

Some Singular Value Inequalities for Convex Functions of Matrices

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Abstract. In this paper, we obtain some upper bounds for singular value inequalities for convex functions of matrices. Some applications involving the spectral norm and numerical radii of matrices were given. Among other inequalities, we prove that for $A, B \in \mathbb{M}_n(\mathbb{C})$ and for any nonnegative increasing convex function f on $[0, \infty)$ with $f(0) = 0$, we have

$$s_j(f(|aA^*B + bB^*A|)) \leq \frac{1}{2} \left[\begin{array}{c} s_j(f(a|A|^2 + b|B|^2) \oplus f(b|A|^2 + a|B|^2)) \\ + s_{j-i+1}(f(|bA^*B + aB^*A|) \oplus f(|aA^*B + bB^*A|)) \end{array} \right],$$

where $a, b \geq 0$ and $j = 1, \dots, n$. Also, an upper bound for $\|ReA\|$ were given.

1. INTRODUCTION

Let $\mathbb{M}_n(\mathbb{C})$ be the C^* -algebra of all $n \times n$ complex matrices. A positive semidefinite (positive) matrix $A \in \mathbb{M}_n(\mathbb{C})$ is the matrix that satisfies $x^*Ax \geq 0$ ($x^*Ax > 0$) for all $x \in \mathbb{C}^n$. This singular values of $A \in \mathbb{M}_n(\mathbb{C})$ are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ denoted by $s_j(A)$, $j = 1, \dots, n$, with $s_1(A) \geq \dots \geq s_n(A)$. If the matrix norm on $\mathbb{M}_n(\mathbb{C})$, denoted by $\|\cdot\|$, satisfies that $\|UAV\| = \|A\|$ for every $A, U, V \in \mathbb{M}_n(\mathbb{C})$ where U and V are unitary, then it is called unitarily invariant norm. The spectral norm is one of the most common examples of unitarily invariant norms, which is defined for every $A \in \mathbb{M}_n(\mathbb{C})$ and $x \in \mathbb{C}^n$ by

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

It is known that $\|A\| = s_1(A)$ for every $A \in \mathbb{M}_n(\mathbb{C})$.

The numerical radius $w(A)$ of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$w(A) = \max_{\|x\|=1} \{|\langle Ax, x \rangle| : x \in \mathbb{C}^n\}. \quad (1.1)$$

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In [24], the author gave a distinguished definition for $w(A)$ by

$$w(A) = \sup_{\theta \in \mathbb{R}} \| \operatorname{Re} e^{i\theta} A \|,$$

where $\operatorname{Re} A = \frac{A+A^*}{2}$ is the real part of $A \in \mathbb{M}_n(\mathbb{C})$.

It is known that $w(\cdot)$ defines a norm on $\mathbb{M}_n(\mathbb{C})$ which is equivalent to the spectral norm $\|\cdot\|$. Moreover, for every $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|. \quad (1.2)$$

It is known that for $A, B \in \mathbb{M}_n(\mathbb{C})$, the direct sum of A and B , denoted by $A \oplus B$, is defined on $\mathbb{M}_{2n}(\mathbb{C})$ by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

It is known that for every $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$s_j(A) = s_j(A^*) = s_j(|A|), \quad j = 1, \dots, n,$$

$$s_j \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = s_j \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right), \quad j = 1, \dots, n,$$

and

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max\{\|A\|, \|B\|\}. \quad (1.3)$$

The previous basic relations on singular values, spectral norms, and numerical radii of matrices will be used in our paper. For more about matrices, the reader may refer to [4], [20], and [25].

In 2007, Hirzallah and Kittaneh [19] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(AB^* + BA^*) \leq s_j \left((|A|^2 + |B|^2) \oplus (|A|^2 + |B|^2) \right), \quad (1.4)$$

for $j = 1, \dots, n$. Also, a related result of the inequality (1.4) was given by the authors in [16], that is for $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$s_j(A^*B + B^*A) \leq s_j \left((|A|^2 + |B|^2) \oplus (|A|^2 + |B|^2) \right), \quad (1.5)$$

for $j = 1, \dots, n$. In [1], Al-Natoor proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $a, b \geq 0$ and if f is a nonnegative increasing concave function on $[0, \infty)$, $f(0) = 0$, then

$$s_j(f(|aA^*B + bB^*A|)) \leq \left[\begin{array}{c} s_j \left(f \left(\frac{a|A|^2 + b|B|^2}{2} \right) \oplus f \left(\frac{b|A|^2 + a|B|^2}{2} \right) \right) \\ + s_{j-i+1} \left(f \left(\frac{b|A^*B + aB^*A|}{2} \right) \oplus f \left(\frac{|aA^*B + bB^*A|}{2} \right) \right) \end{array} \right], \quad (1.6)$$

for $1 \leq i \leq j \leq n$. By taking $f(t) = t$, $t \geq 0$, it can be noticed that the inequality (1.6) is related to the inequality (1.5) and they are equivalent if $j = 1$ and $a = b = 1$.

For more generalizations, counterparts, and recent related results about singular values, numerical radius, and matrix norms, the reader may refer to [2], [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17], and [22].

The United Nations Sustainable Development Goals (SDGs) highlight the importance of mathematical tools for analyzing stability, efficiency, and robustness in complex systems. Such systems are often modeled by matrices or linear operators, where singular values, the spectral norm, and the numerical radius quantify amplification, sensitivity, and stability properties. These spectral quantities arise in applications related to resilient infrastructure and network dynamics (SDGs 9 and 11), as well as in modeling and optimization problems in water distribution and resource management systems (SDG 6), thereby motivating the investigation of relationships between matrix norms and spectral characteristics.

In this paper, we give some singular value inequalities for functions of matrices that related to some inequalities given in [1] and we give some upper bounds for $s_j(A^*B + B^*A)$. In addition, we obtain an upper bound of $\|ReA\|$ involving the numerical radii of matrices.

2. MAIN RESULTS

We begin this section with the following lemma (see [4, p. 291]) which is essential in our work:

Lemma 2.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing function on $[0, \infty)$. Then*

$$s_j(f(|A|)) = f(s_j(A)),$$

for $j = 1, \dots, n$.

To achieve our goal, we need the following two lemmas. The first lemma can be found in [23] and the reader may refer to [25, p. 275] for the second lemma.

Lemma 2.2. *Let $R \in \mathbb{M}_m(\mathbb{C})$ and $T \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} R & S \\ S^* & T \end{bmatrix}$ is positive semidefinite and let $r = \min(m, n)$. Then,*

$$2s_j(S) \leq s_j\left(\begin{bmatrix} R & S \\ S^* & T \end{bmatrix}\right),$$

for $j = 1, \dots, r$.

Lemma 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(A + B) \leq s_j(A) + s_{j-i+1}(B),$$

for $1 \leq i \leq j \leq n$. In particular, if $i = j$, then

$$s_j(A + B) \leq s_j(A) + \|B\|.$$

Theorem 2.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing convex function on $[0, \infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have*

$$s_j(f(|aA^*B + bB^*A|)) \leq \frac{1}{2} \left[\begin{array}{c} s_j(f(a|A|^2 + b|B|^2) \oplus f(b|A|^2 + a|B|^2)) \\ + s_{j-i+1}(f(|bA^*B + aB^*A|) \oplus f(|aA^*B + bB^*A|)) \end{array} \right], \quad (2.1)$$

for $1 \leq i \leq j \leq n$.

Proof. Let $X = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} B & A \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} aX^*X + bY^*Y &= a \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} B^* & 0 \\ A^* & 0 \end{bmatrix} \begin{bmatrix} B & A \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a|A|^2 + b|B|^2 & aA^*B + bB^*A \\ bA^*B + aB^*A & b|A|^2 + a|B|^2 \end{bmatrix}. \end{aligned}$$

Now, using Lemma 2.1, Lemma 2.2, and Lemma 2.3, we have

$$\begin{aligned} s_j(f(|aA^*B + bB^*A|)) &= f(s_j(aA^*B + bB^*A)) \\ &\leq f\left(\frac{1}{2}s_j\left(\begin{bmatrix} a|A|^2 + b|B|^2 & aA^*B + bB^*A \\ bA^*B + aB^*A & b|A|^2 + a|B|^2 \end{bmatrix}\right)\right) \end{aligned} \quad (2.2)$$

$$\begin{aligned} &= f\left(\frac{1}{2}s_j\left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} + \begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix}\right)\right) \\ &\leq f\left(\frac{1}{2}s_j\left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix}\right) + \frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix}\right)\right), \end{aligned} \quad (2.3)$$

by the convexity of f and using Lemma 2.1 on the inequality (2.3), we get

$$\begin{aligned} s_j(f(|aA^*B + bB^*A|)) &\leq \frac{1}{2} \left[\begin{aligned} &f\left(s_j\left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix}\right)\right) \\ &+ f\left(s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix}\right)\right) \end{aligned} \right] \\ &= \frac{1}{2} \left[\begin{aligned} &f\left(s_j\left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix}\right)\right) \\ &+ f\left(s_{j-i+1}\left(\begin{bmatrix} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{bmatrix}\right)\right) \end{aligned} \right] \\ &= \frac{1}{2} \left[\begin{aligned} &s_j\left(\begin{bmatrix} f(a|A|^2 + b|B|^2) & 0 \\ 0 & f(b|A|^2 + a|B|^2) \end{bmatrix}\right) \\ &+ s_{j-i+1}\left(f\left(\begin{bmatrix} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{bmatrix}\right)\right) \end{aligned} \right] \\ &= \frac{1}{2} \left[\begin{aligned} &s_j\left(\begin{bmatrix} f(a|A|^2 + b|B|^2) & 0 \\ 0 & f(b|A|^2 + a|B|^2) \end{bmatrix}\right) \\ &+ s_{j-i+1}\left(\begin{bmatrix} f(|bA^*B + aB^*A|) & 0 \\ 0 & f(|aA^*B + bB^*A|) \end{bmatrix}\right) \end{aligned} \right] \end{aligned} \quad (2.4)$$

$$= \frac{1}{2} \left[\begin{array}{c} s_j \left(f \left(a|A|^2 + b|B|^2 \right) \oplus f \left(b|A|^2 + a|B|^2 \right) \right) \\ + s_{j-i+1} \left(f \left(|bA^*B + aB^*A| \right) \oplus f \left(|aA^*B + bB^*A| \right) \right) \end{array} \right].$$

This proves the inequality (2.1). □

An application on Theorem 2.1 is the following corollary.

Corollary 2.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $a, b \geq 0$, we have*

$$s_j(aA^*B + bB^*A) \leq \frac{1}{2} \left[\begin{array}{c} s_j \left(\left(a|A|^2 + b|B|^2 \right) \oplus \left(b|A|^2 + a|B|^2 \right) \right) \\ + s_{j-i+1} \left(\left(bA^*B + aB^*A \right) \oplus \left(aA^*B + bB^*A \right) \right) \end{array} \right], \tag{2.5}$$

for $1 \leq i \leq j \leq n$. And

$$s_j(aA^*B + bB^*A) \leq \frac{1}{2} \left[\begin{array}{c} s_j \left(\left(a|A|^2 + b|B|^2 \right) \oplus \left(b|A|^2 + a|B|^2 \right) \right) \\ + \|aA^*B + bB^*A\| \end{array} \right], \tag{2.6}$$

for $j = 1, \dots, n$.

Proof. The inequality (2.5) follows from the inequality (2.1) by taking $f(t) = t, t \geq 0$. Also, the inequality (2.6) follows by letting $i = j$ in the inequality (2.5) and using the relation (1.3). □

Theorem 2.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing convex function on $[0, \infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have*

$$s_j(f(|aA^*B + bB^*A|)) \leq \frac{1}{4} \left[\begin{array}{c} s_j \left(f \left(2a|A|^2 \right) \oplus f \left(2a|B|^2 \right) \right) + \max \left\{ \|f(2b|B|^2)\|, \|f(2b|A|^2)\| \right\} \\ + 2s_{j-i+1} \left(f \left(|bA^*B + aB^*A| \right) \oplus f \left(|aA^*B + bB^*A| \right) \right) \end{array} \right], \tag{2.7}$$

for $1 \leq i \leq j \leq n$.

Proof. Starting from the inequality (2.4), we have

$$s_j(f(|aA^*B + bB^*A|)) \leq \frac{1}{2} \left[\begin{array}{c} f \left(s_j \left(\left[\begin{array}{cc} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{array} \right] \right) \right) \\ + f \left(s_{j-i+1} \left(\left[\begin{array}{cc} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{array} \right] \right) \right) \end{array} \right].$$

Applying Lemma 2.3, the convexity of the function f , and Lemma 2.1 on the previous inequality, we get

$$\begin{aligned} s_j(f(|aA^*B + bB^*A|)) &\leq \frac{1}{2} \left[\begin{array}{c} f \left(s_j \left(\left[\begin{array}{cc} a|A|^2 & 0 \\ 0 & a|B|^2 \end{array} \right] \right) + \left\| \left[\begin{array}{cc} b|B|^2 & 0 \\ 0 & b|A|^2 \end{array} \right] \right\| \right) \\ + f \left(s_{j-i+1} \left(\left[\begin{array}{cc} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{array} \right] \right) \right) \end{array} \right] \\ &\leq \frac{1}{4} \left[\begin{array}{c} s_j \left(f \left(\left[\begin{array}{cc} 2a|A|^2 & 0 \\ 0 & 2a|B|^2 \end{array} \right] \right) + f \left(\left\| \left[\begin{array}{cc} 2b|B|^2 & 0 \\ 0 & 2b|A|^2 \end{array} \right] \right\| \right) \right) \\ + 2f \left(s_{j-i+1} \left(\left[\begin{array}{cc} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{array} \right] \right) \right) \end{array} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left[s_j \left(\begin{bmatrix} f(2a|A|^2) & 0 \\ 0 & f(2a|B|^2) \end{bmatrix} \right) + \left\| \begin{bmatrix} f(2b|B|^2) & 0 \\ 0 & f(2b|A|^2) \end{bmatrix} \right\| \right] \\
&\quad + 2s_{j-i+1} \left(\begin{bmatrix} f(|bA^*B + aB^*A|) & 0 \\ 0 & f(|aA^*B + bB^*A|) \end{bmatrix} \right) \\
&= \frac{1}{4} \left[s_j (f(2a|A|^2) \oplus f(2a|B|^2)) + \max \left\{ \left\| f(2b|B|^2) \right\|, \left\| f(2b|A|^2) \right\| \right\} \right] \\
&\quad + 2s_{j-i+1} (f(|bA^*B + aB^*A|) \oplus f(|aA^*B + bB^*A|))
\end{aligned}$$

This proves the inequality (2.7). \square

An application on Theorem 2.2 is the following corollary.

Corollary 2.2. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $a, b \geq 0$, we have

$$s_j(aA^*B + bB^*A) \leq \left[\begin{array}{l} s_j \left(\frac{a|A|^2 \oplus a|B|^2}{2} \right) + \frac{b}{2} \max \{ \|A\|^2, \|B\|^2 \} \\ + s_{j-i+1} \left(\frac{(bA^*B + aB^*A) \oplus (aA^*B + bB^*A)}{2} \right) \end{array} \right], \quad (2.8)$$

for $1 \leq i \leq j \leq n$. And

$$s_j(aA^*B + bB^*A) \leq \left[\begin{array}{l} s_j \left(\frac{a|A|^2 \oplus a|B|^2}{2} \right) + \frac{b}{2} \max \{ \|A\|^2, \|B\|^2 \} \\ + \frac{1}{2} \|aA^*B + bB^*A\| \end{array} \right], \quad (2.9)$$

for $j = 1, \dots, n$.

Proof. The inequality (2.8) follows from the inequality (2.7) by taking $f(t) = t, t \geq 0$. Also, the inequality (2.9) follows by letting $i = j$ in the inequality (2.8) and using the relation (1.3). \square

Theorem 2.3. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing convex function on $[0, \infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have

$$s_j(f(|aA^*B + bB^*A|)) \leq \frac{1}{2} \left[\begin{array}{l} s_j (f(|bA^*B + aB^*A|) \oplus f(|aA^*B + bB^*A|)) \\ + s_{j-i+1} (f(a|A|^2 + b|B|^2) \oplus f(b|A|^2 + a|B|^2)) \end{array} \right], \quad (2.10)$$

for $1 \leq i \leq j \leq n$.

Proof. Starting from the inequality (2.2), we have

$$\begin{aligned}
s_j(f(|aA^*B + bB^*A|)) &\leq f \left(\frac{1}{2} s_j \left(\begin{bmatrix} a|A|^2 + b|B|^2 & aA^*B + bB^*A \\ bA^*B + aB^*A & b|A|^2 + a|B|^2 \end{bmatrix} \right) \right) \\
&= f \left(\frac{1}{2} s_j \left(\begin{array}{l} \begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix} \\ + \begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} \end{array} \right) \right). \quad (2.11)
\end{aligned}$$

Now, applying Lemma 2.3, the convexity of the function f , and Lemma 2.1 on the inequality (2.11), we get

$$\begin{aligned}
 s_j(f(|aA^*B + bB^*A|)) &\leq f \left(\begin{array}{c} \frac{1}{2}s_j \left(\begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix} \right) \\ + \frac{1}{2}s_{j-i+1} \left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} \right) \end{array} \right) \\
 &= f \left(\begin{array}{c} \frac{1}{2}s_j \left(\begin{bmatrix} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{bmatrix} \right) \\ + \frac{1}{2}s_{j-i+1} \left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} \right) \end{array} \right) \\
 &\leq \frac{1}{2} \left[\begin{array}{c} f \left(s_j \left(\begin{bmatrix} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{bmatrix} \right) \right) \\ + f \left(s_{j-i+1} \left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} \right) \right) \end{array} \right] \\
 &= \frac{1}{2} \left[\begin{array}{c} s_j \left(f \left(\begin{bmatrix} |bA^*B + aB^*A| & 0 \\ 0 & |aA^*B + bB^*A| \end{bmatrix} \right) \right) \\ + s_{j-i+1} \left(f \left(\begin{bmatrix} a|A|^2 + b|B|^2 & 0 \\ 0 & b|A|^2 + a|B|^2 \end{bmatrix} \right) \right) \end{array} \right] \\
 &= \frac{1}{2} \left[\begin{array}{c} s_j \left(\begin{bmatrix} f(|bA^*B + aB^*A|) & 0 \\ 0 & f(|aA^*B + bB^*A|) \end{bmatrix} \right) \\ + s_{j-i+1} \left(\begin{bmatrix} f(a|A|^2 + b|B|^2) & 0 \\ 0 & f(b|A|^2 + a|B|^2) \end{bmatrix} \right) \end{array} \right] \\
 &= \frac{1}{2} \left[\begin{array}{c} s_j (f(|bA^*B + aB^*A|) \oplus f(|aA^*B + bB^*A|)) \\ + s_{j-i+1} (f(a|A|^2 + b|B|^2) \oplus f(b|A|^2 + a|B|^2)) \end{array} \right].
 \end{aligned}$$

This proves the inequality (2.10). □

An application on Theorem 2.3 is the following corollary.

Corollary 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $a, b \geq 0$, we have*

$$s_j(aA^*B + bB^*A) \leq \frac{1}{2} \left[\begin{array}{c} s_j ((bA^*B + aB^*A) \oplus (aA^*B + bB^*A)) \\ + s_{j-i+1} ((a|A|^2 + b|B|^2) \oplus (b|A|^2 + a|B|^2)) \end{array} \right], \tag{2.12}$$

for $1 \leq i \leq j \leq n$. And

$$s_j(aA^*B + bB^*A) \leq \frac{1}{2} \left[\begin{array}{c} s_j ((bA^*B + aB^*A) \oplus (aA^*B + bB^*A)) \\ + \max\{\|a|A|^2 + b|B|^2\|, \|b|A|^2 + a|B|^2\|\} \end{array} \right], \tag{2.13}$$

for $j = 1, \dots, n$.

Proof. The inequality (2.12) follows from the inequality (2.10) by taking $f(t) = t, t \geq 0$. Also, the inequality (2.13) follows by letting $i = j$ in the inequality (2.12) and using the relation (1.3). □

Corollary 2.4. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then

$$s_j(A^*B + B^*A) \leq \frac{1}{2}s_j((A^*B + B^*A) \oplus (A^*B + B^*A)) + \|A\| \|B\|,$$

for $j = 1, \dots, n$.

Proof. Taking $a = b = 1$ in the inequality (2.13), we get

$$s_j(A^*B + B^*A) \leq \frac{1}{2} \left[s_j((A^*B + B^*A) \oplus (A^*B + B^*A)) + \left\| |A|^2 + |B|^2 \right\| \right] \quad (2.14)$$

$$\leq \frac{1}{2} \left[s_j((A^*B + B^*A) \oplus (A^*B + B^*A)) + \|A\|^2 + \|B\|^2 \right]. \quad (2.15)$$

Consider $t > 0$, replacing A and B in the inequality (2.15) by $\sqrt{t}A$ and $\frac{1}{\sqrt{t}}B$, respectively, then taking the minimum over t , we get

$$s_j(A^*B + B^*A) \leq \frac{1}{2}s_j((A^*B + B^*A) \oplus (A^*B + B^*A)) + \|A\| \|B\|,$$

this completes the proof. \square

The following corollary gives an application of our work.

Corollary 2.5. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then

$$s_j(2\operatorname{Re}A) \leq \frac{1}{2} \left[s_j(2\operatorname{Re}A \oplus 2\operatorname{Re}A) + \left\| |A|^2 + I \right\| \right], \quad (2.16)$$

for $j = 1, \dots, n$.

Proof. Taking $B = I$ in the inequality (2.14), we get the inequality (2.16). \square

It is easy to show that for $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$\|\operatorname{Re}A\| \leq \|A\|. \quad (2.17)$$

Using Corollary 2.5, a refinement of the inequality (2.17) will be given. This can be shown by taking $j = 1$ in the inequality (2.16) then using the fact that $s_1(A) = \|A\|$ and the relation (1.3), which gives

$$\begin{aligned} 2\|\operatorname{Re}A\| &\leq \frac{1}{2} \left[\|2\operatorname{Re}A \oplus 2\operatorname{Re}A\| + \left\| |A|^2 + I \right\| \right] \\ &= \|\operatorname{Re}A\| + \frac{1}{2} \left\| |A|^2 + I \right\| \\ &\leq \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}A\| + \frac{1}{2} (\|A\|^2 + 1) \\ &= w(A) + \frac{1}{2} (\|A\|^2 + 1), \end{aligned}$$

which implies that

$$\|\operatorname{Re}A\| \leq \frac{1}{2}w(A) + \frac{1}{4}(\|A\|^2 + 1). \quad (2.18)$$

The inequalities (2.17) and (2.18) give a noncomparable upper bounds for $\|\operatorname{Re}A\|$. This can be shown in the following example

Example 2.1. Let $A = \begin{bmatrix} 0.1 & 1 \\ 0.1 & 0.1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, then

$$0.8305 \approx \frac{1}{2}w(A) + \frac{1}{4}(\|A\|^2 + 1) < \|A\| \approx 1.011,$$

and

$$4.1451 \approx \frac{1}{2}w(B) + \frac{1}{4}(\|B\|^2 + 1) > \|B\| \approx 3.193.$$

Corollary 2.6. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then

$$\|ReA\| \leq \min \left\{ \frac{1}{2}w(A) + \frac{1}{4}(\|A\|^2 + 1), \|A\| \right\}.$$

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REFERENCES

- [1] A. Al-Natoor, Singular Value Inequalities for Functions of Matrices, *Filomat* 39 (2025), 6483–6491.
- [2] M. Al-khlyleh, F. Alrimawi, Further Interpolation Inequalities Related to Arithmetic-Geometric Mean, Cauchy-Schwarz and Hölder Inequalities for Unitarily Invariant Norms, *Math. Inequal. Appl.* (2020), 1135–1143. <https://doi.org/10.7153/mia-2020-23-87>.
- [3] M. Al-khlyleh, F. Alrimawi, Generalizations of the Unitarily Invariant Norm Version of the Arithmetic-Geometric Mean Inequality, *Adv. Oper. Theory* 6 (2020), 2. <https://doi.org/10.1007/s43036-020-00106-1>.
- [4] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, Springer, New York, 1997. <https://doi.org/10.1007/978-1-4612-0653-8>.
- [5] M.W. Alomari, Numerical Radius Inequalities for Hilbert Space Operators, *Complex Anal. Oper. Theory* 15 (2021), 111. <https://doi.org/10.1007/s11785-021-01161-z>.
- [6] F. Alrimawi, M. Al-Khlyleh, F.A. Abushaheen, Norm Inequalities for Accretive-Dissipative Block Matrices, *Methods Funct. Anal. Topol.* 26 (2020), 201–215. https://doi.org/10.31392/mfat-npu26_3.2020.02.
- [7] F. Alrimawi, M. Al-Khlyleh, Some Norm Inequalities Involving Convex Functions of Operators, *Positivity* 26 (2022), 79. <https://doi.org/10.1007/s11117-022-00946-6>.
- [8] F. Alrimawi, H. Kawariq, On Some Generalized Numerical Radius Inequalities for Hilbert Space Operators, *J. Math. Comput. Sci.* 32 (2024), 257–262. <https://doi.org/10.22436/jmcs.032.03.06>.
- [9] F. Alrimawi, F.A. Abushaheen, R. Alkhateeb, Improved Lower Bounds for Numerical Radius via Cartesian Decomposition, *J. Math. Comput. Sci.* 33 (2023), 169–175. <https://doi.org/10.22436/jmcs.033.02.05>.
- [10] F. Alrimawi, H. Kawariq, F.A. Abushaheen, Generalized-Weighted Numerical Radius Inequalities for Schatten p -Norms, *Int. J. Math. Comput. Sci.* 17 (2022), 1463–2022.
- [11] F. Alrimawi, Inequalities for Certain Powers of Several Positive Definite Matrices, *Math. Inequal. Appl.* 22 (2019), 1027–1035. <https://doi.org/10.7153/mia-2019-22-70>.
- [12] F. Alrimawi, Some Norm Inequalities of Davidson-Power Type, *Int. J. Math. Comput. Sci.* 16 (2021), 1423–1433.
- [13] F. Alrimawi, Inequalities for Functions of 2×2 Block Matrices, *Acta Sci. Math.* 89 (2023), 23–33. <https://doi.org/10.1007/s44146-023-00082-x>.
- [14] F. Alrimawi, O. Hirzallah, F. Kittaneh, Singular Value Inequalities Involving Convex and Concave Functions of Positive Semidefinite Matrices, *Ann. Funct. Anal.* 11 (2020), 1257–1273. <https://doi.org/10.1007/s43034-020-00084-8>.
- [15] F. Alrimawi, O. Hirzallah, F. Kittaneh, Norm Inequalities Involving the Weighted Numerical Radii of Operators, *Linear Algebr. Appl.* 657 (2023), 127–146. <https://doi.org/10.1016/j.laa.2022.10.018>.

- [16] R. Bhatia, F. Kittaneh, The Matrix Arithmetic–Geometric Mean Inequality Revisited, *Linear Algebr. Appl.* 428 (2008), 2177–2191. <https://doi.org/10.1016/j.laa.2007.11.030>.
- [17] P. Bhunia, K. Paul, Some Improvements of Numerical Radius Inequalities of Operators and Operator Matrices, *Linear Multilinear Algebr.* 70 (2020), 1995–2013. <https://doi.org/10.1080/03081087.2020.1781037>.
- [18] S.S. Dragomir, Power Inequalities for the Numerical Radius of a Product of Two Operators in Hilbert Spaces, *Sarajev. J. Math.* 5 (2024), 269–278. <https://doi.org/10.5644/sjm.05.2.10>.
- [19] O. Hirzallah, F. Kittaneh, Inequalities for Sums and Direct Sums of Hilbert Space Operators, *Linear Algebr. Appl.* 424 (2007), 71–82. <https://doi.org/10.1016/j.laa.2006.03.036>.
- [20] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.
- [21] F. Kittaneh, Numerical Radius Inequalities for Hilbert Space Operators, *Stud. Math.* 168 (2005), 73–80. <https://doi.org/10.4064/sm168-1-5>.
- [22] M. Sattari, M.S. Moslehian, T. Yamazaki, Some Generalized Numerical Radius Inequalities for Hilbert Space Operators, *Linear Algebr. Appl.* 470 (2015), 216–227. <https://doi.org/10.1016/j.laa.2014.08.003>.
- [23] Y. Tao, More Results on Singular Value Inequalities of Matrices, *Linear Algebr. Appl.* 416 (2006), 724–729. <https://doi.org/10.1016/j.laa.2005.12.017>.
- [24] T. Yamazaki, On Upper and Lower Bounds of the Numerical Radius and an Equality Condition, *Stud. Math.* 178 (2007), 83–89. <https://doi.org/10.4064/sm178-1-5>.
- [25] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer New York, 2011. <https://doi.org/10.1007/978-1-4614-1099-7>.