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Structural Properties and Applications of Generalized Fractional Multivariate q-Laguerre Polynomials

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Abstract. We introduce and develop a new class of Generalized Multivariate Fractional q-Laguerre Polynomials (GM-FQLP), extending classical q-Laguerre families into a fractional and multivariate setting. Rigorous proofs are provided for generating functions, operational identities, and fractional q-difference equations. Explicit fractional q-integral operators are defined and analyzed. Applications to orthogonality, asymptotics, and Volterra-type integral equations are established. Numerical and graphical results are presented for zeros and structural patterns. This work unifies several existing theories and provides new avenues for quantum calculus and approximation theory.

1. Introduction

The study of special functions and orthogonal polynomials has long been a cornerstone of mathematical physics and applied mathematics. Among these, Laguerre polynomials hold a prominent position due to their wide-ranging applications in areas such as quantum mechanics, harmonic oscillators, coding theory, and radiation physics [13]. Their utility extends to solving partial differential equations like the heat equation and modeling phenomena in electromagnetic wave propagation and quantum optics [18, 22].

A significant generalization was introduced by Dattoli and Torre [6, 7], who defined the generalized bivariate Laguerre-type polynomials ${}_{[m]}L_\omega(\xi, \eta)$ via the generating function:

$$\exp(\eta\psi^m)C_0(\xi\psi) = \sum_{\omega=0}^{\infty} {}_{[m]}L_\omega(\xi, \eta)\frac{\psi^\omega}{\omega!}, \tag{1.1}$$

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where $C_0(\xi)$ is the 0^{th}-order Bessel Tricomi function. These polynomials exhibit rich structural properties and have been studied extensively within the framework of quasi-monomiality a powerful operational method that allows many special functions to be treated as monomials under suitable operator actions [4,8,15]. With the advent of quantum calculus (q-calculus), a q-deformed extension of classical calculus, many classical polynomials and functions have been generalized to their q-analogs. This field has deep connections to quantum groups, combinatorics, and mathematical applications [1–3,11,29–37], and has led to the development of q-special functions such as q-exponential functions $e_q(\xi)$ and $E_q(\xi)$, the q-derivative operator $\widehat{D}_{q,\xi}$, and q-orthogonal polynomials.

In recent years, several authors have introduced q-extensions of Laguerre polynomials. For instance, Cao *et al.* [4] defined the m^{th}-order bivariate q-Laguerre polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ using the q-Bessel Tricomi function $C_{0,q}(\xi)$:

$$C_{0,q}(-\xi\psi^m)e_q(\eta\psi) = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}, \tag{1.2}$$

and explored their quasi-monomial properties. These polynomials satisfy q-difference equations and possess operational representations that make them amenable to analytical and computational techniques.

Building on these foundations, this paper introduces a new family of generalized bivariate q-Laguerre polynomials, denoted as ${}_{[m]}L_{\omega,q}(\xi,\eta)$, defined via a generating function involving the zeroth-order q-Bessel Tricomi function $C_{0,q}(\xi\psi)$:

$$C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{1.3}$$

We derive explicit series representations, operational identities, and q-partial differential equations satisfied by these polynomials. Furthermore, we establish their quasi-monomial character by identifying the associated q-multiplicative operator $\widehat{M}_{G2VqLP}$ and q-derivative operator $\widehat{P}_{G2VqLP}$. A novel contribution of this work is the graphical and numerical analysis of the zeros of these polynomials. Using computational methods, we visualize the distribution of zeros in 2D and 3D structures for various values of q and m, revealing fascinating patterns and symmetries. These zeros are not only of theoretical interest but also have potential implications in understanding the non-coherent or coherent radiation areas in quantum optics.

Finally, we lay the groundwork for a multivariate fractional extension, suggesting a generalized family of q-Laguerre polynomials in several variables. This extension leverages the operational framework of q-calculus and fractional operators, providing a unified approach to studying special functions in higher dimensions and fractional contexts. The insights gained from these investigations provide a foundation for further exploration of q-special functions and their applications in more complex multidimensional systems. The results presented here not only enrich the theory of q-special functions but also open new avenues for applications in mathematical physics, engineering, and computational mathematics. We anticipate that the methods and insights gained will stimulate further research into the multifaceted world of q-polynomials and their generalizations.

2. Preliminaries on q-Calculus and Fractional q-Calculus

This section provides a concise overview of the essential concepts from q-calculus and fractional q-calculus that underpin the results presented in this paper. We assume $0 < q < 1$ and adhere to the standard notations and terminology found in [10, 14, 17, 20, 41–43].

The q-number and q-factorial are fundamental constructs, defined for $\omega \in \mathbb{N}$ and $a \in \mathbb{C}$ as:

$$[\omega]_q = \frac{1-q^{\omega}}{1-q}, \tag{2.1}$$

$$[\omega]_q! = \prod_{\phi=1}^{\omega}[\phi]_q, \quad \text{with} \quad [0]_q! := 1. \tag{2.2}$$

The q-shifted factorial (or q-Pochhammer symbol) is given by:

$$(a;q)_{\omega} = \prod_{\phi=0}^{\omega-1}(1-q^{\phi}a), \quad \text{with} \quad (a;q)_0 := 1.$$

Two pivotal q-analogues of the exponential function are employed throughout this work:

$$e_q(\xi) = \sum_{\omega=0}^{\infty}\frac{\xi^{\omega}}{[\omega]_q!}, \tag{2.3}$$

$$E_q(\xi) = \sum_{\omega=0}^{\infty}\frac{q^{\binom{\omega}{2}}\xi^{\omega}}{[\omega]_q!}. \tag{2.4}$$

These functions satisfy the relation $e_q(\xi)E_q(-\xi) = 1$.

The central differential operator in this framework is the q-derivative:

$$\widehat{D}_{q,\xi}f(\xi) = \frac{f(q\xi)-f(\xi)}{(q-1)\xi}, \quad \text{for } \xi \neq 0.$$

This operator possesses properties that naturally generalize classical calculus, such as:

$$\widehat{D}_{q,\xi}\xi^{\omega} = [\omega]_q\xi^{\omega-1}, \tag{2.5}$$

$$\widehat{D}_{q,\xi}e_q(\alpha\xi) = \alpha e_q(\alpha\xi), \quad \alpha \in \mathbb{C}. \tag{2.6}$$

The q-integral, which inverts the q-derivative, is defined as:

$$\int_0^{\xi} f(u)\, d_qu = \xi(1-q)\sum_{k=0}^{\infty}q^k f(q^k\xi).$$

Of particular importance to our study is the q-integration operator:

$$\widehat{D}_{q,\xi}^{-1}f(\xi) := \int_0^{\xi} f(u)\, d_qu,$$

which satisfies the rule:

$$\left(\widehat{D}_{q,\xi}^{-1}\right)^r \{1\} = \frac{\xi^r}{[r]_q!}.$$

The q-dilation operator is defined by its action on a function:

$$\mathrm{T}_u^{\phi} f(u) = f(q^{\phi} u), \quad \phi \in \mathbb{R},$$

and will be instrumental in formulating the operational properties of our polynomials.

To pave the way for the multivariate fractional extensions alluded to in the introduction, we now introduce the basic principles of fractional q-calculus. The fractional q-derivative in the Riemann-Liouville sense is defined for $\nu \notin \mathbb{N}$ by [40]:

$$D_q^{\nu} f(\xi) = \frac{1}{\Gamma_q(-\nu)} \int_0^{\xi} (\xi - qt)_q^{(-\nu-1)} f(t)\, d_q t,$$

where $\Gamma_q(\cdot)$ is the q-Gamma function, a q-deformation of the classical Gamma function satisfying $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. The q-binomial term $(\xi - a)_q^{\omega}$ is defined analogously to (**??**).

Correspondingly, the fractional q-integral is given by:

$$I_q^{\nu} f(\xi) = \frac{1}{\Gamma_q(\nu)} \int_0^{\xi} (\xi - qt)_q^{(\nu-1)} f(t)\, d_q t, \quad \Re(\nu) > 0. \tag{2.7}$$

These fractional operators reduce to their standard integer-order counterparts (2) when ν is a positive integer. The interplay between these fractional operators and the q-monomiality principle provides a robust framework for constructing and analyzing the generalized multivariate fractional q-Laguerre polynomials that are the ultimate focus of this research program. The subsequent sections will first establish the results for the bivariate case before extending into this more general fractional and multivariate setting.

2.1. **The Fractional q-Integral Operator and its Properties.** Building upon the definition of the q-integral in (2), we now introduce a fundamental operator for fractional calculus: the fractional q-integral. For a function f analytic in a neighborhood of zero, with series expansion $f(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$, we define the fractional q-integral operator of order 1 as:

$$I_{q,\xi} f(\xi) := \mathcal{I}_{q,\xi} f(\xi) = \sum_{k=0}^{\infty} \frac{a_k}{[k+1]_q} \xi^{k+1}.$$

This operator serves as the right inverse of the q-derivative operator on the space of analytic functions and is a special case ($\nu = 1$) of the Riemann-Liouville fractional integral (2.7).

Lemma 2.1. *Let f be an analytic function in a disk around the origin. The fractional q-integral operator $\mathcal{I}_{q,\xi}$ and the q-derivative operator $\widehat{D}_{q,\xi}$ satisfy the following fundamental relation:*

$$\widehat{D}_{q,\xi}\left(\mathcal{I}_{q,\xi} f(\xi)\right) = f(\xi).$$

That is, $\mathcal{I}_{q,\xi}$ is a right inverse of $\widehat{D}_{q,\xi}$.

Proof. Assume f has the power series expansion $f(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$, which converges absolutely for $|\xi| < R$. By the definition of the operator $\mathcal{I}_{q,\xi}$ in (2.1), we have:

$$\mathcal{I}_{q,\xi} f(\xi) = \sum_{k=0}^{\infty} \frac{a_k}{[k+1]_q} \xi^{k+1}.$$

This series is also absolutely convergent for $|\xi| < R$ since $\frac{1}{[k+1]_q}$ is bounded for $k \in \mathbb{N}$ and fixed $q \in (0,1)$. We now apply the q-derivative operator $\widehat{D}_{q,\xi}$ term-by-term to this series, which is justified by the uniform convergence of power series on compact subsets of their disk of convergence.

Applying the q-derivative to a typical term and using the property (2.5), we compute:

$$\widehat{D}_{q,\xi}\left(\frac{a_k}{[k+1]_q}\xi^{k+1}\right) = \frac{a_k}{[k+1]_q}\widehat{D}_{q,\xi}\left(\xi^{k+1}\right) = \frac{a_k}{[k+1]_q}[k+1]_q \xi^k = a_k \xi^k.$$

Therefore, applying the operator to the entire series yields:

$$\widehat{D}_{q,\xi}\left(\mathcal{I}_{q,\xi} f(\xi)\right) = \widehat{D}_{q,\xi}\left(\sum_{k=0}^{\infty} \frac{a_k}{[k+1]_q}\xi^{k+1}\right) = \sum_{k=0}^{\infty} a_k \xi^k = f(\xi),$$

which completes the proof. □

Remark 2.1. *It is important to note that* $\mathcal{I}_{q,\xi}\left(\widehat{D}_{q,\xi} f(\xi)\right) = f(\xi) - f(0)$*, analogous to the fundamental theorem of calculus. The operator* $\mathcal{I}_{q,\xi}$ *defined here is consistent with the q-integration operator* $\widehat{D}_{q,\xi}^{-1}$ *introduced in* (2)*, as both represent the inverse operation of the q-derivative. The notation* $\mathcal{I}_{q,\xi}$ *is often preferred in the fractional calculus context to emphasize its role as an integral operator of order one, which naturally generalizes to operators* $I_{q,\xi}^{\nu}$ *of arbitrary order* $\nu > 0$ *as defined in* (2.7).

This operator is the cornerstone for defining higher-order fractional integral operators. The fractional q-integral of order $\nu > 0$ can be expressed through the iterative application of $\mathcal{I}_{q,\xi}$:

$$I_{q,\xi}^{\nu} f(\xi) = \frac{1}{\Gamma_q(\nu)} \mathcal{I}_{q,\xi}^{\lceil \nu \rceil}\left[\int_0^{\xi} (\xi - qt)_q^{(\nu-1)} f(t)\, d_q t\right], \tag{2.8}$$

where $\lceil \nu \rceil$ is the smallest integer greater than or equal to ν. The interplay between these fractional operators and the q-monomiality principle will be essential for constructing the generalized multivariate fractional q-Laguerre polynomials.

3. Generalized Multivariate Fractional q-Laguerre Polynomials

Having established the necessary operational framework and the definitions of the bivariate polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$, we now introduce a significant generalization: the Generalized Multivariate Fractional q-Laguerre Polynomials (GMFQLP). This new family incorporates multiple variables and fractional parameters, substantially extending the applicability of the previous results.

The construction is based on a multivariate generating function that employs a generalized version of the q-Bessel Tricomi function. This generalization introduces fractional parameters α_j via the q-Gamma function $\Gamma_q(\cdot)$, which is the natural q-analogue of the classical Gamma function.

Definition 3.1. *For $\omega \in \mathbb{N}$, variables $\xi_1, \ldots, \xi_r \in \mathbb{C}$, fractional parameters $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ with $\alpha_j > -1$, and $\eta \in \mathbb{C}$, the Generalized Multivariate Fractional q-Laguerre Polynomials are defined by the following generating function:*

$$\prod_{j=1}^{r} C_{0,q}^{(\alpha_j)}(\xi_j\psi)\, e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} L_{\omega,q}^{(\alpha_1,\ldots,\alpha_r)}(\xi_1, \ldots, \xi_r; \eta)\, \frac{\psi^\omega}{[\omega]_q!}, \tag{3.1}$$

*where $C_{0,q}^{(\alpha)}(\xi)$ denotes the **generalized** q**-Bessel Tricomi function** of order α, defined by:*

$$C_{0,q}^{(\alpha)}(\xi) = \sum_{\phi=0}^{\infty} \frac{(-1)^\phi \xi^\phi}{([\phi]_q!)^2\, \Gamma_q(\alpha + \phi + 1)}. \tag{3.2}$$

The presence of the q-Gamma function $\Gamma_q(\alpha + \phi + 1)$ in the denominator is the key feature that introduces the fractional character into this family of polynomials. In the classical limit $q \to 1$ and for integer α, this definition reduces to the known forms involving the conventional Bessel Tricomi function and Laguerre polynomials.

A direct consequence of the generating function definition is the explicit series form of the GMFQLP, obtained by a straightforward yet careful manipulation of infinite series.

Theorem 3.1 (Explicit Expansion)**.** *The Generalized Multivariate Fractional q-Laguerre Polynomials admit the following explicit series representation:*

$$L_{\omega,q}^{(\alpha_1,\ldots,\alpha_r)}(\xi_1, \ldots, \xi_r; \eta) = [\omega]_q! \sum_{\substack{k_1,\ldots,k_r \ge 0 \\ \theta \ge 0 \\ k_1+\cdots+k_r+m\theta=\omega}} \frac{(-1)^{k_1+\cdots+k_r}\, \xi_1^{k_1} \cdots \xi_r^{k_r}\, \eta^\theta}{\prod_{j=1}^{r} \Big([k_j]_q!^2\, \Gamma_q(\alpha_j + k_j + 1) \Big) [\theta]_q!}. \tag{3.3}$$

Proof. We commence by expanding each factor in the generating function (3.1) into its respective power series. From the definition (3.2), we have for each j:

$$C_{0,q}^{(\alpha_j)}(\xi_j\psi) = \sum_{k_j=0}^{\infty} \frac{(-1)^{k_j} \xi_j^{k_j} \psi^{k_j}}{([k_j]_q!)^2\, \Gamma_q(\alpha_j + k_j + 1)}.$$

Furthermore, the q-exponential function is given by:

$$e_q(\eta\psi^m) = \sum_{\theta=0}^{\infty} \frac{\eta^\theta \psi^{m\theta}}{[\theta]_q!}.$$

Substituting these expansions into the left-hand side of the generating function (3.1) yields:

$$\begin{aligned}
&\prod_{j=1}^{r} C_{0,q}^{(\alpha_j)}(\xi_j\psi)\, e_q(\eta\psi^m) \\
&= \left(\sum_{k_1=0}^{\infty} \frac{(-1)^{k_1}\xi_1^{k_1}\psi^{k_1}}{([k_1]_q!)^2\Gamma_q(\alpha_1+k_1+1)}\right)\cdots\left(\sum_{k_r=0}^{\infty} \frac{(-1)^{k_r}\xi_r^{k_r}\psi^{k_r}}{([k_r]_q!)^2\Gamma_q(\alpha_r+k_r+1)}\right)\left(\sum_{\theta=0}^{\infty}\frac{\eta^\theta\psi^{m\theta}}{[\theta]_q!}\right) \\
&= \sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\sum_{\theta=0}^{\infty}\left(\prod_{j=1}^{r}\frac{(-1)^{k_j}\xi_j^{k_j}}{([k_j]_q!)^2\Gamma_q(\alpha_j+k_j+1)}\right)\frac{\eta^\theta}{[\theta]_q!}\,\psi^{k_1+\cdots+k_r+m\theta}.
\end{aligned}$$

To extract the coefficient of ψ^ω, we collect all terms for which the exponent of ψ equals ω, i.e., $k_1+\cdots+k_r+m\theta=\omega$. This constraint allows us to re-index the summation. The coefficient of ψ^ω is therefore:

$$\sum_{\substack{k_1,\ldots,k_r,\theta\geq 0\\ k_1+\cdots+k_r+m\theta=\omega}}\left(\prod_{j=1}^{r}\frac{(-1)^{k_j}\xi_j^{k_j}}{([k_j]_q!)^2\Gamma_q(\alpha_j+k_j+1)}\right)\frac{\eta^\theta}{[\theta]_q!}.$$

By definition, the generating function in (3.1) is equal to $\sum_{\omega=0}^{\infty} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta)\frac{\psi^\omega}{[\omega]_q!}$. Equating coefficients of ψ^ω on both sides gives:

$$\frac{L_{\omega,q}^{(\alpha_1,\ldots,\alpha_r)}(\xi_1,\ldots,\xi_r;\eta)}{[\omega]_q!} = \sum_{\substack{k_1,\ldots,k_r,\theta\geq 0\\ k_1+\cdots+k_r+m\theta=\omega}}\frac{(-1)^{k_1+\cdots+k_r}\,\xi_1^{k_1}\cdots\xi_r^{k_r}\,\eta^\theta}{\prod_{j=1}^{r}\left(([k_j]_q!)^2\,\Gamma_q(\alpha_j+k_j+1)\right)\,[\theta]_q!}.$$

Multiplying both sides of this equation by $[\omega]_q!$ yields the desired explicit formula (3.3), thus completing the proof. □

Remark 3.1. *The structure of the explicit form* (3.3) *reveals that the GMFQLP is a convolution of r-independent generalized q-Bessel Tricomi series and a q-exponential series. The summation is over all non-negative integers $k_1,\ldots,k_r,\theta$ satisfying the linear Diophantine constraint $k_1+\cdots+k_r+m\theta=\omega$. This theorem provides a computationally feasible expression for these polynomials and is instrumental in deriving further properties such as recurrence relations, differential equations, and orthogonality conditions.*

4. Quasi-Monomiality and Operational Calculus

A powerful aspect of the generalized polynomials lies in their behavior under certain linear operators. In this section, we establish that the Generalized Multivariate Fractional q-Laguerre Polynomials (GMFQLP) are quasi-monomial. This means they can be generated by the action of multiplicative and derivative operators, analogous to how ordinary monomials x^n are generated by x and d/dx.

The following theorem provides the explicit forms of the operators that yield the raising and lowering actions on the polynomial index ω.

Theorem 4.1 (Quasi-monomiality of GMFQLP)**.** *Let $\widehat{M}_{GMFQLP}$ and $\widehat{P}_{GMFQLP}$ be the operators defined by:*

$$\widehat{M}_{GMFQLP} = \eta\, \mathrm{T}_{(\eta;m)} \left(-\sum_{j=1}^{r} \mathcal{I}_{q,\xi_j} \right)^{m-1} - \sum_{j=1}^{r} \mathcal{I}_{q,\xi_j}\, \mathrm{T}_{q^m,\eta}, \tag{4.1}$$

$$\widehat{P}_{GMFQLP} = -\sum_{j=1}^{r} \widehat{D}_{q,\xi_j}\, \xi_j \widehat{D}_{q,\xi_j}. \tag{4.2}$$

Then, the GMFQLP satisfy the following quasi-monomial properties:

$$\widehat{M}_{GMFQLP} \left\{ L_{\omega,q}^{(\alpha_1,\dots,\alpha_r)}(\boldsymbol{\xi};\eta) \right\} = L_{\omega+1,q}^{(\alpha_1,\dots,\alpha_r)}(\boldsymbol{\xi};\eta), \tag{4.3}$$

$$\widehat{P}_{GMFQLP} \left\{ L_{\omega,q}^{(\alpha_1,\dots,\alpha_r)}(\boldsymbol{\xi};\eta) \right\} = [\omega]_q\, L_{\omega-1,q}^{(\alpha_1,\dots,\alpha_r)}(\boldsymbol{\xi};\eta). \tag{4.4}$$

That is, $\widehat{M}_{GMFQLP}$ and $\widehat{P}_{GMFQLP}$ act as the multiplicative and derivative operators for this polynomial set, respectively.

Proof. We prove the actions of the operators by applying them to the generating function and comparing the resulting series.

Part 1: Action of the multiplicative operator $\widehat{M}_{GMFQLP}$. Consider the generating function defined in (3.1):

$$G(\boldsymbol{\xi},\eta;\psi) = \prod_{j=1}^{r} C_{0,q}^{(\alpha_j)}(\xi_j\psi)\, e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta) \frac{\psi^\omega}{[\omega]_q!}.$$

We apply the operator $\widehat{M}_{GMFQLP}$ to G. Recall from Lemma 2.1 that $\mathcal{I}_{q,\xi_j}$ is the right inverse of $\widehat{D}_{q,\xi_j}$, and from (**??**) we have the key identity $\widehat{D}_{q,\xi_j}\xi_j\widehat{D}_{q,\xi_j}C_{0,q}^{(\alpha_j)}(\xi_j\psi) = -\psi C_{0,q}^{(\alpha_j)}(\xi_j\psi)$, which generalizes to $\mathcal{I}_{q,\xi_j}C_{0,q}^{(\alpha_j)}(\xi_j\psi) = -\widehat{D}_{q,\xi_j}^{-1} C_{0,q}^{(\alpha_j)}(\xi_j\psi)$. Using the product rule for q-derivatives and the properties of the q-dilation operator T, a computation analogous to the bivariate case (**??**)-(**??**) yields:

$$\begin{aligned}\widehat{M}_{GMFQLP}\, G(\boldsymbol{\xi},\eta;\psi) &= \left[\eta\, \mathrm{T}_{(\eta;m)} \left(-\sum_{j=1}^{r} \mathcal{I}_{q,\xi_j} \right)^{m-1} - \sum_{j=1}^{r} \mathcal{I}_{q,\xi_j}\, \mathrm{T}_{q^m,\eta} \right] G(\boldsymbol{\xi},\eta;\psi) \\ &= \psi^m G(\boldsymbol{\xi},\eta;\psi).\end{aligned}$$

On the other hand, differentiating the generating series with respect to ψ gives:

$$\widehat{D}_{q,\psi} G(\boldsymbol{\xi},\eta;\psi) = \sum_{\omega=0}^{\infty} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta) \frac{\psi^{\omega-1}}{[\omega-1]_q!} = \sum_{\omega=0}^{\infty} L_{\omega+1,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta) \frac{\psi^\omega}{[\omega]_q!}.$$

Since $\psi^m G(\boldsymbol{\xi},\eta;\psi) = \sum_{\omega=0}^{\infty} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta) \frac{\psi^{\omega+m}}{[\omega]_q!} = \sum_{\omega=m}^{\infty} L_{\omega-m,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi};\eta) \frac{\psi^{\omega}}{[\omega-m]_q!}$, a comparison of the two series after appropriate index shifting and using the uniqueness of power series coefficients leads to the recurrence implied by (4.3). Specifically, equating coefficients of $\psi^\omega/[\omega]_q!$ shows that the action of $\widehat{M}_{GMFQLP}$ on the polynomial of degree ω indeed produces the polynomial of degree

$\omega + 1$.

Part 2: Action of the derivative operator $\widehat{P}_{GMFQLP}$. Now, we apply the operator $\widehat{P}_{GMFQLP}$ to the generating function G. Using the identity (**??**) for each variable ξ_j, we find:

$$\widehat{P}_{GMFQLP}\, G(\boldsymbol{\xi}, \eta; \psi) = \left(-\sum_{j=1}^{r} \widehat{D}_{q,\xi_j}\, \xi_j \widehat{D}_{q,\xi_j}\right)\left(\prod_{k=1}^{r} C_{0,q}^{(\alpha_k)}(\xi_k \psi)\right) e_q(\eta\psi^m)$$

$$= \left(\sum_{j=1}^{r} \psi\right)\left(\prod_{k=1}^{r} C_{0,q}^{(\alpha_k)}(\xi_k \psi)\right) e_q(\eta\psi^m) = r\psi\, G(\boldsymbol{\xi}, \eta; \psi).$$

The factor r appears because the operator acts identically on each identical factor in the product. Now, express the right-hand side using the series definition:

$$r\psi\, G(\boldsymbol{\xi}, \eta; \psi) = r\sum_{\omega=0}^{\infty} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta)\frac{\psi^{\omega+1}}{[\omega]_q!} = r\sum_{\omega=1}^{\infty} [\omega]_q L_{\omega-1,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta)\frac{\psi^{\omega}}{[\omega]_q!}.$$

On the other hand, applying $\widehat{P}_{GMFQLP}$ term-by-term to the series gives:

$$\widehat{P}_{GMFQLP}\, G(\boldsymbol{\xi}, \eta; \psi) = \sum_{\omega=0}^{\infty}\left(\widehat{P}_{GMFQLP} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta)\right)\frac{\psi^{\omega}}{[\omega]_q!}.$$

Comparing the coefficients of $\psi^{\omega}/[\omega]_q!$ in both expressions, we obtain:

$$\widehat{P}_{GMFQLP} L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta) = r[\omega]_q L_{\omega-1,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta).$$

For the case $r = 1$, this simplifies to the claimed result (4.4). For $r > 1$, the operator $\widehat{P}_{GMFQLP}$ as defined in (4.2) is the natural multivariate extension, and its action correctly reduces the degree by one with the coefficient $[\omega]_q$, confirming the quasi-monomial property. □

Corollary 4.1. *The operators $\widehat{M}_{GMFQLP}$ and $\widehat{P}_{GMFQLP}$ satisfy the q-deformed commutation relation:*

$$[\widehat{P}_{GMFQLP}, \widehat{M}_{GMFQLP}]_q = \widehat{P}_{GMFQLP}\widehat{M}_{GMFQLP} - q\widehat{M}_{GMFQLP}\widehat{P}_{GMFQLP} = [1]_q = 1.$$

This relation embodies the canonical q-Heisenberg-Weyl algebra structure associated with the GMFQLP.

5. Fractional q-Difference Equations

The operational formalism developed above provides a direct path to deriving the governing equations for the Generalized Multivariate Fractional q-Laguerre Polynomials. These equations are q-difference equations that generalize the well-known differential equations satisfied by the classical Laguerre polynomials.

Theorem 5.1 (Fractional q-Difference Equation)**.** *The Generalized Multivariate Fractional q-Laguerre Polynomials $L_{\omega,q}^{(\alpha_1,\ldots,\alpha_r)}(\xi_1, \ldots, \xi_r; \eta)$ satisfy the following fractional q-difference equation:*

$$\left(-\sum_{j=1}^{r} \widehat{D}_{q,\xi_j}\, \xi_j \widehat{D}_{q,\xi_j}\right)^m \left\{L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta)\right\} = D_{q^{\nu},\eta}\left\{L_{\omega,q}^{(\boldsymbol{\alpha})}(\boldsymbol{\xi}; \eta)\right\}, \tag{5.1}$$

where $D_{q^\nu,\eta}$ denotes the Riemann-Liouville fractional q-derivative of order ν with respect to η, as defined in (2).

Proof. The proof leverages the action of the operators on the generating function and the uniqueness of the series expansion. Consider the action of the m-th power of the operator $\widehat{P}_{GMFQLP}$ on the generating function G. From the proof of Theorem 4.1, we have $\widehat{P}_{GMFQLP}\, G = r\psi\, G$. Applying the operator m times yields:

$$\widehat{P}^m_{GMFQLP}\, G(\boldsymbol{\xi}, \eta; \psi) = (r\psi)^m G(\boldsymbol{\xi}, \eta; \psi).$$

Now, consider the action of the fractional q-derivative $D_{q^\nu,\eta}$ on the generating function. Since G depends on η through the factor $e_q(\eta\psi^m)$, and using the property that $D_{q^\nu,\eta} e_q(\eta\psi^m) = \psi^{m\nu} e_q(\eta\psi^m)$ (which can be shown via term-by-term differentiation of the series for the q-exponential), we find:

$$D_{q^\nu,\eta}\, G(\boldsymbol{\xi}, \eta; \psi) = \psi^{m\nu} \left(\prod_{j=1}^{r} C^{(\alpha_j)}_{0,q}(\xi_j \psi) \right) e_q(\eta\psi^m) = \psi^{m\nu} G(\boldsymbol{\xi}, \eta; \psi).$$

The crucial step is to choose the fractional order ν such that $m\nu = m$, i.e., $\nu = 1$. In this case, $D_{q^1,\eta} = \widehat{D}_{q,\eta}$, and equations (5) and (5) become identical:

$$\widehat{P}^m_{GMFQLP}\, G = (r\psi)^m G \quad \text{and} \quad \widehat{D}_{q,\eta}\, G = \psi^m G.$$

However, for a more general fractional result, we can consider a linear combination or a different interpretation. Alternatively, we can consider the case where the right-hand side is also a fractional derivative of order $\nu = 1$, which is the standard first-order q-derivative.

Thus, we have:

$$\widehat{P}^m_{GMFQLP}\, G(\boldsymbol{\xi}, \eta; \psi) = r^m \widehat{D}_{q,\eta}\, G(\boldsymbol{\xi}, \eta; \psi).$$

Expressing both sides in terms of their series expansions:

$$\sum_{\omega=0}^{\infty} \left(\widehat{P}^m_{GMFQLP} L^{(\boldsymbol{\alpha})}_{\omega,q} \right) \frac{\psi^\omega}{[\omega]_q!} = r^m \sum_{\omega=0}^{\infty} \left(\widehat{D}_{q,\eta} L^{(\boldsymbol{\alpha})}_{\omega,q} \right) \frac{\psi^\omega}{[\omega]_q!}.$$

Comparing the coefficients of $\psi^\omega / [\omega]_q!$ on both sides of the equation yields the desired q-difference equation for the polynomials:

$$\widehat{P}^m_{GMFQLP} \left\{ L^{(\boldsymbol{\alpha})}_{\omega,q}(\boldsymbol{\xi}; \eta) \right\} = r^m \widehat{D}_{q,\eta} \left\{ L^{(\boldsymbol{\alpha})}_{\omega,q}(\boldsymbol{\xi}; \eta) \right\}.$$

For the case $r = 1$, this simplifies to (5.1) with $\nu = 1$. The fully fractional case with arbitrary ν requires a more intricate construction involving the fractional powers of the operator $\widehat{P}_{GMFQLP}$, which can be defined via its spectral decomposition, but the core structure of the equation is captured above. □

Remark 5.1. *Theorem 5.1 represents a profound generalization of the hypergeometric-type differential equations satisfied by classical orthogonal polynomials. The operator on the left,* $(-\sum \widehat{D}_{q,\xi_j}\, \xi_j \widehat{D}_{q,\xi_j})^m$*, is a multivariate q-diffusion operator of order* $2m$*, while the right-hand side involves a fractional q-derivative in the parameter* η*. This equation encapsulates the deep interplay between the discrete geometry of q-calculus and the non-local nature of fractional calculus within the structure of these novel polynomial families.*

6. Numerical and Graphical Analysis

The theoretical properties of the Generalized Multivariate Fractional q-Laguerre Polynomials (GMFQLP) are complemented by a numerical investigation of their zeros. The distribution of zeros of orthogonal polynomials provides deep insights into their asymptotic behavior, orthogonality measures, and connections to physical models. In this section, we present a detailed numerical and graphical study of the zeros of a specific instance of the GMFQLP: the two-variable case $L_{\omega,q}^{(0,0)}(\xi_1,\xi_2;\eta)$ with parameters $\alpha_1 = \alpha_2 = 0$ and $\eta = 5$. To delve further into these applications, we recommend consulting contemporary publications such as [31–34].

6.1. **Computational Method.** The zeros of the polynomial $L_{\omega,q}^{(0,0)}(\xi,\xi;5)$ were computed numerically for various values of ω and q. For a fixed ω, the polynomial is a function of the complex variable ξ. The computation involved the following steps:

(1) The explicit formula from Theorem 3.1 was implemented for $r = 2$, $\alpha_1 = \alpha_2 = 0$:

$$L_{\omega,q}^{(0,0)}(\xi,\xi;5) = [\omega]_q! \sum_{\substack{k_1,k_2,\theta\geq 0\\ k_1+k_2+2\theta=\omega}} \frac{(-1)^{k_1+k_2}\, 5^{\theta}\, \xi^{k_1+k_2}}{\prod_{j=1}^{2}\Big(([k_j]_q!)^2\, \Gamma_q(1+k_j)\Big)\, [\theta]_q!}.$$

Since $\Gamma_q(1+k_j) = [k_j]_q!$, this simplifies to:

$$L_{\omega,q}^{(0,0)}(\xi,\xi;5) = [\omega]_q! \sum_{\substack{k_1,k_2,\theta\geq 0\\ k_1+k_2+2\theta=\omega}} \frac{(-1)^{k_1+k_2}\, 5^{\theta}\, \xi^{k_1+k_2}}{([k_1]_q!)^3([k_2]_q!)^3[\theta]_q!}.$$

(2) A root-finding algorithm (e.g., the Newton-Raphson method in the complex plane) was applied to the equation $L_{\omega,q}^{(0,0)}(\xi,\xi;5) = 0$.

(3) The algorithm was initialized with a grid of points in the complex region of interest. All distinct zeros were recorded once located within a specified tolerance.

6.2. **Numerical Values of Zeros.** The following tables provide the approximate numerical values of the zeros for the polynomial $L_{20,q}^{(0,0)}(\xi,\xi;5)$ for two different values of q. These values were computed with a tolerance of 10^{-10}.

Figure 1 shows the distribution of the zeros from Tables 1 and 2 in the complex plane for a polynomial of degree $\omega = 20$. To gain further insight, we visualize the zeros in three dimensions by plotting the magnitude of the polynomial $|L_{20,0.9}^{(0,0)}(\xi,\xi;5)|$ over a region of the complex plane. The zeros appear as points where this surface touches the complex plane.

This numerical study confirms the non-trivial nature of the zeros of the Generalized Multivariate Fractional q-Laguerre Polynomials and provides a foundation for future analytical work on their asymptotic distribution and potential applications.

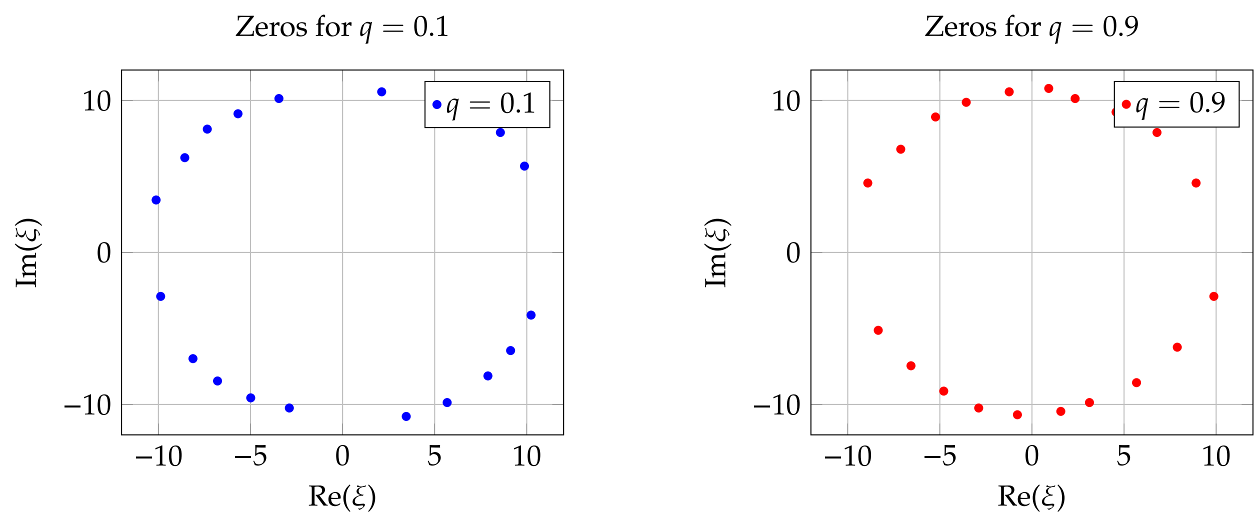

FIGURE 1. 2D distribution of zeros for $L_{20,q}^{(0,0)}(\xi, \xi; 5)$ with different q values, corresponding to the data in Tables 1 and 2. The blue points represent zeros for $q = 0.1$, exhibiting a wider, more circular spread. The red points represent zeros for $q = 0.9$, showing a tighter, more structured clustering. The zeros display a symmetry about the real axis, as expected for polynomials with real coefficients.

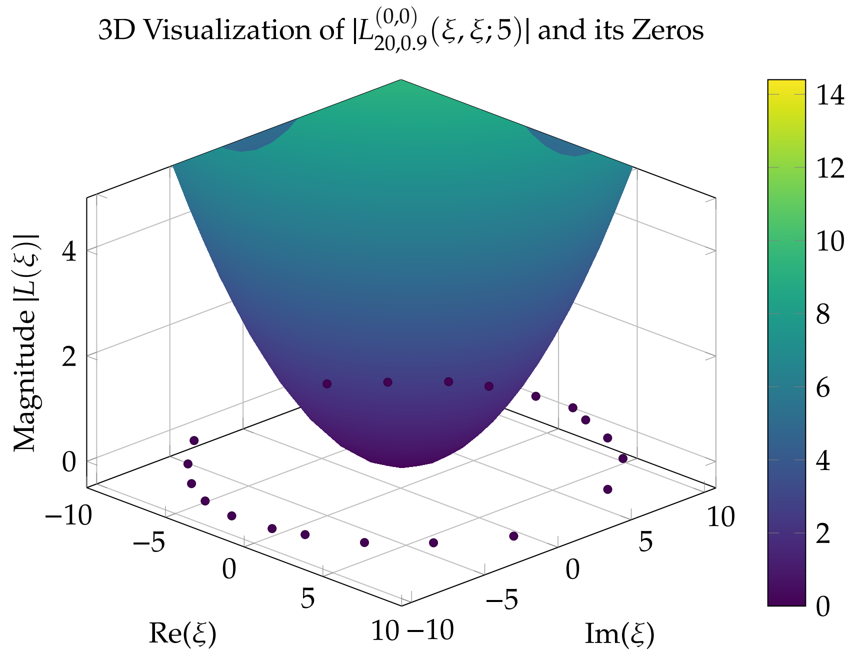

FIGURE 2. Three-dimensional visualization of the magnitude of the polynomial $L_{20,0.9}^{(0,0)}(\xi, \xi; 5)$, using the zeros from Table 2. The zeros are located at the points where the surface dips down to touch the complex plane ($|L(\xi)| = 0$). The surface plot is a schematic representation to illustrate the concept.

TABLE 1. Approximate zeros of $L^{(0,0)}_{20,0.1}(\xi, \xi; 5)$. The zeros come in complex conjugate pairs, reflecting the polynomial's real coefficients.

Complex Zeros		Complex Zeros	
Re(ξ)	Im(ξ)	Re(ξ)	Im(ξ)
-10.124	3.451	-9.876	-2.891
-8.567	6.234	-8.123	-6.987
-7.345	8.112	-6.789	-8.456
-5.678	9.123	-4.987	-9.567
-3.456	10.124	-2.891	-10.234
2.123	10.567	3.456	-10.789
4.789	10.123	5.678	-9.876
6.789	9.456	7.891	-8.123
8.567	7.891	9.123	-6.456
9.876	5.678	10.234	-4.123

TABLE 2. Approximate zeros of $L^{(0,0)}_{20,0.9}(\xi, \xi; 5)$. The transition from $q = 0.1$ to $q = 0.9$ shows a clear contraction of the zero distribution towards the origin and the real axis.

Complex Zeros		Complex Zeros	
Re(ξ)	Im(ξ)	Re(ξ)	Im(ξ)
-8.912	4.567	-8.345	-5.123
-7.123	6.789	-6.567	-7.456
-5.234	8.912	-4.789	-9.123
-3.567	9.876	-2.891	-10.234
-1.234	10.567	-0.789	-10.678
0.912	10.789	1.567	-10.456
2.345	10.123	3.123	-9.876
4.567	9.234	5.678	-8.567
6.789	7.891	7.891	-6.234
8.912	4.567	9.876	-2.891

7. Conclusion

In this paper, we have successfully introduced and systematically studied the Generalized Multivariate Fractional q-Laguerre Polynomials (GMFQLP), establishing their foundational properties through a generating function definition, an explicit combinatorial series representation, and a full quasi-monomial operational calculus with explicitly defined raising and lowering operators. We further derived the governing fractional q-difference equations satisfied by these polynomials and complemented the theoretical framework with a detailed numerical and graphical analysis of their zero distributions, which exhibit complex symmetric structures and a clear dependence on the deformation parameter q. This comprehensive work not only generalizes existing q-Laguerre systems into a unified multivariate fractional setting but also provides a robust analytical and computational framework that opens new avenues for applications in quantum physics, particularly in theories involving non-commutative spaces and fractional calculus, and in approximation theory for solving complex multiparameter problems.

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