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Soft Co-Compact Sets

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Abstract. In this paper, we introduce and investigate the notion of soft co-compact spaces as a natural generalization of classical co-compact spaces within the framework of soft topology. The aim of this new concept is to provide researchers with a flexible structure through which advanced topological properties-such as generalized compactness, separation axioms, and continuity-can be studied in the context of soft sets. We begin by presenting the formal definition of a soft co-compact space and demonstrating how it extends the classical idea of co-compactness to parameterized environments. Several fundamental properties of this new class are established, and we show that soft co-compact spaces form a distinct category that is not reducible to previously known soft topological constructs. We further explore the interaction between soft co-compactness and various soft separation axioms, thereby revealing new characterizations and criteria that govern their relationships. Motivated by these findings, we introduce associated soft operators-such as soft co-compact interior and soft co-compact closure-and describe their behaviors and structural roles. The developed framework opens a new approach in soft topology, allowing for refined analysis of soft continuity, decomposition theorems, and transitions between different soft topological settings.

1. Introduction

Soft set theory was first proposed by Molodtsov [21] in 1999 as a powerful method for dealing with uncertainty and imprecision. In his original work, Molodtsov [21] illustrated how soft sets could be applied in a variety of contexts. Unlike earlier mathematical tools such as fuzzy sets and rough sets, the soft-set approach avoids many of their inherent limitations because it does not rely on predefined membership functions or equivalence relations. Over the years, numerous studies

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and monographs have shown that soft set theory is highly effective in modeling and analyzing complex real-world situations in fields such as information theory [6], and more.

To establish the foundational concepts of soft-set theory, Maji et al. [20] introduced the notions of empty and universal soft sets, along with the operations of union, intersection, and difference in the soft environment. However, several of these initial definitions exhibited certain shortcomings. In 2011, Shabir and Naz [22] pioneered the concept of soft topology and presented its fundamental notions. Since then, numerous researchers have extended classical topological ideas into the framework of soft topology, soft mappings [16], and soft continuity and soft homeomorphisms [7].

In parallel, several generalizations of soft-open sets were developed by following similar methods to those used in classical topology, thereby generating many new forms of soft-topological concepts. Some foundational contributions include soft semi-open sets, soft α-open sets [10], and soft somewhere dense sets [9]. For more see [1, 8, 10–12, 14–16, 23].

Motivated by these observations, we aim in this work to contribute a new direction in soft topology by introducing the concept of co-compact spaces that defined in [5], which serves as a soft generalization of the classical co-compact space. Classical co-compactness has played a role in topology due to its connections with compactness, and open sets. Therefore, extending this idea to the soft environment opens the possibility of generating new soft-topological tools that are stronger band better adapted to parameterized uncertainty. For more about cocompact spaces and co-Lindelöf, see [1–4, 18].

In this article, we establish the essential definitions and foundational properties of soft co-compact spaces, present their relationships with well-known soft-topological notions, and introduce several operators inspired by the classical co-compact. The rest of the paper is organized as follows. In Section 2, we recall the basic definitions and preliminary results from soft set theory and soft topology required for our development. Section 3 is dedicated to introducing the definition of soft co-regular and soft co-normality and related results .

2. Preliminaries

Definition 2.1. *[21] A mapping $\mathcal{F} : E \to 2^X$ is called a* soft set*, denoted by $(\mathcal{F}, E)$, where $X \neq \emptyset$ is the universal set and $E \neq \emptyset$ is a set of parameters. We write $(\mathcal{F}, E)$ as $(\mathcal{F}, E) = \{(e, \mathcal{F}(e)) : e \in E, \mathcal{F}(e) \in 2^X\}$.*

Definition 2.2. *[21] The complement of a soft set $(\mathcal{F}, E)$, as $(\mathcal{F}^c, E)$, is defined by:*

$$\mathcal{F}^c(e) = X - \mathcal{F}(e) \quad \textit{for each } e \in E.$$

Definition 2.3. *[20] An soft set $(\mathcal{F}, E)$ is called absolute set defined by $\mathcal{F}(e) = U$ for each $e \in E$, and its complement is called a* null *soft-set. They are symbolized by $\tilde{X}$ and $\tilde{\phi}$, respectively.*

If $(\mathcal{F}, E)$ is defined as $\mathcal{F}(e) = \{x\} \subseteq X$ and for each $e^* \in E - \{e\}$ we have $\mathcal{F}(e^*) = \emptyset$, then $(\mathcal{F}, E)$ is called a soft point and is denoted by $\tilde{x}_e$.

Definition 2.4. *[22]*

Let $\mathcal{F}$ be a subfamily of $(X, \mathcal{M}, E)$.

The family $\mathcal{F}$ forms a soft topology *over X with respect to E if:*

- *The absolute soft set and the null soft set belong to $\mathcal{F}$.*
- *$\mathcal{F}$ is closed under finite soft intersections.*
- *$\mathcal{F}$ is closed under arbitrary soft unions.*

In this case, the triple $(X, \mathcal{F}, E)$ is called a soft topological space *(briefly, an* ST-space*).*

Every element of $\mathcal{F}$ is called a soft-open set *(briefly,* s-open*), and its complement is called a* soft-closed set *(briefly,* s-closed*).*

Definition 2.5. *Let $(\mathcal{F}_1, E)$ and $(\mathcal{F}_2, E)$ be soft subset. Then :*

(1) $(\mathcal{F}_1, E)\tilde{\subseteq}(\mathcal{F}_2, E)$, *if for every $e \in E$ we have $\mathcal{F}_1(e) \subseteq \mathcal{F}_2(e)$*
(2) $(\mathcal{F}_1, E)\tilde{\cup}(\mathcal{F}_2, E) = (\mathcal{Z}, E)$, *if for every $e \in E$ we have $\mathcal{Z}(e) = \mathcal{F}_1(e) \cup \mathcal{F}_2(e)$*
(3) $(\mathcal{F}_1, E)\tilde{\cap}(\mathcal{F}_2, E) = (\mathcal{Z}, E)$, *if for every $e \in E$ we have $\mathcal{Z}(e) = \mathcal{F}_1(e) \cap \mathcal{F}_2(e)$*
(4) $(\mathcal{F}_1, E)\Delta(\mathcal{F}_2, E) = (\mathcal{Z}, E)$, *if for every $e \in E$ we have $\mathcal{Z}(e) = \mathcal{F}_1(e) - \mathcal{F}_2(e)$*
(5) $(\mathcal{F}_1, E) \times (\mathcal{F}_2, E) = (\mathcal{Z}, E)$, *where $\mathcal{Z}(e_1, e_2) = \mathcal{F}_1(e_1) \times \mathcal{F}_2(e_2)$ for every $(e_1, e_2) \in E \times E$.*

Definition 2.6. *[22] Let $x \in X$ and $(\mathcal{F}_1, E)$ be a soft subset. Then :*

(1) *$\tilde{x}_e \in (\mathcal{F}_1, E)$ whenever $x \in \mathcal{F}_1(e)$;*
(2) *$x \in (\mathcal{F}_1, E)$ (resp., $u \notin (\mathcal{F}_1, E)$) if $x \in \mathcal{F}_1(e)$ for every (resp., some) $e \in E$.*

Definition 2.7. *[22] Let $(X, \mathcal{F}, E)$ be a soft topological space, and let $(\mathcal{F}, E)$ be a soft set in $(X, \mathcal{F}, E)$. Then:*

(1) *The* soft closure set *of $(\mathcal{F}, E)$, denoted by $CL(\mathcal{F}, E)$, is defined as*

$$CL(\mathcal{F}, E) = \tilde{\cap}\{(C, E) \in (\mathcal{F}, E) : \tilde{\subseteq} (C, E)\}.$$

(2) *The* soft interior set *of $(\mathcal{F}, E)$, denoted by $int(\mathcal{F}, E)$, is defined as*

$$int(\mathcal{F}, E) = \tilde{\cup}\{(O, E) \in \mathcal{F} : (O, E) \tilde{\subseteq} (\mathcal{F}, E)\}.$$

Definition 2.8. *[22]*

(1) *A collection $\mathcal{A}$ of S-sets in $(X, \mathcal{F}, E)$ is said to be a soft open cover (SO-cover) of a soft set $(\mathcal{F}_1, E)$ if*

$$(\mathcal{F}_1, E)\tilde{\subseteq}\tilde{\cup}\{(\mathcal{F}_i, E) : (F_i, E) \in \mathcal{A} \text{ and } i \in I\}.$$

(2) *A soft topological space $(X, \mathcal{F}, E)$ is said to be soft compact (s-compact, in short) if every SO-cover of $\mathcal{U}$ possess a finite sub-cover.*

Definition 2.9. *[8] Let $(X, \mathcal{F}, E)$ be a soft topological space and let $x_e, y_{e'}$ be two distinct soft points in X.*

(1) *$(X, \mathcal{F}, E)$ is called a* soft T_0 space *if there exists a soft open set $(F, E) \in \mathcal{F}$ such that*

$$x_e \widetilde{\in} (F, E) \quad \text{and} \quad y_{e'} \widetilde{\notin} (F, E),$$

or

$$y_{e'} \widetilde{\in} (F, E) \quad \text{and} \quad x_e \widetilde{\notin} (F, E).$$

(2) $(X, \mathcal{F}, E)$ *is called a* soft T_1 space *if there exist soft open sets* $(F_1, E), (F_2, E) \in \mathcal{F}$ *such that*

$$x_e \widetilde{\in} (F_1, E), \quad y_{e'} \widetilde{\notin} (F_1, E),$$

and

$$y_{e'} \widetilde{\in} (F_2, E), \quad x_e \widetilde{\notin} (F_2, E).$$

(3) $(X, \mathcal{F}, E)$ *is called a* soft T_2 space *(or a* soft Hausdorff space*) if there exist disjoint soft open sets* $(F_1, E), (F_2, E) \in \mathcal{F}$ *such that*

$$x_e \widetilde{\in} (F_1, E), \quad y_{e'} \widetilde{\in} (F_2, E),$$

and

$$(F_1, E) \widetilde{\cap} (F_2, E) = \widetilde{\emptyset}.$$

Definition 2.10. *Let* $(X, \mathcal{F}, E)$ *be a soft topological space. A soft set* $(\mathcal{F}_1, E)$ *is called a* soft co-compact set *if for every* $a_x \widetilde{\in} (\mathcal{F}_1, E)$ *there exist a soft open set* $(\mathcal{F}_2, E)$ *and an s-compact set* $(\mathcal{K}, E)$ *such that*

$$a_x \widetilde{\in} (\mathcal{F}_2, E) \,\Delta\, (\mathcal{K}, E) \,\widetilde{\subseteq}\, (\mathcal{F}_1, E).$$

The set of all soft co-compact sets is called soft co-compact space SCC-space and is denoted by $(X, \mathcal{F}^k, E)$.

Theorem 2.1. *The SCC-space* $(X, \mathcal{F}^k, E)$ *is a soft topology on* $(X, \mathcal{F}, E)$.

Proof. (1) $\tilde{\phi} \widetilde{\in} (X, \mathcal{F}^k, E)$ since $\phi \Delta \phi = \tilde{\phi}$, and $\tilde{X} \widetilde{\in} (X, \mathcal{F}^k, E)$ since $X \Delta \varnothing = X$.

(2) Suppose $(\mathcal{F}_1, E)$ and $(\mathcal{F}_2, E)$ are in $\mathcal{F}^k$, and

$$\mathcal{F}_1 = (\mathcal{U}_1, E)\Delta(\mathcal{K}_1, E), \qquad \mathcal{F}_2 = (\mathcal{U}_2, E)\Delta(\mathcal{K}_2, E),$$

where $(\mathcal{U}_1, E)$ and $(\mathcal{U}_2, E)$ are s-open sets, and $(\mathcal{K}_1, E)$ and $(\mathcal{K}_2, E)$ are s-compact sets.

Now, for

$$a_x \widetilde{\in} (\mathcal{F}_1, E) \cap (\mathcal{F}_2, E),$$

we have

$$a \in \mathcal{F}_1(x) \cap \mathcal{F}_2(x).$$

Define

$$(\mathcal{U}, E) = (\mathcal{U}_1, E)\tilde{\cap}(\mathcal{U}_2, E), \qquad (\mathcal{K}, E) = (\mathcal{K}_1, E)\tilde{\cup}(\mathcal{K}_2, E).$$

Then $(\mathcal{U}, E)$ is s-open sets and $(\mathcal{K}, E)$ is s-compact set, and

$$a_x \widetilde{\in} (\mathcal{U}, E)\Delta(\mathcal{K}, E) \,\widetilde{\subseteq}\, (\mathcal{F}_1, E)\Delta(\mathcal{F}_2, E).$$

(3) Let $\{(\mathcal{F}_\alpha, E) : \alpha \in \Lambda\}$ be a collection of elements of $\mathcal{F}^k$, and suppose

$$a_x \widetilde{\in} \tilde{\bigcup}_{\alpha \in \Lambda} (\mathcal{F}_\alpha, E).$$

Then there exists $\alpha_0 \in \Lambda$ such that

$$a_x \tilde{\in} (\mathcal{F}_{\alpha_0}, E).$$

Since $(\mathcal{F}_{\alpha_0}, E) \in \mathcal{F}^k$, there exists an s-open set $(\mathcal{U}, E)$ and an s-compact set $(\mathcal{K}, E)$ such that

$$a_x \widetilde{\in} (\mathcal{U}, E)\Delta(\mathcal{K}, E) \widetilde{\subseteq} (\mathcal{F}_{\alpha_0}, E) \widetilde{\subseteq} \tilde{\bigcup}_{\alpha\in\Lambda} (\mathcal{F}_\alpha, E)$$

Hence $(X, \mathcal{F}^k, E)$ is a soft topology.

□

Definition 2.11. *Let $(\mathcal{F}, E)$ be an soft co-compact set in soft topological space $(X, \mathcal{F}, E)$. Then:*

(1) *The soft co-compact closure of $(\mathcal{F}, E)$, denoted by $CL_{\mathcal{F}^k}(\mathcal{F}, E)$ is:*

$$CL_{\mathcal{F}^k}(\mathcal{F}, E) = \tilde{\cap}\{(C, E) : (C, E) \in \mathcal{F}^k \quad with (\mathcal{F}, E) \tilde{\subseteq} (C, E)\}.$$

(2) *The soft co-compact interior of $(\mathcal{F}, E)$,denoted by $int_{\mathcal{F}^k}(\mathcal{F}, E)$ is:*

$$int_{\mathcal{F}^k}(\mathcal{F}, E) = \tilde{\cup}\{(O, E) : (O, E) \in \mathcal{F}^k \quad with (O, E) \tilde{\subseteq} (\mathcal{F}, E)\}.$$

For the family of soft co-compact subsets of $(X, \mathcal{F}, E)$,

$$\mathcal{B}^k(\mathcal{F}) = \{(\mathcal{U}, E)\Delta(\mathcal{K}, E) \mid (\mathcal{U}, E) \text{ is } s\text{-open and } (\mathcal{K}, E) \text{ is } s\text{-compact}\},$$

this forms a soft base for the soft co-compact space.

Definition 2.12. *A soft space $(X, \mathcal{F}, E)$ is said to have property C-soft if every s-compact set is s-closed.*

Theorem 2.2. *For a soft space $(X, \mathcal{F}, E)$, the following are equivalent:*

(1) *$(X, \mathcal{F}, E)$ has property C-soft.*
(2) *$(X, \mathcal{F}, E) = \mathcal{B}^k(\mathcal{F})$.*
(3) *$(X, \mathcal{F}, E) = (X, \mathcal{F}^k, E)$.*

Proof. (1 ⇒ 2) For any $(\mathcal{F}, E) \widetilde{\in} (X, \mathcal{F}, E)$, $(\mathcal{F}, E) = (\mathcal{F}, E)\Delta\tilde{\phi}$, so
$(\mathcal{F}, E) \widetilde{\subseteq} \mathcal{B}^k(\mathcal{F})$.

For the converse. Let $(\mathcal{F}_1, E) = (\mathcal{F}, E)\Delta(\mathcal{K}, E) \tilde{\in} \mathcal{B}^k(\mathcal{F})$. Then the set $(\mathcal{K}, E)$ is s-closed and hence $(\mathcal{F}_1, E) \tilde{\in} (X, \mathcal{F}, E)$.

(2 ⇒ 3)

Enough to show

$$(X, \mathcal{F}^k, E) \widetilde{\subseteq} (X, \mathcal{F}, E),$$

note that

$$(X, \mathcal{F}, E) = \mathcal{B}^k(\mathcal{F}),$$

and since $\mathcal{B}^k(\mathcal{F})$ is a soft base for $(X, \mathcal{F}^k, E)$, we have

$$(X, \mathcal{F}^k, E) \widetilde{\subseteq} (X, \mathcal{F}, E).$$

(3 ⇒ 1) Let $(\mathcal{K}, E)$ be s-compact. Then $\widetilde{X}\Delta(\mathcal{K}, E)$ is SCC-open, and hence

$$\widetilde{X}\Delta(\mathcal{K}, E) \widetilde{\in} (X, \mathcal{F}, E),$$

and by (3) we obtain the result. □

Corollary 2.1. *If $(X,\mathcal{F},E)$ is a soft -T_2- space, then*

$$(X,\mathcal{F},E) = (X,\mathcal{F}^k,E).$$

Proof. Since $(X,\mathcal{F},E)$ is a soft- T_2-space, every s-compact set is s-closed, and the result follows from Theorem 2.2. □

Definition 2.13. *A soft space $(X,\mathcal{F},E)$ is called a* soft co-T_2 space *if for each distinct soft points $\widetilde{x}_e \neq \widetilde{y}_e$, there exist soft co-compact disjoint subsets $(\mathcal{F}_1,E),(\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F}^k,E)$ such that*

$$\Big\{(\mathcal{F}_1,E),(\mathcal{F}_2,E)\Big\}\tilde{\cap}(X,\mathcal{F},E) \neq \tilde{\phi},$$

and

$$\widetilde{x}_e\widetilde{\in}(\mathcal{F}_1,E) \quad \textit{and} \quad \widetilde{y}_e\widetilde{\in}(\mathcal{F}_2,E).$$

Theorem 2.3. *Every soft T_2-space is a soft co-T_2 space.*

Theorem 2.4. *If $(X,\mathcal{F},E)$ is hereditarily s-compact, then $(X,\mathcal{F}^k,E)$ is a soft discrete topology.*

Proof. For every soft point $\widetilde{x}_e\widetilde{\in}(X,\mathcal{F},E)$. The set $\tilde{X}\Delta\widetilde{x}_e$ is a soft co-compact subset, and hence

$$\widetilde{x}_e = \tilde{X}\Delta(\tilde{X}\Delta\widetilde{x}_e)\tilde{\in}(X,\mathcal{F}^k,E),$$

and therefore $\widetilde{x}_e$ is soft discrete. □

Theorem 2.5. *If $(X,\mathcal{F},E)$ is a hereditarily s-compact space, then $(X,\mathcal{F}^k,E)$ is a soft T_2-space.*

Proof. By Theorem 2.4, $(X,\mathcal{F}^k,E)$ is a soft discrete, hence it is a soft T_2-space. □

Theorem 2.6. *Let $(X,\mathcal{F},E)$ be hereditarily compact and soft T_1-space. Then $(X,\mathcal{F},E)$ is a soft co-T_2 space.*

Proof. Let $\widetilde{x}_e,\widetilde{y}_e\tilde{\in}(X,\mathcal{F},E)$ with $\widetilde{x}_e \neq \widetilde{y}_e$. Take

$$(\mathcal{F}_1,E) = \{\widetilde{x}_e\}\tilde{\in}(X,\mathcal{F}^k,E)$$

and

$$(\mathcal{F}_2,E) = \tilde{X}\Delta(\mathcal{F}_1,E)\widetilde{\in}(X,\mathcal{F},E).$$

By Theorem 2.5, $(\mathcal{F}_1,E)\widetilde{\in}(X,\mathcal{F}^k,E)$.

Since $(X,\mathcal{F},E)$ is soft- T_1- space, then

$$(\mathcal{F}_1,E),(\mathcal{F}_2,E) \in (X,\mathcal{F}^k,E),$$

and

$$\Big\{(\mathcal{F}_1,E),(\mathcal{F}_2,E)\Big\}\tilde{\cap}(X,\mathcal{F},E) \neq \tilde{\phi}$$

with

$$\widetilde{x}_e\widetilde{\in}(\mathcal{F}_1,E), \qquad \widetilde{y}_e\widetilde{\in}(\mathcal{F}_2,E),$$

and

$$(\mathcal{F}_1,E)\tilde{\cap}(\mathcal{F}_2,E) = \tilde{\phi},$$

the result follows. □

Theorem 2.7. *A soft space $(X,\mathcal{F},E)$ is soft co-T_2 if and only if for any soft points $\widetilde{x}_e,\widetilde{y}_e \in X$ with $\widetilde{x}_e \neq \widetilde{y}_e$, there exist $(\mathcal{F}_1,E),(\mathcal{F}_2,E) \in (X,\mathcal{F},E)$ and a soft compact set $(\mathcal{K},E)$ such that*

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1,E)\Delta(\mathcal{K},E) \quad \text{and} \quad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E)$$

with

$$\Big\{(\mathcal{F}_1,E)\Delta(\mathcal{K},E)\Big\}\tilde{\cap}(\mathcal{F}_2,E) = \tilde{\phi}.$$

or

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1,E), \qquad \text{and} \quad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E)\Delta(\mathcal{K},E),$$

with

$$\Big\{(\mathcal{F}_2,E)\Delta(\mathcal{K},E)\Big\}\tilde{\cap}(\mathcal{F}_1,E) = \tilde{\phi}.$$

Proof. ($\Rightarrow$) Assume that $(X,\mathcal{F},E)$ is a soft co-T_2 space, and let $\widetilde{x}_e \neq \widetilde{y}_e$. Then there exist

$$(\mathcal{F}_1,E),(\mathcal{F}_2,E) \widetilde{\in} (X,\mathcal{F}^k,E)$$

such that

$$\Big\{(\mathcal{F}_1,E),(\mathcal{F}_2,E)\Big\}\tilde{\cap}(X,\mathcal{F},E) \neq \tilde{\phi},$$

with

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1,E), \qquad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E).$$

Assume that

$$(\mathcal{F}_1,E)\tilde{\cap}(\mathcal{F}_2,E) = \tilde{\phi} \quad \text{and} \quad (\mathcal{F}_1,E)\widetilde{\in}(X,\mathcal{F},E).$$

Since $(\mathcal{F}_1,E) \in (X,\mathcal{F}^k,E)$, there exists a soft compact set $(\mathcal{K},E)$ such that

$$\widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E)\Delta(\mathcal{K},E)$$

and

$$\Big\{(\mathcal{F}_2,E)\Delta(\mathcal{K},E)\Big\}\tilde{\cap}(\mathcal{F}_1,E) = \tilde{\phi}.$$

Hence, $(\mathcal{F}_1,E)\tilde{\cap}\Big((\mathcal{F}_2,E)\Delta(\mathcal{K},E)\Big) = \tilde{\phi}$.

($\Leftarrow$) Let $\widetilde{x}_e,\widetilde{y}_e \tilde{\in} (X,\mathcal{F},E)$ with $\widetilde{x}_e \neq \widetilde{y}_e$. So, there exist $(\mathcal{F}_1,E),(\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F},E)$ and a compact soft set $(\mathcal{K},E)$ of $(X,\mathcal{F},E)$ such that

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1,E)\Delta(\mathcal{K},E), \qquad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E),$$

with

$$\Big((\mathcal{F}_1,E)\Delta(\mathcal{K},E)\Big)\tilde{\cap}(\mathcal{F}_2,E) = \tilde{\phi}. \qquad (*)$$

Or

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1,E), \qquad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_2,E)\Delta(\mathcal{K},E),$$

and

$$(\mathcal{F}_1,E) \cap \Big((\mathcal{F}_2,E)\Delta(\mathcal{K},E)\Big) = \tilde{\phi}. \qquad (**)$$

For either (*) or (**), then we have $(X,\mathcal{F},E)$ is a soft co-T_2 space. □

Theorem 2.8. *A soft space $(X, \mathcal{F}, E)$ is soft co-T_2 iff for any soft points $\widetilde{x}_e, \widetilde{y}_e \in X$ with $\widetilde{x}_e \neq \widetilde{y}_e$, there exist a soft open set $(\mathcal{F}, E)$ and a soft compact set $(\mathcal{K}, E)$ such that*

$$\widetilde{x}_e \,\widetilde{\in}\, (\mathcal{F}, E)\Delta(\mathcal{K}, E), \qquad \widetilde{y}_e \,\widetilde{\notin}\, \mathrm{CL}\Big[(\mathcal{F}, E)\Delta(\mathcal{K}, E)\Big],$$

or

$$\widetilde{x}_e \,\widetilde{\notin}\, \mathrm{CL}\Big((\mathcal{F}, E)\Delta(\mathcal{K}, E)\Big), \qquad \widetilde{y}_e \,\widetilde{\in}\, (\mathcal{F}, E)\Delta(\mathcal{K}, E).$$

Proof. $(\Rightarrow)$

Let $(X, \mathcal{F}, E)$ be a soft co-T_2 space and let $\widetilde{x}_e, \widetilde{y}_e \,\widetilde{\in}\, X$ with $\widetilde{x}_e \neq \widetilde{y}_e$. Since $(X, \mathcal{F}, E)$ is soft co-T_2, by the previous theorem there exist a soft open set $(\mathcal{F}_1, E)$, a soft open set $(\mathcal{F}_2, E)$, and a soft compact set $(\mathcal{K}, E)$ such that

$$\widetilde{x}_e \,\widetilde{\in}\, (\mathcal{F}_1, E)\Delta(\mathcal{K}, E), \qquad \widetilde{y}_e \,\widetilde{\in}\, (\mathcal{F}_2, E),$$

and

$$\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big) \cap (\mathcal{F}_2, E) = \widetilde{\varnothing},$$

or

$$\widetilde{x}_e \,\widetilde{\in}\, (\mathcal{F}_1, E), \qquad \widetilde{y}_e \,\widetilde{\in}\, (\mathcal{F}_2, E)\Delta(\mathcal{K}, E),$$

and

$$(\mathcal{F}_1, E) \cap \Big((\mathcal{F}_2, E)\Delta(\mathcal{K}, E)\Big) = \widetilde{\varnothing}.$$

Case I.

Assume

$$\widetilde{x}_e \,\widetilde{\in}\, (\mathcal{F}_1, E)\Delta(\mathcal{K}, E), \qquad \widetilde{y}_e \,\widetilde{\in}\, (\mathcal{F}_2, E),$$

and

$$\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big) \cap (\mathcal{F}_2, E) = \widetilde{\varnothing}.$$

We have

$$(\mathcal{F}_1, E)\Delta(\mathcal{K}, E) \,\widetilde{\subseteq}\, \widetilde{X}\Delta(\mathcal{F}_2, E),$$

so

$$\mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big) \subseteq \mathrm{CL}\Big(\widetilde{X}\Delta(\mathcal{F}_2, E)\Big) = \widetilde{X}\Delta(\mathcal{F}_2, E).$$

Since

$$\widetilde{y}_e \widetilde{\notin} \widetilde{X}\Delta(\mathcal{F}_2, E),$$

we obtain

$$\widetilde{y}_e \widetilde{\notin} \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big).$$

Case II

Assume

$$\widetilde{x}_e \,\widetilde{\in}\, (\mathcal{F}_1, E), \qquad \widetilde{y}_e \,\widetilde{\in}\, (\mathcal{F}_2, E)\Delta(\mathcal{K}, E).$$

This is analogous to Case I.

(⇐)

Let $\widetilde{x}_e, \widetilde{y}_e \tilde{\in} X$ with $\widetilde{x}_e \neq \widetilde{y}_e$. Suppose there exist a soft open set $(\mathcal{F}_1, E)$ and a soft compact set $(\mathcal{K}, E)$ such that

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1, E)\Delta(\mathcal{K}, E) \quad \text{and} \quad \widetilde{y}_e \tilde{\notin} \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big),$$

or

$$\widetilde{x}_e \tilde{\notin} \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big), \quad \widetilde{y}_e \widetilde{\in} (\mathcal{F}_1, E)\Delta(\mathcal{K}, E).$$

Case I.

Assume

$$\widetilde{x}_e \tilde{\in} (\mathcal{F}_1, E)\Delta(\mathcal{K}, E), \qquad \widetilde{y}_e \tilde{\notin} \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big).$$

Take

$$(\mathcal{F}_2, E) = \tilde{X}\Delta\, \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big).$$

Then $(\mathcal{F}_2, E)$ is soft open, and $\widetilde{y}_e \in (\mathcal{F}_2, E)$, and $\Big\{(\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big\}\tilde{\cap}(\mathcal{F}_2, E) = \widetilde{\phi}$.

Case II.

Assume

$$\widetilde{x}_e \tilde{\notin} \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big), \qquad \widetilde{y}_e \tilde{\in} (\mathcal{F}_1, E)\Delta(\mathcal{K}, E).$$

Set

$$(\mathcal{F}_2, E) = \tilde{X}\Delta\, \mathrm{CL}\Big((\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big).$$

Then $(\mathcal{F}_2, E)$ is soft open, and $\widetilde{x}_e \in (\mathcal{F}_2, E)$, $\Big\{(\mathcal{F}_1, E)\Delta(\mathcal{K}, E)\Big\}\tilde{\cap}(\mathcal{F}_2, E) = \widetilde{\phi}$. Therefore $(X, \mathcal{F}, E)$ is a soft co-T_2 space.

□

3. Soft Co-Regular and Soft Co-Normality

Definition 3.1. *A soft space $(X, \mathcal{F}, E)$ is called* soft co-regular *if for each soft closed set (C, E) and each soft point $\widetilde{x}_e \widetilde{\in} X\Delta(C, E)$, there exist soft sets $(\mathcal{F}_1, E), (\mathcal{F}_2, E) \tilde{\in} (X, \mathcal{F}^k, E)$ such that*

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1, E), \qquad (C, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad (\mathcal{F}_1, E)\tilde{\cap}(\mathcal{F}_2, E) = \widetilde{\phi}.$$

Theorem 3.1. *If $(X, \mathcal{F}, E)$ is a soft topological space such that $(X, \mathcal{F}^k, E)$ is the soft discrete topology, then $(X, \mathcal{F}, E)$ is soft co-regular.*

Proof. Let (C, E) be a soft closed subset in $(X, \mathcal{F}, E)$ and let $\widetilde{x}_e \tilde{\in} X\Delta(C, E)$.
Let $(\mathcal{F}_1, E) = \widetilde{x}_e$, $\quad (\mathcal{F}_2, E) = (C, E)$. Then $(\mathcal{F}_1, E), (\mathcal{F}_2, E) \widetilde{\in} (X, \mathcal{F}^k, E)$ and

$$\widetilde{x}_e \widetilde{\in} (\mathcal{F}_1, E), \qquad (C, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad (\mathcal{F}_1, E) \cap (\mathcal{F}_2, E) = \widetilde{\phi}.$$

Therefore, $(X, \mathcal{F}, E)$ is soft co-regular. □

Theorem 3.2. *Every soft regular topological space is soft co-regular.*

Proof. Let $(X, \mathcal{F}, E)$ be a soft regular space. Let (C, E) be a soft closed set in $(X, \mathcal{F}, E)$, and let $\widetilde{x_e} \in X\Delta(C, E)$. Since $(X, \mathcal{F}, E)$ is soft regular, there exist $(\mathcal{F}_1, E), (\mathcal{F}_2, E) \widetilde{\in} (X, \mathcal{F}, E)$ such that

$$\widetilde{x_e} \widetilde{\in} (\mathcal{F}_1, E), \qquad (C, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad (\mathcal{F}_1, E) \tilde{\cap} (\mathcal{F}_2, E) = \widetilde{\phi}.$$

Since $(\mathcal{F}_1, E) \in (X, \mathcal{F}^k, E)$, it follows that $(X, \mathcal{F}, E)$ is soft co-regular. □

Theorem 3.3. *Let $(X, \mathcal{F}, E)$ be a soft space. If $(X, \mathcal{F}^k, E)$ is soft regular, then $(X, \mathcal{F}, E)$ is soft co-regular.*

Theorem 3.4. *A soft space $(X, \mathcal{F}, E)$ is soft co-regular space iff for every soft open set $(\mathcal{F}_1, E)$ and every soft point $\widetilde{x_e} \widetilde{\in} (\mathcal{F}_1, E)$, there exists a soft co-compact open set $(\mathcal{F}_2, E)$ such that*

$$\widetilde{x_e} \widetilde{\in} (\mathcal{F}_2, E), \qquad \mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_2, E) \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Proof. $(\Rightarrow)$ Suppose $(X, \mathcal{F}, E)$ is soft co-regular space. Let $(\mathcal{F}_1, E)$ be a soft open set with $\widetilde{x_e} \widetilde{\in} (\mathcal{F}_1, E)$. Since $\tilde{X}\Delta(\mathcal{F}_1, E)$ is a soft closed set, we have $\widetilde{x_e} \widetilde{\in} \tilde{X}\Delta(\tilde{X}\Delta(\mathcal{F}_1, E))$. Thus, by soft co-regularity, there exist a soft co-compact open set $(\mathcal{F}_2, E)$ and a soft compact set $(\mathcal{K}, E)$ such that

$$\widetilde{x_e} \widetilde{\in} (\mathcal{F}_2, E), \qquad \tilde{X}\Delta(\mathcal{F}_2, E) \widetilde{\subseteq} (\mathcal{F}_1, E), \qquad (\mathcal{F}_2, E) \tilde{\cap} (\mathcal{F}_3, E) = \widetilde{\varnothing}.$$

Now $\widetilde{x_e} \widetilde{\in} (\mathcal{F}_2, E)$, and

$$(\mathcal{F}_2, E) \widetilde{\subseteq} \mathrm{CL}_{F^k}(\mathcal{F}_2, E) \tilde{\subseteq} \mathrm{CL}_{F^k}(\tilde{X}\Delta(\mathcal{F}_3, E)) = \tilde{X}\Delta(\mathcal{F}_3, E) \tilde{\subseteq} (\mathcal{F}_1, E).$$

Thus the required $(\mathcal{F}_2, E)$ exists.

$(\Leftarrow)$ Let (C, E) be soft closed in $(X, \mathcal{F}, E)$ and $\widetilde{x_e} \widetilde{\in} \tilde{X}\Delta(C, E)$. Then $\tilde{X}\Delta(C, E) \in (X, \mathcal{F}, E)$, and hence there exists a soft co-compact open set $(\mathcal{F}_1, E)$ such that

$$\widetilde{x_e} \widetilde{\in} (\mathcal{F}_1, E), \qquad \mathrm{CL}(\mathcal{F}_2, E) \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Let $(\mathcal{F}_2, E) = \tilde{X}\Delta\, \mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_1, E)$. Then $(\mathcal{F}_1, E), (\mathcal{F}_2, E) \widetilde{\in} (\tilde{X}, \mathcal{F}^k, E)$ and

$$(C, E) \widetilde{\subseteq} (\mathcal{F}_1, E), \qquad (\mathcal{F}_1, E) \tilde{\cap} (\mathcal{F}_2, E) = \widetilde{\phi}.$$

Thus $(X, \mathcal{F}, E)$ is soft co-regular. □

Theorem 3.5. *If $(X, \mathcal{F}, E)$ is a soft co-regular topological space, then for every soft closed set (C, E) in $(X, \mathcal{F}, E)$, we have*

$$(C, E) = \tilde{\cap}\{\, (\mathcal{F}_1, E) \widetilde{\in} (X, \mathcal{F}^k, E) \mid (C, E) \widetilde{\subseteq} (\mathcal{F}_1, E) \,\} \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Proof. Suppose that $(X, \mathcal{F}, E)$ is soft co-regular. Clearly,

$$(C, E) \widetilde{\subseteq} \tilde{\cap}\{(\mathcal{F}_1, E) \widetilde{\in} (X, \mathcal{F}^k, E)\}, (C, E) \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Let

$$\widetilde{x_e} \widetilde{\in} \tilde{X}\Delta(C, E).$$

Since $(X, \mathcal{F}, E)$ is soft co-regular, there exists $(\mathcal{F}_1, E), (\mathcal{F}_2, E) \widetilde{\in} (X, \mathcal{F}^k, E)$ such that

$$\widetilde{x_e} \widetilde{\in} (\mathcal{F}_1, E), \qquad (C, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad (\mathcal{F}_1, E) \tilde{\cap} (\mathcal{F}_2, E) = \widetilde{\varnothing}.$$

Thus

$$(\mathcal{C},E)\widetilde{\subseteq}(\mathcal{F}_1,E), \qquad (\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F}^k,E),$$

and we have $\widetilde{x_e}\tilde{\notin}(\mathcal{F}_2,E)$, which implies that

$$\widetilde{x_e}\tilde{\notin}\tilde{\cap}\Big\{(\mathcal{F}_1,E)\tilde{\in}\mathcal{F}^k, \quad (\mathcal{C},E)\widetilde{\subseteq}(\mathcal{F}_1,E)\Big\}$$

as required. □

Definition 3.2. *A soft topological space $(X,\mathcal{F},E)$ is called* soft s-regular *if for each soft closed set $(\mathcal{C},E)$ in $(X,\mathcal{F}^k,E)$ and each $\widetilde{x_e}\in X\Delta(\mathcal{C},E)$, there exist $(\mathcal{F}_1,E),(\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F},E)$ such that*

$$\widetilde{x_e}\widetilde{\in}(\mathcal{F}_1,E), \qquad (\mathcal{C},E)\widetilde{\subseteq}(\mathcal{F}_2,E), \qquad (\mathcal{F}_1,E)\tilde{\cap}(\mathcal{F}_2,E)=\widetilde{\phi}.$$

Theorem 3.6. *Every soft s-regular topological space is soft regular.*

Theorem 3.7. *Let $(X,\mathcal{F},E)$ be a soft T_2-space. Then $(X,\mathcal{F},E)$ is soft s-regular if and only if $(X,\mathcal{F}^k,E)$ is soft regular.*

Proof. The result comes from Theorem 3.6 and Corollary 2.1. □

Definition 3.3. *Let $(X,\mathcal{F},E)$ be a soft topological space. Then $(X,\mathcal{F},E)$ is called* soft co-normal *if for each pair of disjoint soft closed sets $(\mathcal{C}_1,E),(\mathcal{C}_2,E)$ of $(X,\mathcal{F},E)$, there exist $(\mathcal{F}_1,E),(\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F}^k,E)$ such that*

$$(\mathcal{C}_1,E)\widetilde{\subseteq}(\mathcal{F}_1,E), \qquad (\mathcal{C}_1,E)\widetilde{\subseteq}(\mathcal{F}_2,E), \qquad (\mathcal{F}_1,E)\tilde{\cap}(\mathcal{F}_2,E)=\widetilde{\varnothing}.$$

Theorem 3.8. *Every soft normal space $(X,\mathcal{F},E)$ is soft co-normal.*

Proof. Let $(\mathcal{C}_1,E)$ and $(\mathcal{C}_2,E)$ be two disjoint soft closed sets. Since $(X,\mathcal{F},E)\widetilde{\subseteq}(X,\mathcal{F}^k,E)$ and soft closed sets $(\mathcal{C}_1,E)$ and $(\mathcal{C}_2,E)$ are soft closed in $(X,\mathcal{F}^k,E)$, and since $(X,\mathcal{F}^k,E)$ is soft normal, there exist

$$(\mathcal{F}_1,E),(\mathcal{F}_2,E)\widetilde{\in}(X,\mathcal{F}^k,E)$$

such that

$$(\mathcal{C}_1,E)\widetilde{\subseteq}(\mathcal{F}_1,E), \qquad (\mathcal{C}_2,E)\widetilde{\subseteq}(\mathcal{F}_2,E), \qquad (\mathcal{F}_1,E)\tilde{\cap}(\mathcal{F}_2,E)=\widetilde{\varnothing}.$$

Therefore, $(X,\mathcal{F},E)$ is soft co-normal. □

Theorem 3.9. *Let $(X,\mathcal{F},E)$ be a soft topological space. If $(X,\mathcal{F}^k,E)$ is soft normal, then $(X,\mathcal{F},E)$ is soft co-normal.*

Corollary 3.1. *If $(X,\mathcal{F},E)$ is a soft topological space and $(X,\mathcal{F}^k,E)$ is soft discrete, then $(X,\mathcal{F},E)$ is soft co-normal.*

Theorem 3.10. *A topological space $(X,\mathcal{F},E)$ is soft co-normal if and only if for every soft open set $(\mathcal{F}_1,E)$ and any soft closed set $(\mathcal{C}_2,E)$ in $(X,\mathcal{F},E)$ with $(\mathcal{C}_1,E)\widetilde{\subseteq}(\mathcal{F}_1,E)$, there exists $(\mathcal{F}_2,E)$ soft open in $(X,\mathcal{F}^k,E)$ such that*

$$(\mathcal{C}_2,E)\widetilde{\subseteq}(\mathcal{F}_2,E), \qquad \mathrm{CL}_{\mathcal{F}^k}\Big((\mathcal{F}_2,E)\Big)\widetilde{\subseteq}(\mathcal{F}_1,E).$$

Proof. ($\Rightarrow$) Assume $(X, \mathcal{F}, E)$ is soft co-normal. Let $(\mathcal{F}_1, E)$ be a soft open set in $(X, \mathcal{F}, E)$ and let (C_1, E) be a soft closed subset of $(X, \mathcal{F}, E)$ with

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Thus (C_1, E) and $\tilde{X}\Delta(\mathcal{F}_1, E)$ are disjoint soft closed sets in $(X, \mathcal{F}, E)$. Since $(X, \mathcal{F}, E)$ is soft co-normal, there exist

$$(\mathcal{F}_2, E), (\mathcal{F}_3, E) \widetilde{\in} (X, \mathcal{F}^k, E)$$

such that

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad \tilde{X}\Delta(\mathcal{F}_1, E) \widetilde{\subseteq} (\mathcal{F}_3, E), \qquad (\mathcal{F}_2, E) \tilde{\cap} (\mathcal{F}_3, E) = \widetilde{\varnothing}.$$

Hence,

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_2, E) \widetilde{\subseteq} \mathrm{CL}_{\mathcal{F}^k}\Big((\mathcal{F}_2, E)\Big) \widetilde{\subseteq} (\mathcal{F}_1, E),$$

as required.

($\Leftarrow$) Let (C_1, E) and (C_2, E) be two disjoint soft closed sets in $(X, \mathcal{F}, E)$. Put

$$(\mathcal{F}_1, E) = \tilde{X}\Delta(C_2, E).$$

Then $(\mathcal{F}_1, E) \widetilde{\in} (X, \mathcal{F}^k, E)$ with

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_1, E).$$

Since $(X, \mathcal{F}, E)$ is soft co-normal, there exists $(\mathcal{F}_2, E) \widetilde{\in} (X, \mathcal{F}^k, E)$ such that

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_2, E), \qquad (\mathcal{F}_2, E) \widetilde{\subseteq} \mathrm{CL}_{\mathcal{F}^k}\Big((\mathcal{F}_1, E)\Big).$$

Let

$$(\mathcal{F}_3, E) = \tilde{X}\Delta\, \mathrm{CL}_{\mathcal{F}^k}\Big((\mathcal{F}_2, E)\Big).$$

Then

$$(\mathcal{F}_3, E) \widetilde{\in} (X, \mathcal{F}^k, E), \qquad (C_2, E) \widetilde{\subseteq} (\mathcal{F}_3, E),$$

and

$$(\mathcal{F}_3, E) \cap (\mathcal{F}_2, E) = \Big(\tilde{X}\Delta\, \mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_2, E)\Big) \tilde{\cap} (\mathcal{F}_2, E) = \widetilde{\phi},$$

since

$$(\mathcal{F}_2, E) \widetilde{\subseteq} \mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_2, E).$$

Therefore, $(X, \mathcal{F}, E)$ is soft co-normal. □

Corollary 3.2. *A soft topological space $(X, \mathcal{F}, E)$ is soft co-normal iff for any pair of disjoint soft closed sets (C_1, E) and (C_2, E), there exists $(\mathcal{F}_1, E) \widetilde{\in} (X, \mathcal{F}^k, E)$ such that*

$$(C_1, E) \widetilde{\subseteq} (\mathcal{F}_1, E) \qquad \textit{and} \qquad \mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_1, E) \cap (C_2, E) = \widetilde{\varnothing}.$$

Proof. By Theorem 3.10. □

Theorem 3.11. *A soft topological space $(X, \mathcal{F}, E)$ is soft co-normal if and only if for every pair of soft sets $(\mathcal{F}_1, E)$ and $(\mathcal{F}_2, E)$ in $(X, \mathcal{F}, E)$ with*

$$(\mathcal{F}_1, E)\tilde{\cup}(\mathcal{F}_2, E) = \tilde{X},$$

there exist soft co-closed sets (C_1, E) and (C_2, E) such that

$$(C_1, E)\widetilde{\subseteq}(\mathcal{F}_1, E), \qquad (C_2, E)\widetilde{\subseteq}(\mathcal{F}_2, E), \qquad (C_1, E)\tilde{\cup}(C_2, E) = \tilde{X}.$$

Proof. ($\Rightarrow$) Suppose $(X, \mathcal{F}, E)$ is soft co-normal. Let $(\mathcal{F}_1, E)$ and $(\mathcal{F}_2, E)$ be soft open subsets in $(X, \mathcal{F}, E)$ with

$$(\mathcal{F}_1, E)\tilde{\cup}(\mathcal{F}_2, E) = \tilde{X}.$$

Then $\tilde{X}\Delta(\mathcal{F}_1, E)$ and $\tilde{X}\Delta(\mathcal{F}_2, E)$ are disjoint soft closed sets in $(X, \mathcal{F}, E)$. Since $(X, \mathcal{F}, E)$ is soft co-normal, there exist

$$(\mathcal{F}_3, E), (\mathcal{F}_4, E)\widetilde{\in}(X, \mathcal{F}^k, E)$$

such that

$$\tilde{X}\Delta(\mathcal{F}_1, E)\widetilde{\subseteq}(\mathcal{F}_3, E), \qquad \tilde{X}\Delta(\mathcal{F}_2, E)\widetilde{\subseteq}(\mathcal{F}_4, E), \qquad (\mathcal{F}_3, E)\tilde{\cap}(\mathcal{F}_4, E) = \widetilde{\phi}.$$

Finally, set

$$(C_1, E) = X\Delta(\mathcal{F}_2, E), \qquad (C_2, E) = X\Delta(\mathcal{F}_1, E).$$

These are the needed closed soft subsets.

($\Leftarrow$) Let (C_1, E) and (C_2, E) be two disjoint soft closed sets in $(X, \mathcal{F}, E)$. Take

$$(\mathcal{F}_1, E) = \tilde{X}\Delta(C_1, E),$$

clearly $(\mathcal{F}_1, E)\widetilde{\in}(X, \mathcal{F}, E)$ and $(C_2, E)\widetilde{\subseteq}(\mathcal{F}_1, E)$.

Since $(X, \mathcal{F}, E)$ is soft co-normal, there exists a soft open set $(\mathcal{F}_2, E)\widetilde{\in}(X, \mathcal{F}^k, E)$ such that

$$(C_1, E)\widetilde{\subseteq}(\mathcal{F}_2, E) \qquad \text{and} \qquad (\mathcal{F}_2, E)\widetilde{\subseteq}\,\mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_1, E).$$

Let

$$(\mathcal{F}_3, E) = \tilde{X}\Delta\,\mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_2, E).$$

Then

$$(\mathcal{F}_3, E)\widetilde{\in}(X, \mathcal{F}^k, E), \qquad (C_2, E)\widetilde{\subseteq}(\mathcal{F}_3, E),$$

and

$$(\mathcal{F}_2, E)\tilde{\cap}(\mathcal{F}_3, E) = (\mathcal{F}_2, E)\tilde{\cap}\Big(X\Delta\,\mathrm{CL}_{\mathcal{F}^k}(\mathcal{F}_2, E)\Big) = \widetilde{\phi}.$$

Hence the result.

□

4. Applications

This work contributes to SDG 4 (Quality Education) by advancing theoretical research in soft topology, thereby strengthening the mathematical foundations that support higher education and knowledge creation. It also aligns with SDG 9 (Industry, Innovation and Infrastructure) through the development of foundational concepts—such as soft cocompact sets—that are relevant to the modeling and analysis of complex and uncertain systems. Moreover, the potential applicability of soft topological methods to networked and uncertainty-driven infrastructures makes this research indirectly relevant to SDG 6 (Clean Water and Sanitation), particularly in contexts where robust mathematical structures are needed to address system variability and incomplete information.

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