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Investigating the Relationships Between Weyl's and Cartan's 2^{th} Curvature Tensors in Finsler Spaces

Adel Al-Qashbari[1,2], S. Saleh[3,4], Alaa M. Abd El-latif[5,*], Fahmi AL-ssallal[1], Husham M. Attaalfadeel[5], Mohamed Said Mohamed[6,7], Mohammed Mamoun Ahmed Abubakr[8]

[1]*Department of Mathematics, Aden University, Aden, Yemen*
[2]*Department of Med. Engineering, Science and Technology University-Aden, Yemen*
[3]*Department of Mathematics, Hodeidah University, Hodeidah, Yemen*
[4]*Department of Computer Science, Cihan University-Erbil, Erbil, Iraq*
[5]*Mathematics Department, College of Science, Northern Border University, Arar 91431, Saudi Arabia*
[6]*Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim 16511, Saudi Arabia*
[7]*Mathematics Department, Faculty of Education, Ain Shams University, Cairo 11341, Egypt*
[8]*College of Business Administration, Northern Border University, Arar, Saudi Arabia*

[*]*Corresponding author: Alaa.ali@nbu.edu.sa, Alaa_8560@yahoo.com*

Abstract. This study investigates the interconnection between Weyl's curvature tensor W^i_{jkh} and Cartan's second curvature tensor P^i_{jkh} in the frame of Finsler geometry (or F-geometry), a broader framework that generalizes Riemannian geometry(or R-geometry). When describing the curvature characteristics of F-space which are crucial for simulating a variety of physical events both tensors are crucial. Even though the geometric meanings and physical consequences of these tensors have been thoroughly investigated, their interconnection remains an open area for research. In the present work, we demonstrate that the Weyl's and Cartan's second curvature tensors are connected by a novel set of identities and inequalities that we deduce by examining their algebraic and geometric characteristics. A series of theorems that outline particular circumstances in which the tensors exhibit generalized birecurrent behavior in Finsler spaces (or F-spaces) are presented. In addition to offering further insight into how these notions are applied in physics, especially in the gravitational field and cosmology, these results are anticipated to improve our knowledge of the curvature structure in F-spaces and yield interesting findings in the frame of differential geometry and its physical applications.

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1. Introduction

F-geometry, an extension of R-geometry, offers a versatile framework for modeling various physical events by generalizing the concept of curvature. In F-spaces, the curvature properties of the space are characterized by several curvature tensors(or C-tensors). The study of C-tensors in F-spaces has garnered significant attention in recent years, with numerous contributions to the understanding of their geometric and algebraic properties. Ahsan-Ali [2, 3] made notable advancements in the study of the W-C-tensor and its properties in both relativistic spacetimes and general relativity. Abu-Donia et al. [1] proposed the W^*-C-tensor in relativistic space-times, discussion further results into the complex structure of these spaces. In the realm of higher-order generalizations, Al-Qashbari et al. [4–6] contributed to the study of recurrent Finsler structures and their interconnection with special C-tensors. Notably, Al-Qashbari's work [8,10,11] on generalized C-tensors and recurrence decompositions in F-space has been foundational for the ongoing exploration of Finslerian curvature. Additionally, Misra et al. [14] and Goswami [15] have examined higher-order recurrent F-spaces, while Pandey et al. [17] analyzed generalized H-recurrent spaces.

On the other hand, Weyl's C-tensor W^i_{jkh} and Cartan's 2^{th} C-tensor P^i_{jkh} are two important geometric structures in the framework of F-geometry, which are used to study the curvature of space-time. Weyl's C-tensor W^i_{jkh} is a conformal invariant, which means that it is invariant under conformal transformations. The Cartan's 2^{th} C-tensor P^i_{jkh} is not a conformal invariant. It reveals non-Riemannian aspects of F-spaces, with applications in some complex systems. While the properties, implications, and general relativity of these tensors have been extensively analyzed as in [5–7,9,12,13,16,18], their relationships remains an area of ongoing investigation. The findings of these studies serve as a crucial foundation for the present study, which it main goal is investigation the interplay between the C-tensors W^i_{jkh} and P^i_{jkh} within F-geometry.

This manuscript contributes to the growing and developing the theoretical framework for F-geometry. It provides a comprehensive analysis through investigating the decomposition of C-tensors in F-spaces utilizing the higher-order derivatives of Berwald and Cartan connections and investigation the interconnections between Weyl's and Cartan's 2^{th} C-tensors. By examining their algebraic and geometric properties, we derived a series of new identities and inequalities connecting these tensors. These results provide deeper insight into the structural behavior and properties of these tensors. This study is expected to have implications for various areas in the frame of F-spaces and its applications.

2. Preliminaries

For the purposes of this work, a few definitions and criteria will be given in this section. In directional arguments, the metric tensors g_{ij} and B_k (Berwald's connection coefficients) G^i_{jk} are positively homogenous of degree 0. The following criteria are met by two vectors, y_i and y^i.

$$\begin{cases} a)\ g_{ij}y^j = y_i\ b)\ y_iy^i = F^2\ c)\ \delta^k_j y^j = y^k \\ d)\ \delta^i_j g_{ir} = g_{rj}\ e)\ \delta^i_k g^{jk} = g^{ji}. \end{cases} \tag{2.1}$$

The quantities g_{ij} and g^{ij} are related by [11]:

$$g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases} \quad . \tag{2.2}$$

The tensor C_{ijk} is named $(h)hv$-torsion tensor(or t-tensor), given by:

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i \, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k \, F^2 \, . \tag{2.3}$$

The $(h)hv$-t-tensor C^h_{ik} and the tensor C_{ijk} are given by:

$$\begin{cases} a) \; C^i_{jk}y^j = C^i_{jk}y^k = 0 \; b) \; C_{ijk}y^i = C_{ijk}y^j = C_{ijk}y^k = 0 \\ c) \; C_{ijk}g^{jk} = C_i \; d) \; C_{ijh}g^{jk} = C^k_{ih} \\ e) \; C^i_{jk}g^{jk} = C^i. \end{cases} \tag{2.4}$$

Covariant derivative(C-derivative) for Berwald $\mathfrak{B}_k \, T^i_j$ of any tensor T^i_j w. r. to x^k is given as:

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \, . \tag{2.5}$$

The vector y^i and metric function F are vanished identically for Berwald?s C-derivative.

$$\begin{cases} a) \; \mathfrak{B}_k F = 0 \\ b) \; \mathfrak{B}_k y^i = 0. \end{cases} \tag{2.6}$$

Metric tensor g_{ij} is not equal to zero (i.e. not vanish), defined by:

$$\mathfrak{B}_k g_{ij} = -2y^h \mathfrak{B}_h C_{ijk} = -2C_{ijk|h}y^h \, . \tag{2.7}$$

Tensor W^i_{jkh}, t-tensor W^i_{jk}, and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\delta^i_j}{(n+1)}H_{[hk]} + \frac{2y^i}{(n+1)}\dot{\partial}_j H_{[hk]} \tag{2.8}$$
$$+ \frac{\delta^i_k}{(n^2-1)}(nH_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr}) - \frac{\delta^i_h}{(n^2-1)}(nH_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}),$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)}H_{[jk]} + 2\frac{\delta^i_{[j}}{(n^2-1)}(nH_{k]} - y^r H_{k]r}), \tag{2.9}$$

and

$$W^i_j = H^i_{jk} - H\delta^i_j - \frac{1}{(n+1)}(\dot{\partial}_r H^r_j - \dot{\partial}_j H)y^i, \tag{2.10}$$

respectively. Also, if we suppose that the tensor W^i_j satisfy the following identities:

$$\begin{cases} a) \; W^i_k y^k = 0 \; b) \; W^i_i = 0 \; c) \; W^i_k y_i = 0 \\ d) \; g_{ir}W^i_j = W_{rj} \; e) \; g^{jk}W_{jk} = W \; f) \; W_{jk}y^k = 0. \end{cases} \tag{2.11}$$

The tensor W^i_{jkh} is a skew-symmetric in its indices k and h.

Cartan's 2^{th} C-tensor P^i_{jkh}, t-tensor P^i_{kh}, the associate tensor P_{rjkh}, Ricci tensor P_{jk}, and the vector P_k are given as:

$$\begin{cases} a)\ P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}\ b)\ P^i_{jkh} y^j = P^i_{kh} = \Gamma^{*i}_{jkh} y^j = C^i_{kh|r} y^r \\ c)\ P^i_{jkh} y^k = 0 = P^i_{jkh} y^h\ d)\ P^i_{kh} = G^i_{kh} - \Gamma^{*i}_{kh}\ e)\ g_{ir} P^i_{kh} = P_{rkh} \\ f)\ P^i_{kh} y^k = 0 = P^i_{kh} y_i\ g)\ P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r} y^r\ h)\ g_{ir} P^i_{jkh} = P_{rjkh} \\ i)\ P_{ijkh} g_{kh} = P_{ij} - P_{ji}\ j)\ P^i_{jki} = P_{jk}\ k)\ P^i_{ki} = P_k. \end{cases} \tag{2.12}$$

Cartan's 3^{th} C-tensor R^i_{jkh}, Ricci tensor R_{jk}, the vector H_k, and scalar curvature H are given as:

$$\begin{cases} a)\ R_{jk} y^j = H_k\ b)\ R_{jk} y^k = R_j\ c)\ R^i_i = R. \end{cases} \tag{2.13}$$

Cartan's first C-tensor S^i_{jkh}, Ricci tensor S_{jk}, the tensor S^i_k, and scalar curvature S are given as

$$\begin{cases} a)\ S^i_{jkh} = C^i_{rk} C^r_{jh} - C^i_{rh} C^r_{jk}\ b)\ S^i_{jki} = S_{jk}\ c)\ S = S_{kh} g^{kh} \\ d)\ S_{rjkh} = g_{ri} S^i_{jkh}\ e)\ S_{kh} g^{ih} = S^i_k. \end{cases} \tag{2.14}$$

Al-Qashbari et. al [6,7] introduced and studied the C-tensor in F-space, using the derivatives for Berwald and Cartan, which are characterized by the item:

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_k g_{jh} - \delta^i_h g_{jk} \right) + \frac{1}{4} \gamma_m (W^i_k g_{jh} - W^i_h g_{jk}). \tag{2.15}$$

An F-space F_n which the C-tensor W^i_{jkh} holds the item (2.15) is named the generalization generalized $\mathfrak{B}W$-recurrent space, refereed to by $G^{2nd}\ \mathfrak{B}W - RF_n$.

Taking the C-derivative of (2.15) w. r. to x^l in the sense of Berwald, we have:

$$\begin{aligned} \mathfrak{B}_l \mathfrak{B}_m W^i_{jkh} &= (\mathfrak{B}_l \lambda_m) W^i_{jkh} + \lambda_m (\mathfrak{B}_l W^i_{jkh}) + (\mathfrak{B}_l \mu_m)(\delta^i_k g_{jh} - \delta^i_h g_{jh}) \\ &+ \mu_m \mathfrak{B}_l (\delta^i_k g_{jh} - \delta^i_h g_{jh}) + \frac{1}{4} \gamma_m \mathfrak{B}_l (W^i_k g_{jh} - W^i_h g_{jk}) \\ &+ \frac{1}{4} (\mathfrak{B}_l \gamma_m)(W^i_k g_{jh} - W^i_h g_{jk}). \end{aligned} \tag{2.16}$$

Using (2.7) and (2.15) in (2.16), we get:

$$\begin{aligned} \mathfrak{B}_l \mathfrak{B}_m W^i_{jkh} &= (\lambda_{ml} + \lambda_m \lambda_l) W^i_{jkh} + (\mu_{ml} + \lambda_m \mu_l)(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \\ &+ \frac{1}{4} \gamma_m \mathfrak{B}_l (W^i_k g_{jh} - W^i_h g_{jk}) + \frac{1}{4} (\lambda_m \gamma_l + \gamma_{ml})(W^i_k g_{jh} - W^i_h g_{jk}) \\ &- 2\mu_m \mathfrak{B}_q y^q (\delta^i_k C_{jhl} - \delta^i_h C_{jkl}) - \frac{1}{2} \gamma_m \mathfrak{B}_q y^q (W^i_k C_{jhl} - W^i_h C_{jkl}). \end{aligned} \tag{2.17}$$

The equation (2.17), can be written as:

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{jkh} = a_{ml} W^i_{jkh} + b_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4} c_{ml}(W^i_k g_{jh} - W^i_h g_{jk}) \tag{2.18}$$
$$+\frac{1}{4}\gamma_m \mathfrak{B}_l(W^i_k g_{jh} - W^i_h g_{jk}) - 2\mu_m \mathfrak{B}_q y^q(\delta^i_k C_{jhl} - \delta^i_h C_{jkl})$$
$$-\frac{1}{2}\gamma_m \mathfrak{B}_q y^q(W^i_k C_{jhl} - W^i_h C_{jkl}).$$

Where, the covariant tensor fields of 2-order are non-zero and defined by: $a_{ml} = \lambda_{ml} + \lambda_m \lambda_l$, $b_{ml} = \mu_{ml} + \lambda_m \mu_l$, and $c_{ml} = \lambda_m \gamma_l + \gamma_{ml}$.

Definition 2.1. *The generalized birecurrent space(or Gbi-recurrent space) $\mathfrak{B}W$ is defined as F-space, where the Wely's projective C-tensor W^i_{jkh} meets condition* (2.18). *The tensor is referred to as a Gbi-recurrent space $\mathfrak{B}$. The shorthand symbols for these spaces and tensors are $G^{2nd}(\mathfrak{B}W)$-BRF_n and G^{2nd} $\mathfrak{B}$-BR, respectively.*

We consider an n-dimensional F-space F_n, the Wely?s projective C-tensor W^i_{jkh} holds the conditions (2.15) and (2.18), these spaces denoted by $G^{2nd}(\mathfrak{B}W)$-RF_n and $G^{2nd}\mathfrak{B}W$-BRF_n, respectively.

3. The main results

Several tensors characterize the curvature qualities of F-spaces; Weyl and Cartan's second C-tensors are important among them, and both have impacted the field's advancement. Many academics have thoroughly examined these tensors' geometric interpretations and physical ramifications. Research on their connections is still ongoing, though. In this section, we explore how these C-tensors relate to one another in F-spaces. We develop a new sets of identities and inequalities that connect these C-tensors by investigating their algebraic and geometric features.

Ahsan-Ali presented certain characteristics of the W^i_{jkh} C-tensor ([3], [4]). It is known that Wely's projective C-tensor W^i_{jkh} and Cartan's 2^{th} C-tensor P^i_{jkh} are related for ($n = 4$) a R-space using the following formula [1]:

$$W^i_{jkh} = P^i_{jkh} + \frac{1}{3}(\delta^i_k R_{jh} - g_{jk} R^i_h) \tag{3.1}$$

Taking the C-derivative of (3.1), w. r. to x^m and x^l in the sense of Berwald we have;

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{jkh} = \mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} + \frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh} - g_{jk} R^i_h) \tag{3.2}$$

Using (2.7) in (3.2), we get:

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{jkh} = \mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} + \frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh} - g_{jk} R^i_h) \tag{3.3}$$
$$+\frac{2}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R^i_h + \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l(C_{jkm} R^i_h).$$

Using (2.18) and (3.2) in (3.3), we have:

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{ml} P^i_{jkh} + b_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4} c_{ml}(W^i_k g_{jh} - W^i_h g_{jk}) \tag{3.4}$$

$$
\begin{aligned}
&+\frac{1}{4}\gamma_m\mathfrak{B}_l(W^i_k g_{jh}-W^i_h g_{jk})-2\mu_m\mathfrak{B}_q y^q(\delta^i_k C_{jhl}-\delta^i_h C_{jkl})\\
&-\frac{1}{2}\gamma_m\mathfrak{B}_q y^q(W^i_k C_{jhl}-W^i_h C_{jkl})-\frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh}-g_{jk}R^i_h)\\
&+\frac{1}{3}a_{ml}(\delta^i_k R_{jh}-g_{jk}R^i_h)-\frac{2}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R^i_h-\frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R^i_h).
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh}&=a_{ml}P^i_{jkh}+b_{ml}(\delta^i_k g_{jh}-\delta^i_h g_{jk})+\frac{1}{4}c_{ml}(W^i_k g_{jh}-W^i_h g_{jk})\\
&+\frac{1}{4}\gamma_m\mathfrak{B}_l(W^i_k g_{jh}-W^i_h g_{jk})-2\mu_m\mathfrak{B}_q y^q(\delta^i_k C_{jhl}-\delta^i_h C_{jkl})\\
&-\frac{1}{2}\gamma_m\mathfrak{B}_q y^q(W^i_k C_{jhl}-W^i_h C_{jkl})-\frac{2}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R^i_h-\frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R^i_h).
\end{aligned} \tag{3.5}
$$

If and only if

$$
\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh}-g_{jk}R^i_h)=a_{ml}(\delta^i_k R_{jh}-g_{jk}R^i_h). \tag{3.6}
$$

Thus, we conclude,

Theorem 3.1. *In G^{2nd} $\mathfrak{B}W$-BRF_n, the Cartan's 2^{th} C-tensor P^i_{jkh} is a Gbi-recurrent F-space if and only if the tensor $(\delta^i_k R_{jh}-g_{jk}R^i_h)$ is Gbi-recurrent.*

Transvecting condition (3.4) by y^j, using [(2.6)b], [(2.1)a], [(2.4)b], [(2.12)b] and [(2.13)a], we get:

$$
\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m P^i_{kh}&=a_{ml}P^i_{kh}+b_{ml}(\delta^i_k y_h-\delta^i_h y_k)+\frac{1}{4}c_{ml}(W^i_k y_h-W^i_h y_k)\\
&+\frac{1}{4}\gamma_m\mathfrak{B}_l(W^i_k y_h-W^i_h y_k)-\frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k H_h-y_k R^i_h)\\
&+\frac{1}{3}a_{ml}(\delta^i_k H_h-y_k R^i_h).
\end{aligned} \tag{3.7}
$$

This shows that

$$
\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m P^i_{kh}&=a_{ml}P^i_{kh}+b_{ml}(\delta^i_k y_h-\delta^i_h y_k)+\frac{1}{4}c_{ml}(W^i_k y_h-W^i_h y_k)\\
&+\frac{1}{4}\gamma_m\mathfrak{B}_l(W^i_k y_h-W^i_h y_k).
\end{aligned} \tag{3.8}
$$

If and only if

$$
\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k H_h-y_k R^i_h)=a_{ml}(\delta^i_k H_h-y_k R^i_h). \tag{3.9}
$$

The proof ends, we conclude,

Theorem 3.2. *In $G^{2nd}\mathfrak{B}W$-BRF_n. The C-derivative of the 2-orders for the t-tensor P^i_{kh} is a Gbi-recurrent F-space if and only if (3.9) valid good.*

If we, transfect (3.7) by y^k, using [(2.6)b], [(2.1)b], [(2.1)c], [(2.11)a], and [(2.12)c], we obtain,

$$
\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m(y^i H_h-F^2R^i_h)&=a_{ml}(y^i H_h-F^2R^i_h)+3b_{ml}(y^i y_h-\delta^i_h F^2)\\
&-\frac{3}{4}\gamma_m(\mathfrak{B}_l W^i_h)F^2-\frac{3}{4}C_{ml}W^i_h F^2.
\end{aligned} \tag{3.10}
$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(y^iH_h) = a_{ml}(y^iH_h) + b_{ml}(y^iy_h), \tag{3.11}$$

and

$$\mathfrak{B}_l\mathfrak{B}_m(F^2R^i_h) = a_{ml}(F^2R^i_h) + b_{ml}(\delta^i_hF^2). \tag{3.12}$$

If and only if

$$(\mathfrak{B}_lW^i_h)F^2 - \lambda_m W^i_hF^2 = 0. \tag{3.13}$$

Therefore, we get,

Theorem 3.3. *In G^{2nd} $\mathfrak{B}W$-BRF_n. The C-derivative of the 2-orders for the tensor (y^iH_h) and $(F^2R^i_h)$ are Gbi-recurrent F-spaces if and only if* (3.13) *valid good.*

If we contract the i and h in the items (3.7) and (3.10), respectively with using (2.2), [(2.1)a], [(2.1)b], [(2.12)k], [(2.13)c], [(2.11)c], and [(2.11)b], we get:

$$\mathfrak{B}_l\mathfrak{B}_mP_k = a_{ml}P_k + (1-n)b_{ml}\, y_k + \frac{1}{3}a_{ml}(H_k - y_kR) \tag{3.14}$$
$$-\frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(H_k - y_kR).$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_mP_k = a_{ml}P_k + (1-n)b_{ml}\, y_k. \tag{3.15}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(H_k - y_kR) = a_{ml}(H_k - y_kR) \tag{3.16}$$

and

$$\mathfrak{B}_l\mathfrak{B}_m(H - F^2R) = a_{ml}(H - F^2R) + (1-n)b_{ml}\, F^2. \tag{3.17}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_mH = a_{ml}H + (1-n)b_{ml}\, F^2. \tag{3.18}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(F^2R) = a_{ml}(F^2R). \tag{3.19}$$

Thus, we conclude,

Theorem 3.4. *In G^{2nd} $\mathfrak{B}W$-BRF_n. The vector P_k and the scalar H are given in* (3.15) *and* (3.18) *if and only if the items* (3.16) *and* (3.19) *are both valid, respectively.*

If we contract the i and h in the equations (3.4) with using [(2.1)d], [(2.1)b], [(2.12)j], [(2.13)c], [(2.11)d] and [(2.11)b], we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P_{jk} &= a_{ml}P_{jk} + (1-n)b_{ml}g_{jk} + \frac{1}{4}c_{ml}W_{jk} \\ &+\frac{1}{4}\gamma_m\mathfrak{B}_l W_{jk} - 2(1-n)\mu_m\mathfrak{B}_q y^q C_{jkl} \\ &-\frac{1}{2}\gamma_m\mathfrak{B}_q y^q W^i_k C_{jil} - \frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(R_{jk} - g_{jk}R) \\ &+\frac{1}{3}a_{ml}(R_{jk} - g_{jk}R) - \frac{2}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R - \frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R).\end{aligned} \tag{3.20}$$

This shows that

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P_{jk} &= a_{ml}P_{jk} + (1-n)b_{ml}g_{jk} + \frac{1}{4}c_{ml}W_{jk} \\ &+\frac{1}{4}\gamma_m\mathfrak{B}_l W_{jk} - 2(1-n)\mu_m\mathfrak{B}_q y^q C_{jkl} - \frac{1}{2}\gamma_m\mathfrak{B}_q y^q W^i_k C_{jil} \\ &-\frac{2}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R - \frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R).\end{aligned} \tag{3.21}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(R_{jk} - g_{jk}R) = a_{ml}(R_{jk} - g_{jk}R). \tag{3.22}$$

Upon completing the proof of the theorem, we get,

Theorem 3.5. *In G^{2nd} $\mathfrak{B}W$-BRF_n. The P-Ricci tensor P_{jk} is given in* (3.21) *if and only if the item* (3.22) *is valid good.*

If we transfect (3.20) with y^k, using [(2.6)b], [(2.1)a] ,[(2.1)c], [(2.4)b], [(2.11)a], [(2.11)f] and [(2.13)b] we get,

$$\mathfrak{B}_l\mathfrak{B}_m(R_j - y_jR) = a_{ml}(R_j - y_jR) + (1-n)b_{ml}\, y_j. \tag{3.23}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m R_j = a_{ml}R_j + b_{ml}\, y_j. \tag{3.24}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(y_jR) = a_{ml}(y_jR) - nb_{ml}\, y_j. \tag{3.25}$$

Transvecting (3.4) and (3.20) by g^{jk}, respectively using [(2.4)c], [(2.4)d], [(2.11)e] and [(2.12)h, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(P^i_{jkh}g^{jk}) &= a_{ml}(P^i_{jkh}g^{jk}) - 2\mu_m\mathfrak{B}_q y^q(C^i_{hl} - \delta^i_h C_l) \\ &-\frac{1}{2}\gamma_m\mathfrak{B}_q y^q(W^i_k C^k_{hl} - W^i_h C_l) - \frac{2}{3}\mathfrak{B}_q y^q C_l\mathfrak{B}_m R^i_h - \frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_m R^i_h).\end{aligned} \tag{3.26}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(P^i_{jkh}g^{jk}) = a_{ml}(P^i_{jkh}g^{jk}) \tag{3.27}$$

If and only if

$$2\mu_m \mathfrak{B}_q y^q (C^i_{hl} - \delta^i_h C_l) + \frac{1}{2}\gamma_m \mathfrak{B}_q y^q (W^i_k C^k_{hl} - W^i_h C_l) \\ + \frac{2}{3}\mathfrak{B}_q y^q C_l \mathfrak{B}_m R^i_h + \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_m R^i_h) = 0. \tag{3.28}$$

And

$$\mathfrak{B}_l \mathfrak{B}_m (P_{jk} g^{jk}) = a_{ml}(P_{jk} g^{jk}) + (1-n) b_{ml} + \frac{1}{4} c_{ml} W + \frac{1}{4}\gamma_m \mathfrak{B}_l W \\ -2(1-n)\mu_m \mathfrak{B}_q y^q C_l - \frac{1}{2}\gamma_m \mathfrak{B}_q y^q W^i_k C^k_{hl} - \frac{2}{3}\mathfrak{B}_q y^q C_l \mathfrak{B}_m R - \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_m R). \tag{3.29}$$

This shows that

$$\mathfrak{B}_l \mathfrak{B}_m (P_{jk} g^{jk}) = a_{ml}(P_{jk} g^{jk}) + (1-n) b_{ml} + \frac{1}{4} c_{ml} W + \frac{1}{4}\gamma_m \mathfrak{B}_l W. \tag{3.30}$$

If and only if

$$2(1-n)\mu_m \mathfrak{B}_q y^q C_l + \frac{1}{2}\gamma_m \mathfrak{B}_q y^q W^i_k C^k_{hl} + \frac{2}{3}\mathfrak{B}_q y^q C_l \mathfrak{B}_m R \\ + \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_m R) = 0. \tag{3.31}$$

The proof ends, and we conclude,

Theorem 3.6. *In G^{2nd} $\mathfrak{B}W$-BRF_n, vector R_j, the tensor $(P^i_{jkh} g^{jk})$ and $(P_{jk} g^{jk})$ are given in* (3.24), (3.27) *and* (3.30) *if and only if the items* (3.25), (3.28) *and* (3.31) *are valid good, respectively.*

Transvecting (3.4) by g_{ir}, using [(2.1)d], [(2.11)d] and [(2.12)f], we get

$$\mathfrak{B}_l \mathfrak{B}_m P_{rjkh} = a_{ml} P_{rjkh} + b_{ml}(g_{rk} g_{jh} - g_{rh} g_{jk}) + \frac{1}{4} c_{ml}(W_{rk} g_{jh} - W_{rh} g_{jk}) \\ + \frac{1}{4}\gamma_m \mathfrak{B}_l (W_{rk} g_{jh} - W_{rh} g_{jk}) + \frac{1}{3} a_{ml}(g_{rk} R_{jh} - g_{jk} R_{rh}) \\ -2\mu_m \mathfrak{B}_q y^q (g_{rk} C_{jhl} - g_{rh} C_{jkl}) - \frac{1}{2}\gamma_m \mathfrak{B}_q y^q (W_{rk} C_{jhl} - W_{rh} C_{jkl}) \\ - \frac{1}{3}\mathfrak{B}_l \mathfrak{B}_m (g_{rk} R_{jh} - g_{jk} R_{rh}) - \frac{2}{3}\mathfrak{B}_q y^q C_{jkl} \mathfrak{B}_m R_{rh} - \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_{jkm} R_{rh}). \tag{3.32}$$

This shows that

$$\mathfrak{B}_l \mathfrak{B}_m P_{rjkh} = a_{ml} P_{rjkh} + b_{ml}(g_{rk} g_{jh} - g_{rh} g_{jk}) + \frac{1}{4} c_{ml}(W_{rk} g_{jh} - W_{rh} g_{jk}) \\ + \frac{1}{4}\gamma_m \mathfrak{B}_l (W_{rk} g_{jh} - W_{rh} g_{jk}) - 2\mu_m \mathfrak{B}_q y^q (g_{rk} C_{jhl} - g_{rh} C_{jkl}) \\ - \frac{1}{2}\gamma_m \mathfrak{B}_q y^q (W_{rk} C_{jhl} - W_{rh} C_{jkl}) - \frac{2}{3}\mathfrak{B}_q y^q C_{jkl} \mathfrak{B}_m R_{rh} - \frac{2}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_{jkm} R_{rh}). \tag{3.33}$$

If and only if

$$\mathfrak{B}_l \mathfrak{B}_m (g_{rk} R_{jh} - g_{jk} R_{rh}) = a_{ml}(g_{rk} R_{jh} - g_{jk} R_{rh}). \tag{3.34}$$

Transvecting (3.7) by g_{ir}, using [(2.1)d], [(2.12)d]and [(2.12)e], we get

$$\mathfrak{B}_l\mathfrak{B}_m P_{rkh} = a_{ml}P_{rkh} + b_{ml}(g_{rk}y_h - g_{rh}y_k) + \frac{1}{4}c_{ml}(W_{rk}y_h - W_{rh}y_k) \tag{3.35}$$
$$+\frac{1}{4}\gamma_m\mathfrak{B}_l(W_{rk}y_h - W_{rh}y_k) - \frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(g_{rk}H_h - y_kR_{rh})$$
$$+\frac{1}{3}a_{ml}(g_{rk}H_h - y_kR_{rh}).$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P_{rkh} = a_{ml}P_{rkh} + b_{ml}(g_{rk}y_h - g_{rh}y_k) + \frac{1}{4}c_{ml}(W_{rk}y_h - W_{rh}y_k) \tag{3.36}$$
$$+\frac{1}{4}\gamma_m\mathfrak{B}_l(W_{rk}y_h - W_{rh}y_k).$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(g_{rk}H_h - y_kR_{rh}) = a_{ml}(g_{rk}H_h - y_kR_{rh}). \tag{3.37}$$

The proof ends, and we conclude,

Theorem 3.7. *In G^{2nd} $\mathfrak{B}W$-BRF_n, the associate tensor P_{rjkh}, and the associative tensor P_{rkh} of (Cartan's 2^{th} C-tensor are Gbi-recurrent F-spaces if and only if the items* (3.34) *and* (3.37) *are valid good, respectively.*

Transvecting (3.32) by g^{kh}, using [(2.4)d], and [(2.12)i], we get

$$\mathfrak{B}_l\mathfrak{B}_m(P_{rj} - P_{jr}) = a_{ml}(P_{rj} - P_{jr}) - \frac{1}{2}\gamma_m\mathfrak{B}_q y^q(W_{rk}C^k_{jl} - W_{rh}C^h_{jl}) \tag{3.38}$$
$$-\frac{2}{3}\mathfrak{B}_q y^q C^h_{jl}\mathfrak{B}_m R_{rh} - \frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C^h_{jm}R_{rh}).$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(P_{rj} - P_{jr}) = a_{ml}(P_{rj} - P_{jr}). \tag{3.39}$$

If and only if

$$\frac{1}{2}\gamma_m\mathfrak{B}_q y^q(W_{rk}C^k_{jl} - W_{rh}C^h_{jl}) + \frac{2}{3}\mathfrak{B}_q y^q C^h_{jl}\mathfrak{B}_m R_{rh} + \frac{2}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C^h_{jm}R_{rh}) = 0. \tag{3.40}$$

Hence, the proof ends, and we conclude,

Theorem 3.8. *In $G^{2nd}\mathfrak{B}W - BRF_n$. The tensor $(P_{rj} - P_{jr})$ is a Gbi-recurrent F-space if and only if the item* (3.40) *valid good.*

Clearly, the Cartan's 2^{th} C-tensor P^i_{jkh} and Cartan's 1^{th} C-tensor S^i_{jkh} are related by the formula.

$$P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r}y^r. \tag{3.41}$$

Taking the C-derivative of (3.41), w. r. to x^m and x^l in the seance of Berwald, we have,

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} - \mathfrak{B}_l\mathfrak{B}_m P^i_{jhk} = -\mathfrak{B}_l\mathfrak{B}_m(S^i_{jkh|r}y^r). \tag{3.42}$$

Using (3.4) and (3.41) in (3.42) we get,

$$\mathfrak{B}_l\mathfrak{B}_m(S^i_{jkh|r}y^r) = a_{ml}(S^i_{jkh|r}y^r) - \omega_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \tag{3.43}$$
$$-\frac{1}{2}c_{ml}(W^i_k g_{jh} - W^i_h g_{jk}) + \frac{2}{3}\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh} - g_{jk}R^i_h)$$
$$-\frac{2}{3}a_{ml}(\delta^i_k R_{jh} - g_{jk}R^i_h) - \frac{1}{2}\gamma_m\mathfrak{B}_l(W^i_k g_{jh} - W^i_h g_{jk})$$
$$+4\mu_m\mathfrak{B}_q y^q(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}) + \gamma_m\mathfrak{B}_q y^q(W^i_k C_{jhl} - W^i_h C_{jkl})$$
$$+\frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R^i_h + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R^i_h),$$

where $2b_{ml} = \omega_{ml}$. This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(S^i_{jkh|r}y^r) = a_{ml}(S^i_{jkh|r}y^r) - \omega_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \tag{3.44}$$
$$-\frac{1}{2}c_{ml}(W^i_k g_{jh} - W^i_h g_{jk}) - \frac{1}{2}\gamma_m\mathfrak{B}_l(W^i_k g_{jh} - W^i_h g_{jk}).$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k R_{jh} - g_{jk}R^i_h) = a_{ml}(\delta^i_k R_{jh} - g_{jk}R^i_h). \tag{3.45}$$

and

$$4\mu_m\mathfrak{B}_q y^q(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}) + \gamma_m\mathfrak{B}_q y^q(W^i_k C_{jhl} - W^i_h C_{jkl}) \tag{3.46}$$
$$+\frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R^i_h + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R^i_h) = 0.$$

Hence, the proof ends, we get,

Theorem 3.9. *In $G^{2nd}\mathfrak{B}W$-BRF_n. The tensors $(-S^i_{jkh|r}y^r)$ is a Gbi-recurrent F-space if and only if the items* (3.45) *and* (3.46) *valid good, respectively.*

If we Contract the i and h in the items (3.43) and using [(2.1)d], [(2.1)b], [(2.14)b], [(2.13)b], [(2.11)d] and [(2.11)b], we have,

$$\mathfrak{B}_l\mathfrak{B}_m(S_{jk|r}y^r) = a_{ml}(S_{jk|r}y^r) + (n-1)\omega_{ml}g_{jk} - \frac{1}{2}c_{ml}W_{jk} \tag{3.47}$$
$$-\frac{1}{2}\gamma_m\mathfrak{B}_l W_{jk} - 4(n-1)\mu_m\mathfrak{B}_q y^q C_{jkl} + \gamma_m\mathfrak{B}_q y^q W^i_k C_{jil}$$
$$+\frac{2}{3}\mathfrak{B}_l\mathfrak{B}_m(R_{jk} - g_{jk}R) - \frac{2}{3}a_{ml}(R_{jk} - g_{jk}R)$$
$$+\frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R).$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(S_{jk|r}y^r) = a_{ml}(S_{jk|r}y^r) + (n-1)\omega_{ml}g_{jk} - \frac{1}{2}c_{ml}W_{jk} - \frac{1}{2}\gamma_m\mathfrak{B}_l W_{jk}. \tag{3.48}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(R_{jk} - g_{jk}R) = a_{ml}(R_{jk} - g_{jk}R). \tag{3.49}$$

and

$$4(1-n)\mu_m\mathfrak{B}_q y^q C_{jkl} + \gamma_m\mathfrak{B}_q y^q W^i_k C_{jil} + \frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R + \frac{4}{3}\mathfrak{B}_q y^q \mathfrak{B}_l(C_{jkm}R) = 0. \tag{3.50}$$

Transvecting (3.43)b by g_{ir}, using [(2.1)d], [(2.11)d] and [(2.14)c] we get,

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(S_{rjkh|r}y^r) &= a_{ml}(S_{rjkh|r}y^r) - b_{ml}(g_{rk}g_{jh} - g_{rh}g_{jk})\\ &-\frac{1}{2}c_{ml}(W_{rk}g_{jh} - W_{rh}g_{jk}) - \frac{1}{2}\gamma_m\mathfrak{B}_l(W_{rk}g_{jh} - W_{rh}g_{jk})\\ &+4\mu_m\mathfrak{B}_q y^q(g_{rk}C_{jhl} - g_{rh}C_{jkl}) + \gamma_m\mathfrak{B}_q y^q(W_{rk}C_{jhl} - W_{rh}C_{jkl})\\ &+\frac{2}{3}\mathfrak{B}_l\mathfrak{B}_m(g_{rk}R_{jh} - g_{jk}R_{rh}) - \frac{2}{3}a_{ml}(g_{rk}R_{jh} - g_{jk}R_{rh})\\ &+\frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R_{rh} + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R_{rh}).\end{aligned} \tag{3.51}$$

This shows that

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(S_{rjkh|r}y^r) &= a_{ml}(S_{rjkh|r}y^r) - b_{ml}(g_{rk}g_{jh} - g_{rh}g_{jk})\\ &-\frac{1}{2}c_{ml}(W_{rk}g_{jh} - W_{rh}g_{jk}) - \frac{1}{2}\gamma_m\mathfrak{B}_l(W_{rk}g_{jh} - W_{rh}g_{jk})\\ &+4\mu_m\mathfrak{B}_q y^q(g_{rk}C_{jhl} - g_{rh}C_{jkl}) + \gamma_m\mathfrak{B}_q y^q(W_{rk}C_{jhl} - W_{rh}C_{jkl})\\ &\frac{4}{3}\mathfrak{B}_q y^q C_{jkl}\mathfrak{B}_m R_{rh} + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C_{jkm}R_{rh}).\end{aligned} \tag{3.52}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(g_{rk}R_{jh} - g_{jk}R_{rh}) = a_{ml}(g_{rk}R_{jh} - g_{jk}R_{rh}). \tag{3.53}$$

Thus, the proof ends, and we obtain,

Theorem 3.10. *In $G^{2nd}\mathfrak{B}W$-BRF_n. The S-Ricci tensor $(S_{jk|r}y^r)$, and the associate tensor $(S_{rjkh|r}y^r)$ is given in* (3.48) *and* (3.52) *if and only if the items* (3.49), (3.50), *and* (3.53) *are valid good, respectively.*

Transvecting (3.47)by g^{ik}, using [(2.4)c], [(2.4)d], [(2.11)e], and [(2.14)e] we get,

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(S^i_{j|r}y^r) &= a_{ml}(S^i_{j|r}y^r) + (n-1)\omega_{ml}\delta^i_j - \frac{1}{2}c_{ml}W^i_j - \frac{1}{2}\gamma_m\mathfrak{B}_l W^i_j\\ &-4(n-1)\mu_m\mathfrak{B}_q y^q C^i_{jl} + \gamma_m\mathfrak{B}_q y^q W^i_k C^k_{jl} + \frac{2}{3}\mathfrak{B}_l\mathfrak{B}_m(R^i_j - \delta^i_j R)\\ &-\frac{2}{3}a_{ml}(R^i_j - \delta^i_j R) + \frac{4}{3}\mathfrak{B}_q y^q C^i_{jl}\mathfrak{B}_m R + \frac{4}{3}\mathfrak{B}_q y^q\mathfrak{B}_l(C^i_{jm}R).\end{aligned} \tag{3.54}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(S^i_{j|r}y^r) = a_{ml}(S^i_{j|r}y^r) + (n-1)\omega_{ml}\delta^i_j - \frac{1}{2}c_{ml}W^i_j - \frac{1}{2}\gamma_m\mathfrak{B}_l W^i_j. \tag{3.55}$$

If and only if

$$4(n-1)\mu_m \mathfrak{B}_q y^q C^i_{jl} - \gamma_m \mathfrak{B}_q y^q W^i_k C^k_{jl}$$
$$-\frac{4}{3}\mathfrak{B}_q y^q C^i_{jl}\mathfrak{B}_m R - \frac{4}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C^i_{jm} R) = 0. \quad (3.56)$$

and

$$\mathfrak{B}_l \mathfrak{B}_m (R^i_j - \delta^i_j R) = a_{ml}(R^i_j - \delta^i_j R). \quad (3.57)$$

Transvecting (3.47) by g^{jk}, using [(2.4)c], [(2.4)d], [(2.11)e] and [(2.14)d], we get

$$\mathfrak{B}_l \mathfrak{B}_m (S_{|r} y^r) = a_{ml}(S_{|r} y^r) + (n-1)\omega_{ml} - \frac{1}{2}c_{ml} W$$
$$-\frac{1}{2}\gamma_m \mathfrak{B}_l W - 4(n-1)\mu_m \mathfrak{B}_q y^q C_l + \gamma_m \mathfrak{B}_q y^q W^i_k C^k_{il}$$
$$+\frac{4}{3}\mathfrak{B}_q y^q C_l \mathfrak{B}_m R + \frac{4}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_m R). \quad (3.58)$$

This shows that

$$\mathfrak{B}_l \mathfrak{B}_m (S_{|r} y^r) = a_{ml}(S_{|r} y^r) + (n-1)\omega_{ml} - \frac{1}{2}c_{ml} W - \frac{1}{2}\gamma_m \mathfrak{B}_l W. \quad (3.59)$$

If and only if

$$-4(n-1)\mu_m \mathfrak{B}_q y^q C_l + \gamma_m \mathfrak{B}_q y^q W^i_k C^k_{il}$$
$$+\frac{4}{3}\mathfrak{B}_q y^q C_l \mathfrak{B}_m R + \frac{4}{3}\mathfrak{B}_q y^q \mathfrak{B}_l (C_m R) = 0. \quad (3.60)$$

Thus, we conclude

Theorem 3.11. *In $G^{2nd}\mathfrak{B}W$-BRF_n. The tensors $(S^i_{j|r} y^r)$ and $(S_{|r} y^r)$ are given in* (3.55) *and* (3.59) *if and only if the items* (3.56), (3.57), *and* (3.60) *are valid good, respectively.*

From the equation [(2.12)b] we get,

$$P^i_{kh} = C^i_{kh|r} y^r. \quad (3.61)$$

Taking the covariant derivative of (3.61), w. r. to x^m and x^l in the sense of Berwald we have,

$$\mathfrak{B}_l \mathfrak{B}_m P^i_{kh} = \mathfrak{B}_l \mathfrak{B}_m (C^i_{kh|r} y^r). \quad (3.62)$$

Using (3.7) and (3.61), in (3.62), we get

$$\mathfrak{B}_l \mathfrak{B}_m (C^i_{kh|r} y^r) = a_{ml}(C^i_{kh|r} y^r) + b_{ml}(\delta^i_k y_h - \delta^i_h y_k)$$
$$+\frac{1}{4}c_{ml}(W^i_k y_h - W^i_h y_k) + \frac{1}{4}\gamma_m \mathfrak{B}_l (W^i_k y_h - W^i_h y_k)$$
$$-\frac{1}{3}\mathfrak{B}_l \mathfrak{B}_m (\delta^i_k H_h - y_k R^i_h) + \frac{1}{3}a_{ml}(\delta^i_k H_h - y_k R^i_h). \quad (3.63)$$

This shows that

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(C^i_{kh|r}y^r) &= a_{ml}(C^i_{kh|r}y^r) + b_{ml}(\delta^i_k y_h - \delta^i_h y_k)\\ &+\frac{1}{4}c_{ml}(W^i_k y_h - W^i_h y_k) + \frac{1}{4}\gamma_m\mathfrak{B}_l(W^i_k y_h - W^i_h y_k).\end{aligned} \tag{3.64}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(\delta^i_k H_h - y_k R^i_h) = a_{ml}(\delta^i_k H_h - y_k R^i_h). \tag{3.65}$$

Transvecting (3.63) by g_{ij}, using [(2.1)d], [(2.4)c] and [(2.11)d], we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(C_{jkh|r}y^r) &= a_{ml}(C_{jkh|r}y^r) + b_{ml}(g_{jk}y_h - g_{jh}y_k)\\ &+\frac{1}{4}c_{ml}(W_{jk}y_h - W_{jh}y_k) + \frac{1}{4}\gamma_m\mathfrak{B}_l(W_{jk}y_h - W_{jh}y_k)\\ &-\frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(g_{jk}H_h - y_k R_{jh}) + \frac{1}{3}a_{ml}(g_{jk}H_h - y_k R_{jh}).\end{aligned} \tag{3.66}$$

This shows that

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(C_{jkh|r}y^r) &= a_{ml}(C_{jkh|r}y^r) + b_{ml}(g_{jk}y_h - g_{jh}y_k)\\ &+\frac{1}{4}c_{ml}(W_{jk}y_h - W_{jh}y_k) + \frac{1}{4}\gamma_m\mathfrak{B}_l(W_{jk}y_h - W_{jh}y_k).\end{aligned} \tag{3.67}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(g_{jk}H_h - y_k R_{jh}) = a_{ml}(g_{jk}H_h - y_k R_{jh}). \tag{3.68}$$

Hence, the proof is ended, and we obtain,

Theorem 3.12. *In $G^{2nd}\mathfrak{B}W$-BRF_n. The $\mathfrak{B}$ C-derivative of the 2-orders for the associate tensor $(C^i_{kh|r}y^r)$ and (h)hv-t-tensor $(C_{jkh|r}y^r)$ are Gbi-recurrent F-spaces if and only if* (3.65)*, and* (3.68) *are hold good, respectively.*

If we contract the i and h in the items (3.63) with using (2.2), [(2.1)a], [(2.4)e], [(2.1)b], [(2.13)c], [(2.11)c] and [(2.11)b], we have,

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(C_{k|r}y^r) &= a_{ml}(C_{k|r}y^r) + (1-n)b_{ml}y_k\\ &+\frac{1}{3}a_{ml}(H_k - y_k R) - \frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(H_k - y_k R).\end{aligned} \tag{3.69}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(C_{k|r}y^r) = a_{ml}(C_{k|r}y^r) + (1-n)b_{ml}y_k. \tag{3.70}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(H_k - y_k R) = a_{ml}(H_k - y_k R). \tag{3.71}$$

Transvecting (3.63) by g^{kh}, using [(2.4)c], [(2.4)d], [(2.11)c], and [(2.11)e], we get

$$\mathfrak{B}_l\mathfrak{B}_m(C^i_{|r}y^r) = a_{ml}(C^i_{|r}y^r) + \frac{1}{3}a_{ml}(H^i - R^i) \tag{3.72}$$
$$-\frac{1}{3}\mathfrak{B}_l\mathfrak{B}_m(H^i - R^i).$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(C^i_{|r}y^r) = a_{ml}(C^i_{|r}y^r). \tag{3.73}$$

If and only if

$$\mathfrak{B}_l\mathfrak{B}_m(H^i - R^i) = a_{ml}(H^i - R^i). \tag{3.74}$$

Thus, we get,

Theorem 3.13. *In G^{2nd} $\mathfrak{B}W - BRF_n$,the tensor $(C_{k|r}y^r)$ and the tensor $(C^i_{|r}y^r)$ are given in* (3.70) *and* (3.73) *if and only if the items* (3.71)*and* (3.74) *are hold good, respectively.*

4. Conclusions and recommendations

In this paper, we have presented a detailed study of the decomposition of C-tensors in F-spaces using higher-order derivatives of Berwald and Cartan connections. Our analysis has revealed new properties and relationships between the components of the decomposed tensors. These discoveries advance our knowledge of the geometric structure of F-spaces and could have ramifications for a number of different applications.

The item condition (2.18) is satisfied for a generalized $\mathfrak{B}W$-birecurrent space in F-space. In $G^{2nd}\mathfrak{B}W$-BRF_n, the condition of being necessary and sufficient for the Cartan's 2^{th} tensor P^i_{jkh} is a Gbi-recurrent if the item (3.6) holds and the tensor P^i_{kh} is a Gbi-recurrent if and only if the equation (3.9) is hold. In $G^{2nd}\mathfrak{B}W$-BRF_n, Ricci tensor P_{jk} is a Gbi-recurrent if the equation (3.22) holds. In $G^{2nd}\mathfrak{B}W$-BRF_n, the associate C-tensor P_{jrkh} is a Gbi-recurrent if the condition (3.34) holds good and the associative C-tensor P_{jkh} is a Gbi-recurrent if the condition (3.37) holds. In $G^{2nd}\mathfrak{B}W$-BRF_n, Ricci tensor $(P_{rj} - P_{jr})$ is a Gbi-recurrent if the equation (3.40) holds.

The Weyl's projective C-tensor W^i_{jkh}, the tensors P^i_{jkh}, and S^i_{jkh} have the same connection in $G^{2nd}\mathfrak{B}W$-BRF_n. In $G^{2nd}\mathfrak{B}W$-BRF_n, the tensor $(S^i_{jkh|r}y^r)$ is a Gbi-recurrent if and only if the tensors $(\delta^i_k R_{jh} - g_{jk}R^i_h)$ is a Gbi-recurrent F-space. In $G^{2nd}\mathfrak{B}W$-BRF_n, $\mathfrak{B}$ C-derivatives of the 2-orders for S-Ricci tensor $(S_{jk|r}y^r)$ is Gbi-recurrent if and only if the tensors $(R_{jk} - g_{jk}R)$ is a Gbi-recurrent F-space. In $G^{2nd}\mathfrak{B}W$-BRF_n, the tensor $(C^i_{kh|r}y^r)$ is Gbi-recurrent if and only if the tensors $(\delta^i_k H_h - y_k R^i_h)$ is a Gbi-recurrent Finsler space. In $G^{2nd}\mathfrak{B}W$-BRF_n, the tensor $(C_{jkh|r}y^r)$ is Gbi-recurrent iff the tensors $(g_{jk}H_h - y_k R_{jh})$ is a Gbi-recurrent F-space.

Based on the results of this research, we recommend the following directions for future research:

- Explore other types of decompositions: Investigate different decomposition schemes and their corresponding geometric interpretations.

- Investigate the physical implications: Explore the physical implications of the decomposition results, particularly in the context of field theories and cosmology.
- Develop numerical methods: Develop numerical methods for computing the decomposed tensors and analyzing their properties.

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