B$  under  $f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .

*Proof.* Let  $B = (M_B, N_B)$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_2$  and let  $x, y \in S_1$ . Then

$$\begin{aligned} f^{-1}(M_B(x + y)) &= M_{f^{-1}(B)}(x + y) \\ &= M_B(f(x + y)), && \text{by Definition 4.1} \\ &= M_B(f(x) + f(y)), && \text{by Definition 4.2} \\ &\geq T(M_B(f(x)), M_B(f(y))), && \text{by Definition 3.1} \\ &= T(f^{-1}(M_B(x)), f^{-1}(M_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(M_B(x + y)) \geq T(f^{-1}(M_B(x)), f^{-1}(M_B(y)))$ . And we have

$$\begin{aligned} f^{-1}(M_B(xy)) &= M_{f^{-1}(B)}(xy) \\ &= M_B(f(xy)) && \text{by Definition 4.1} \\ &= M_B(f(x)f(y)), && \text{by Definition 4.2} \\ &\geq T(M_B(f(x)), M_B(f(y))), && \text{by Definition 3.1} \\ &= T(f^{-1}(M_B(x)), f^{-1}(M_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(M_B(xy)) \geq T(f^{-1}(M_B(x)), f^{-1}(M_B(y)))$ . Again,

$$\begin{aligned} f^{-1}(N_B(x + y)) &= N_{f^{-1}(B)}(x + y) \\ &= N_B(f(x + y)) && \text{by Definition 4.1} \\ &= N_B(f(x) + f(y)), && \text{by Definition 4.2} \\ &\leq S(N_B(f(x)), N_B(f(y))), && \text{by Definition 3.1} \\ &= S(f^{-1}(N_B(x)), f^{-1}(N_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(N_B(x + y)) \leq S(f^{-1}(N_B(x)), f^{-1}(N_B(y)))$ . And we have

$$\begin{aligned} f^{-1}(N_B(xy)) &= N_{f^{-1}(B)}(xy) \\ &= N_B(f(xy)) && \text{by Definition 4.1} \\ &= N_B(f(x)f(y)), && \text{by Definition 4.2} \\ &\leq S(N_B(f(x)), N_B(f(y))), && \text{by Definition 3.1.} \\ &= S(f^{-1}(N_B(x)), f^{-1}(N_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(N_B(xy)) \leq S(f^{-1}(N_B(x)), f^{-1}(N_B(y)))$ .

Therefore  $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$  of  $B$  under  $f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ . □

**Theorem 4.2.** Let  $f : S_1 \rightarrow S_2$  be a anti-homomorphism from a semiring  $S_1$  to a semiring  $S_2$ . If  $B = (M_B, N_B)$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_2$ , then the inverse image  $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$  of  $B$  under  $f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .

*Proof.* Let  $B = (M_B, N_B)$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_2$  and let  $x, y \in S_1$ . Then

$$\begin{aligned} f^{-1}(M_B(x + y)) &= M_{f^{-1}(B)}(x + y) \\ &= M_B(f(x + y)), && \text{by Definition 4.1} \\ &= M_B(f(y) + f(x)), && \text{by Definition 4.3} \\ &\geq T(M_B(f(y)), M_B(f(x))), && \text{by Definition 3.1} \\ &= T(M_B(f(x)), M_B(f(y))) \\ &= T(f^{-1}(M_B(x)), f^{-1}(M_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(M_B(x + y)) \geq T(f^{-1}(M_B(x)), f^{-1}(M_B(y)))$ . And we have

$$\begin{aligned} f^{-1}(M_B(xy)) &= M_{f^{-1}(B)}(xy) \\ &= M_B(f(xy)), && \text{by Definition 4.1} \\ &= M_B(f(y)f(x)), && \text{by Definition 4.3} \\ &\geq T(M_B(f(y)), M_B(f(x))), && \text{by Definition 3.1} \\ &= T(M_B(f(x)), M_B(f(y))) \\ &= T(f^{-1}(M_B(x)), f^{-1}(M_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(M_B(xy)) \geq T(f^{-1}(M_B(x)), f^{-1}(M_B(y)))$ . Again,

$$\begin{aligned} f^{-1}(N_B(x + y)) &= N_{f^{-1}(B)}(x + y) \\ &= N_{f^{-1}(B)}(x + y) \\ &= N_B(f(x + y)) && \text{by Definition 4.1} \\ &= N_B(f(y) + f(x)), && \text{by Definition 4.3} \\ &\leq S(N_B(f(y)), N_B(f(x))), && \text{by Definition 3.1} \\ &\leq S(N_B(f(x)), N_B(f(y))) \\ &= S(f^{-1}(N_B(x)), f^{-1}(N_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(N_B(x + y)) \leq S(f^{-1}(N_B(x)), f^{-1}(N_B(y)))$ . And we have

$$\begin{aligned} f^{-1}(N_B(xy)) &= N_B(f(xy)) && \text{by Definition 4.1} \\ &= N_B(f(y)f(x)), && \text{by Definition 4.3} \\ &\leq S(N_B(f(y)), N_B(f(x))), && \text{by Definition 3.1} \\ &\leq S(N_B(f(x)), N_B(f(y))) \\ &= S(f^{-1}(N_B(x)), f^{-1}(N_B(y))) && \text{by Definition 4.1.} \end{aligned}$$

Thus  $f^{-1}(N_B(xy)) \leq S(f^{-1}(N_B(x)), f^{-1}(N_B(y)))$ .

Therefore  $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$  of  $B$  under  $f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ . □

**Theorem 4.3.** Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S_2$  and  $f$  is an homomorphism from a semiring  $S_1$  onto  $S_2$ . Then  $A \circ f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .

*Proof.* Let  $x, y$  in  $S$  and  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S_2$ . Then we have,

$$\begin{aligned} (M_A \circ f)(x + y) &= M_A(f(x + y)) \\ &= M_A(f(x) + f(y)), && \text{by Definition 4.2} \\ &\geq T(M_A(f(x)), M_A(f(y))), && \text{by Definition 3.1} \\ &= T((M_A \circ f)(x), (M_A \circ f)(y)). \end{aligned}$$

Hence  $(M_A \circ f)(x + y) \geq T((M_A \circ f)(x), (M_A \circ f)(y))$ . And

$$\begin{aligned} (M_A \circ f)(xy) &= M_A(f(xy)) \\ &= M_A(f(x)f(y)), && \text{by Definition 4.2} \\ &\geq T(M_A(f(x)), M_A(f(y))), && \text{by Definition 3.1} \\ &= T((M_A \circ f)(x), (M_A \circ f)(y)). \end{aligned}$$

Hence  $(M_A \circ f)(xy) \geq T((M_A \circ f)(x), (M_A \circ f)(y))$ . Again,

$$\begin{aligned} (N_A \circ f)(x + y) &= N_A(f(x + y)) \\ &= N_A(f(x) + f(y)), && \text{by Definition 4.2} \\ &\leq S(N_A(f(x)), N_A(f(y))), && \text{by Definition 3.1} \\ &= S((N_A \circ f)(x), (N_A \circ f)(y)). \end{aligned}$$

Hence  $(N_A \circ f)(x + y) \leq S((N_A \circ f)(x), (N_A \circ f)(y))$ . And

$$\begin{aligned} (N_A \circ f)(xy) &= N_A(f(xy)) \\ &= N_A(f(x)f(y)), && \text{by Definition 4.2} \\ &\leq S(N_A(f(x)), N_A(f(y))), && \text{by Definition 3.1} \\ &= S((N_A \circ f)(x), (N_A \circ f)(y)). \end{aligned}$$

Hence  $(N_A \circ f)(xy) \leq S((N_A \circ f)(x), (N_A \circ f)(y))$ .

Therefore  $A \circ f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .  $\square$

**Theorem 4.4.** Let  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S_2$  and  $f$  is an anti-homomorphism from a semiring  $S_1$  onto  $S_2$ . Then  $A \circ f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .

*Proof.* Let  $x, y$  in  $S$  and  $A$  be an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of semiring  $S_2$ . Then we have,

$$\begin{aligned} (M_A \circ f)(x + y) &= M_A(f(x + y)) \\ &= M_A(f(y) + f(x)), && \text{by Definition 4.3} \end{aligned}$$

$$\begin{aligned}
&\geq T(M_A(f(y)), M_A(f(x))), && \text{by Definition 3.1} \\
&= T(M_A(f(x)), M_A(f(y))) \\
&= T((M_A \circ f)(x), (M_A \circ f)(y)).
\end{aligned}$$

Hence  $(M_A \circ f)(x + y) \geq T((M_A \circ f)(x), (M_A \circ f)(y))$ . And

$$\begin{aligned}
(M_A \circ f)(xy) &= M_A(f(xy)) \\
&= M_A(f(y)f(x)), && \text{by Definition 4.3} \\
&\geq T(M_A(f(y)), M_A(f(x))), && \text{by Definition 3.1} \\
&= T(M_A(f(x)), M_A(f(y))) \\
&= T((M_A \circ f)(x), (M_A \circ f)(y)).
\end{aligned}$$

Hence  $(M_A \circ f)(xy) \geq T((M_A \circ f)(x), (M_A \circ f)(y))$ . Again,

$$\begin{aligned}
(N_A \circ f)(x + y) &= N_A(f(x + y)) \\
&= N_A(f(y) + f(x)), && \text{by Definition 4.2} \\
&\leq S(N_A(f(y)), N_A(f(x))), && \text{by Definition 3.1} \\
&= S(N_A(f(x)), N_A(f(y))) \\
&= S((N_A \circ f)(x), (N_A \circ f)(y)).
\end{aligned}$$

Hence  $(N_A \circ f)(x + y) \leq S((N_A \circ f)(x), (N_A \circ f)(y))$ . And

$$\begin{aligned}
(N_A \circ f)(xy) &= N_A(f(xy)) \\
&= N_A(f(y)f(x)), && \text{by Definition 4.2} \\
&\leq S(N_A(f(y)), N_A(f(x))), && \text{by Definition 3.1} \\
&= S(N_A(f(x)), N_A(f(y))) \\
&= S((N_A \circ f)(x), (N_A \circ f)(y)).
\end{aligned}$$

Hence  $(N_A \circ f)(xy) \leq S((N_A \circ f)(x), (N_A \circ f)(y))$ .

Therefore  $A \circ f$  is an interval-valued intuitionistic  $(T, S)$ -fuzzy subsemiring of  $S_1$ .  $\square$

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