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# New Properties of Generalized Fusion Frames in Hilbert $C^*$-Modules

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**Abstract.** In this paper, we provide some generalizations of the concept of fusion frames following that evaluate their representability via a linear operator in Hilbert $C^*$-modules. We assume that $\Upsilon_\xi$ is self-adjoint and $\Upsilon_\xi(\mathfrak{R}_\xi) = \mathfrak{R}_\xi$ for all $\xi \in \mathfrak{S}$, and show that if a $g-$fusion frame $\{(\mathfrak{R}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ is represented via a linear operator $\mathcal{T}$ on $\operatorname{span}\{\mathfrak{R}_\xi\}_{\xi\in\mathfrak{S}}$, then $\mathcal{T}$ is bounded. Moreover, if $\{(\mathfrak{R}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ is a tight $g-$fusion frame, then $\Upsilon_\xi$ is not represented via an invertible linear operator on $\operatorname{span}\{\mathfrak{R}_\xi\}_{\xi\in\mathfrak{S}}$, We show that, under certain conditions, a linear operator may also be used to express the perturbation of representable fusion frames. Finally, we investigate the stability of this type of fusion frames.

## 1. Introduction and Preliminaries

The concept of frames has become an active and ongoing area of research in mathematics, as well as in fields such as signal processing and computer science. The foundational work on frames for Hilbert spaces was laid out in 1952 by Duffin and Schaeffer, particularly for the analysis of nonharmonic Fourier series, as cited in [3]. This field gained renewed momentum in 1986 through the significant contributions of Daubechies, Grossmann and Meyer [2], leading to a broader recognition and wide range of applications.

In more recent developments, numerous mathematicians have expanded the theory of frames beyond Hilbert spaces to encompass Hilbert $C^*$-modules. In particular, A. Khosravi and B. Khosravi have been instrumental in introducing and developing the notions of fusion frames and $g$-frames in Hilbert $C^*$-modules [14]. For more detailed information, readers are recommended to consult [4–12, 15–28].

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This paper aims to explore $g$-fusion frames, denoted by $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathbb{Z}}$, through the lens of (possibly unbounded) linear operators.

Throughout the paper, we let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{H}$ a countably generated Hilbert $\mathcal{A}$-module. We denote by $\{\mathcal{K}_\xi\}_{\xi\in\mathbb{Z}}$ a family of Hilbert $\mathcal{A}$-modules, and by $\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}$ a sequence of closed orthogonally complemented submodules of $\mathcal{H}$. For each $\xi \in \mathbb{Z}$, we denote by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi)$ the set of all adjointable $\mathcal{A}$-linear mappings from $\mathcal{H}$ into $\mathcal{K}_\xi$, and we write $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}) := \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$. The orthogonal projection onto the closed submodule $\mathfrak{N}_\xi \subset \mathcal{H}$ (which is orthogonally complemented in $\mathcal{H}$) is denoted by $\mathcal{P}_{\mathfrak{N}_\xi}$. For a given operator $\mathcal{T}$, we denote by $\mathcal{N}(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$ its kernel and range, respectively.

Let $\Upsilon_0 \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ be such that

$$\Upsilon_0(\mathcal{H}) \subseteq \mathrm{span}\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}},$$

and consider the Hilbert $\mathcal{A}$-module

$$\ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}\Big) := \left\{\{f_\xi\}_{\xi\in\mathbb{Z}} : f_\xi \in \mathfrak{N}_\xi,\ \Big\|\sum_{\xi\in\mathbb{Z}}\langle f_\xi, f_\xi\rangle\Big\| < \infty\right\}.$$

For $f = \{f_\xi\}_{\xi\in\mathbb{Z}}$ and $g = \{g_\xi\}_{\xi\in\mathbb{Z}}$ in this module, the inner product is given by

$$\langle f, g\rangle = \sum_{\xi\in\mathbb{Z}}\langle f_\xi, g_\xi\rangle,$$

which makes $\ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}\Big)$ a Hilbert $\mathcal{A}$-module.

We define the (bilateral) shift operator $\mathcal{T}$ on $\ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}\Big)$ by

$$\mathcal{T}\Big(\{f_\xi\}_{\xi\in\mathbb{Z}}\Big) = \{f_{\xi-1}\}_{\xi\in\mathbb{Z}}, \qquad \{f_\xi\}_{\xi\in\mathbb{Z}} \in \ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}\Big).$$

In later sections, we relate suitable families $\{\Upsilon_\xi\}_{\xi\in Z}$ to $\Upsilon_0$ and $T$ via iteration. . More precisely, in the sequel we will often consider families satisfying $\Upsilon_\xi = \mathcal{T}^\xi \Upsilon_0$.

We now recall some standard notions concerning C*-algebras and Hilbert $\mathcal{A}$-modules.

**Definition 1.1.** *[1] Let $\mathcal{A}$ be a Banach algebra. An* involution *on $\mathcal{A}$ is a map $u \mapsto u^*$ from $\mathcal{A}$ to itself such that, for all $u, v \in \mathcal{A}$ and $\alpha \in \mathbb{C}$,*

(1) $(u^*)^* = u;$
(2) $(uv)^* = v^*u^*;$
(3) $(\alpha u + v)^* = \bar{\alpha}\, u^* + v^*.$

**Definition 1.2.** *[1] A* C*-algebra *is a Banach algebra $\mathcal{A}$ endowed with an involution $u \mapsto u^*$ such that*

$$\|u^*u\| = \|u\|^2, \qquad u \in \mathcal{A}.$$

**Definition 1.3.** *[13] Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{H}$ a left $\mathcal{A}$-module with compatible linear structures. We say that $\mathcal{H}$ is a* pre-Hilbert $\mathcal{A}$-module *if it is equipped with an $\mathcal{A}$-valued inner product*

$$\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathcal{A}$$

*which is sesquilinear, positive definite and compatible with the module structure, that is:*

(i) $\langle u, u\rangle \geq 0$ *for all* $u \in \mathcal{H}$*, and* $\langle u, u\rangle = 0$ *if and only if* $u = 0$;

(ii) $\langle \eta u + v, w\rangle = \eta\langle u, w\rangle + \langle v, w\rangle$ *for all* $\eta \in \mathcal{A}$ *and* $u, v, w \in \mathcal{H}$;

(iii) $\langle u, v\rangle = \langle v, u\rangle^*$ *for all* $u, v \in \mathcal{H}$.

*For* $u \in \mathcal{H}$ *we define* $\|u\| := \|\langle u, u\rangle\|^{1/2}$. *If* $\mathcal{H}$ *is complete with respect to this norm, then* $\mathcal{H}$ *is called a* Hilbert $\mathcal{A}$-module *(or Hilbert* $C^*$*-module over* $\mathcal{A}$*). For each* $\eta \in \mathcal{A}$ *we also set* $|\eta| := (\eta^*\eta)^{1/2}$.

In the next definition the family $\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}$ consists of closed $\mathcal{A}$-submodules of $\mathcal{H}$, in the sense of [14].

**Definition 1.4.** *[14] A family* $\{\mathfrak{N}_\xi\}_{\xi\in\mathbb{Z}}$ *of closed submodules of* $\mathcal{H}$ *is called a* fusion frame *for* $\mathcal{H}$ *if there exist constants* $0 < A \leq B < \infty$ *and a family of positive weights* $\{v_\xi\}_{\xi\in\mathbb{Z}}$ *such that, for every* $f \in \mathcal{H}$,

$$A\,\langle f, f\rangle \;\leq\; \sum_{\xi\in\mathbb{Z}} v_\xi^2\,\langle \mathcal{P}_{\mathfrak{N}_\xi}(f), \mathcal{P}_{\mathfrak{N}_\xi}(f)\rangle \;\leq\; B\,\langle f, f\rangle.$$

Now let $\mathcal{H}$ and $\mathcal{K}_\xi$ $(\xi \in \mathfrak{S})$ be Hilbert $\mathcal{A}$-modules and let $\mathfrak{S}$ be a subset of $\mathbb{Z}$.

**Definition 1.5.** *[14] Let* $\mathcal{A}$ *be a unital* $C^*$*-algebra and let* $\mathcal{H}$ *be a Hilbert* $\mathcal{A}$*-module. Let* $\{\mathcal{K}_\xi\}_{\xi\in\mathfrak{S}}$ *be a family of Hilbert* $\mathcal{A}$*-modules. A family of adjointable operators*

$$\{\Lambda_\xi\}_{\xi\in\mathfrak{S}} \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi)$$

*is called a* $g$-frame *for* $\mathcal{H}$ *(with respect to* $\{\mathcal{K}_\xi\}_{\xi\in\mathfrak{S}}$*) if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A\,\langle f, f\rangle \;\leq\; \sum_{\xi\in\mathfrak{S}} \langle \Lambda_\xi f, \Lambda_\xi f\rangle \;\leq\; B\,\langle f, f\rangle, \qquad \text{for all } f \in \mathcal{H}.$$

Fusion frames arise as a particular case of $g$-frames: indeed, taking $\mathcal{K}_\xi = \mathcal{H}$ and

$$\Lambda_\xi = v_\xi \mathcal{P}_{\mathfrak{N}_\xi}, \qquad \xi \in \mathbb{Z},$$

we recover the above notion of fusion frame. More generally, any family of operators whose ranges are contained in $\mathfrak{N}_\xi$ for $\xi \in \mathfrak{S}$ may be used in place of the orthogonal projections to provide an extension of the idea of fusion frame.

Motivated by the notion of $g$-frames in Hilbert $C^*$-modules, we now focus on the particular situation where the range of each operator is contained in a closed orthogonally complemented submodule. This leads to the notion of $g$-fusion frames.

**Definition 1.6.** *Let* $\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$ *be a family of closed orthogonally complemented submodules of* $\mathcal{H}$*, and let* $\Upsilon_\xi \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathfrak{N}_\xi)$ *for each* $\xi \in \mathfrak{S}$*. The family*

$$\left\{(\mathfrak{N}_\xi, \Upsilon_\xi)\right\}_{\xi\in\mathfrak{S}}$$

*is called a* $g$-fusion frame *for* $\mathcal{H}$ *(with respect to* $\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$*) if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A\,\langle f, f\rangle \;\leq\; \sum_{\xi\in\mathfrak{S}} \langle \Upsilon_\xi(f), \Upsilon_\xi(f)\rangle \;\leq\; B\,\langle f, f\rangle, \qquad f \in \mathcal{H}.$$

*If* $A = B$*, the g-fusion frame is called* tight.

When $\mathfrak{N}_\xi = \mathcal{H}$ for all $\xi \in \mathfrak{S}$, the above definition reduces to the classical notion of $g$-frames.

The *synthesis operator*

$$\mathcal{U} : \ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}\Big) \longrightarrow \mathcal{H}$$

associated with a $g$-fusion frame $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ is defined by

$$\mathcal{U}\Big(\{f_\xi\}_{\xi\in\mathfrak{S}}\Big) = \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^*(f_\xi).$$

The corresponding *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is given by

$$Sf = \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^* \Upsilon_\xi f, \qquad f \in \mathcal{H}.$$

A $g$-fusion frame $\{(\mathfrak{N}_\xi, \Gamma_\xi)\}_{\xi\in\mathfrak{S}}$ is called a *dual g-fusion frame* of $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ if

$$f = \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^* \Gamma_\xi f, \qquad f \in \mathcal{H}.$$

In particular, the family $\{(\mathfrak{N}_\xi, \Upsilon_\xi S^{-1})\}_{\xi\in\mathfrak{S}}$ is a dual $g$-fusion frame of $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$, called the *canonical dual g-fusion frame*. Indeed, for every $f \in \mathcal{H}$ we have

$$f = S^{-1}Sf = S^{-1} \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^* \Upsilon_\xi f = \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^* \Upsilon_\xi S^{-1} f.$$

Finally, let $\mathcal{T}$ be a linear operator on $\operatorname{span}\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$ and let $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}} \subset \operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ be such that $\mathcal{R}(\Upsilon_\xi) \subseteq \mathfrak{N}_\xi$ for each $\xi \in \mathfrak{S}$.

**Definition 1.7.** *We say that the family* $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *is* represented via $\mathcal{T}$ *if*

$$\mathcal{T}\Upsilon_\xi = \Upsilon_{\xi+1}, \qquad \xi \in \mathfrak{S}.$$

**Definition 1.8.** *A g-fusion frame* $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ *is said to be* associated with *a linear operator* $\mathcal{T}$ *if the family* $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *is represented via* $\mathcal{T}$ *in the above sense.*

## 2. On the Boundedness of the Operator $\mathcal{T}$

In the following section, let $\Upsilon_\xi$ be a sequence in $\operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ such that for every $\xi \in \mathfrak{S}$, $\mathcal{R}(\Upsilon_\xi)$ is contained in $\mathfrak{N}_\xi$.

**Theorem 2.1.** *Assume that for each* $\xi \in \mathfrak{S}$, $\Upsilon_\xi(\mathfrak{N}_\xi) = \mathfrak{N}_\xi$ *and* $\Upsilon_\xi \in \operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ *is self-adjoint. If* $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ *constitutes a g-fusion frame represented via a linear operator* $\mathcal{T}$ *acting on* $\operatorname{span}\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$*, then* $\mathcal{T}$ *is bounded and*

$$1 \leq \|\mathcal{T}\| \leq \sqrt{\frac{B}{A}},$$

*where A and B are the frame bounds. Moreover, if L denotes the shift operator on* $\ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}\Big)$ *then* $\mathcal{N}(\mathcal{U})$ *is invariant under L, i.e.* $L(\mathcal{N}(\mathcal{U})) \subset \mathcal{N}(\mathcal{U})$.

*Proof.* If $\{g_\xi\}_{\xi\in\mathfrak{S}}$ is a sequence in $\mathcal{N}(\mathcal{U})^\perp$ such that $g_\xi = 0$ except for finitely many $\xi$, then

$$\begin{aligned}\langle \mathcal{T}\mathcal{U}(\{g_\xi\}), \mathcal{T}\mathcal{U}(\{g_\xi\})\rangle &\leq B\sum_{\xi\in\mathfrak{S}}\langle g_\xi, g_\xi\rangle \\ &\leq \frac{B}{A}\langle \mathcal{U}(\{g_\xi\}), \mathcal{U}(\{g_\xi\})\rangle.\end{aligned}$$

This implies that for every $\{h_\xi\}_{\xi\in\mathfrak{S}} \in \mathcal{N}(\mathcal{U})$,

$$\langle \mathcal{T}\mathcal{U}(\{g_\xi\}+\{h_\xi\}), \mathcal{T}\mathcal{U}(\{g_\xi\}+\{h_\xi\})\rangle \leq \frac{B}{A}\|\mathcal{U}(\{g_\xi\}+\{h_\xi\})\|^2.$$

Since $\Upsilon_\xi(N_\xi) = N_\xi$ and $\Upsilon_\xi$ is self-adjoint, we have $N_\xi \subset \mathcal{R}(\Upsilon_\xi^*) \subset \mathcal{R}(\mathcal{U})$, which implies span$\{\mathfrak{N}_\xi\} \subset \mathcal{R}(\mathcal{U})$.

Since span$\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}} \subseteq \mathcal{R}(\mathcal{U})$, we conclude that for all $f \in$ span$\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$,

$$\langle \mathcal{T}(f), \mathcal{T}(f)\rangle \leq \frac{B}{A}\langle f, f\rangle.$$

Therefore, $\mathcal{T}$ is bounded and $\|\mathcal{T}\| \leq \sqrt{\frac{B}{A}}$.

Since $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ is a $g$-fusion frame represented via $\mathcal{T}$, there exists $f_0 \in \mathcal{H}$ such that $\sum_{\xi\in\mathfrak{S}}\langle \Upsilon_\xi(f_0), \Upsilon_\xi(f_0)\rangle \neq 0$. On the other hand,

$$\begin{aligned}\sum_{\xi\in\mathfrak{S}}\langle \Upsilon_\xi(f_0), \Upsilon_\xi(f_0)\rangle &= \sum_{\xi\in\mathfrak{S}}\langle \Upsilon_{\xi+1}(f_0), \Upsilon_{\xi+1}(f_0)\rangle \\ &= \sum_{\xi\in\mathfrak{S}}\langle \mathcal{T}\Upsilon_\xi(f_0), \mathcal{T}\Upsilon_\xi(f_0)\rangle \\ &\leq \|\mathcal{T}\|^2\sum_{\xi\in\mathfrak{S}}\langle \Upsilon_\xi(f_0), \Upsilon_\xi(f_0)\rangle.\end{aligned}$$

This shows that $\|\mathcal{T}\| \geq 1$.

To finish the proof, let $\{f_\xi\}_{\xi\in\mathfrak{S}} \in \mathcal{N}(\mathcal{U})$. Then

$$\mathcal{U}(\{f_\xi\}) = \sum_{\xi\in\mathfrak{S}}\Upsilon_\xi^*(f_\xi) = 0.$$

Define the shift operator $L$ on $\ell^2\Big(\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}\Big)$ by $L(\{f_\xi\}) = \{f_{\xi-1}\}$. Then

$$\begin{aligned}\mathcal{U}(L(\{f_\xi\})) &= \sum_{\xi\in\mathfrak{S}}\Upsilon_\xi^*(f_{\xi-1}) = \sum_{\xi\in\mathfrak{S}}\Upsilon_{\xi+1}^*(f_\xi) \\ &= \sum_{\xi\in\mathfrak{S}}\tilde{\mathcal{T}}\Upsilon_\xi^*(f_\xi) = \tilde{\mathcal{T}}\sum_{\xi\in\mathfrak{S}}\Upsilon_\xi^*(f_\xi) = \tilde{\mathcal{T}}\mathcal{U}(\{f_\xi\}) = 0.\end{aligned}$$

Hence $L(\{f_\xi\}) \in \mathcal{N}(\mathcal{U})$, and therefore $\mathcal{N}(\mathcal{U})$ is invariant under $L$. □

**Theorem 2.2.** *Let $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ be a tight $g-$fusion frame such that $\Upsilon_\xi$ is self-adjoint and $\Upsilon_\xi(\mathfrak{N}_\xi) = \mathfrak{N}_\xi$ for all $\xi \in \mathfrak{S}$. Then $\Upsilon_\xi$ is not represented via an invertible linear operator $\mathcal{T}$ on span$\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$.*

*Proof.* suppose by the absurd that $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ be a $g-$fusion frame represented via an invertible linear operator $\mathcal{T}$ on span$\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$.

So $\mathcal{T}^{\xi}\Upsilon_0 = \Upsilon_\xi$ and so $\Upsilon_{-\xi} = \mathcal{T}^{-\xi}\Upsilon_0$ for all $\xi \in \mathfrak{S}$. Similarly, replacing $\mathcal{T}$ by $\mathcal{T}^{-1}$ in the proof of Theorem 2.1, we get

$$1 \leq \|\mathcal{T}^m\| \leq \sqrt{\frac{B}{A}}, \tag{2.1}$$

where $A$ and $B$ are frame bounds of $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ and $m \in \{-1, 1\}$.

Since $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ is a tight $g$−fusion frame, then $A = B$. So by 2.1

$$\|\mathcal{T}\| = \|\mathcal{T}^{-1}\| = 1.$$

Therefore for every $f \in \mathcal{H}$,

$$\begin{aligned}\|\langle f, f\rangle\|^{\frac{1}{2}} &= \|\langle \mathcal{T}^{-1}\mathcal{T}f, \mathcal{T}^{-1}\mathcal{T}f\rangle\|^{\frac{1}{2}} \\ &\leq \|\langle Tf, Tf\rangle\|^{\frac{1}{2}} \\ &\leq \|\langle f, f\rangle\|^{\frac{1}{2}}.\end{aligned}$$

This shows that $\mathcal{T}$ is an isometry. Thus for every $f \in \mathcal{H}$ and $\xi \in \mathfrak{S}$,

$$\|\langle \Upsilon_\xi f, \Upsilon_\xi f\rangle\|^{\frac{1}{2}} = \|\langle \mathcal{T}^{\xi}\Upsilon_0 f, \mathcal{T}^{\xi}\Upsilon_0 f\rangle\|^{\frac{1}{2}} = \|\langle \Upsilon_0 f, \Upsilon_0 f\rangle\|^{\frac{1}{2}}.$$

Hence

$$\sum_{\xi\in\mathfrak{S}} \langle \Upsilon_0 f, \Upsilon_0 f\rangle = \sum_{\xi\in\mathfrak{S}} \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \leq B\langle f, f\rangle.$$

It follows that $\sum_{\xi\in\mathfrak{S}} \langle \Upsilon_0 f, \Upsilon_0 f\rangle$ is a convergent series. So

$$\|\langle \Upsilon_\xi f, \Upsilon_\xi f\rangle\|^{\frac{1}{2}} = \|\langle \Upsilon_0 f, \Upsilon_0 f\rangle\|^{\frac{1}{2}} = 0$$

for all $f \in \mathcal{H}$ and $\xi \in \mathfrak{S}$. Therefore for every $f \in \mathcal{H}$ we have

$$f = \sum_{\xi\in\mathfrak{S}} \Upsilon_\xi^* \Gamma_\xi f = 0,$$

where $\{(\mathfrak{N}_\xi, \Gamma_\xi)\}_{\xi\in\mathfrak{S}}$ is a dual $g$−fusion frame of $\{(\mathfrak{N}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$, a contradiction. □

## 3. Stability and Linear Independence

**Lemma 3.1.** *Let* $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *be a sequence in* $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ *with* $\mathcal{R}(\Upsilon_\xi) \subset \mathfrak{N}_\xi$ *for all* $\xi \in \mathfrak{S}$. *If* $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *is represented via a linear operator* $\mathcal{T}$ *on span*$\{\mathfrak{N}_\xi\}_{\xi\in\mathfrak{S}}$ *and span*$\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *is infinite dimensional, then* $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ *is linearly independent and infinite.*

*Proof.* Assume that $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ is linearly dependent. Hence there exist constants $\delta_p, \ldots, \delta_q$ such that

$$\sum_{\xi=p}^{q} \delta_\xi \Upsilon_\xi = 0$$

and $\delta_{\xi_0} \neq 0$, for some $p \leq \xi_0 \leq q$. Set

$$\alpha = \min\left\{\xi : \ p \leq \xi \leq q, \delta_\xi \neq 0\right\} \ \text{ and } \ b = \max\left\{\xi : \ p \leq \xi \leq q, \delta_\xi \neq 0\right\}.$$

Then

$$\Upsilon_b = \sum_{\xi=\alpha}^{b-1} d_\xi \Upsilon_\xi \qquad \text{and} \qquad \Upsilon_\alpha = \sum_{\xi=\alpha+1}^{b} d'_\xi \Upsilon_\xi$$

for certain constants $\alpha_\xi$ and $\alpha'_\xi$, $\xi = \alpha, \ldots, b$. Thus for every $i \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{T}^i \Upsilon_b &= \sum_{\xi=\alpha}^{b-1} d_\xi \mathcal{T}^i \Upsilon_\xi \\ &= \sum_{\xi=\alpha}^{b-1} d_\xi \Upsilon_{\xi+i} \\ &= \sum_{\xi=\alpha+i}^{b+i-1} d_{\xi-i} \Upsilon_\xi. \end{aligned}$$

Similarly,

$$\mathcal{T}^{-i} \Upsilon_b = \sum_{\xi=\alpha-i}^{b-1-i} d'_{\xi-i} \Upsilon_\xi.$$

Therefore, in addition to $\Upsilon_\xi = \mathcal{T}^\xi \Upsilon_0$ show that $\text{span}\left\{\Upsilon_\xi\right\}_{\alpha,\ldots,b}$ is invariant under $\mathcal{T}$ and $\mathcal{T}^{-1}$. Also, we have

$$\text{span}\left\{\Upsilon_\xi\right\}_{\xi\in\mathfrak{S}} = \text{span}\left\{\Upsilon_\xi\right\}_{\alpha,\ldots,b}.$$

Hence, $\text{span}\left\{\Upsilon_\xi\right\}_{\xi\in\mathfrak{S}}$ is finite dimensional, a contradiction. □

As an obvious consequence of Lemma 3.1, we get the following results.

**Proposition 3.1.** *Let $\{(\mathfrak{R}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ be a $g-$fusion frame such that $\Upsilon_\xi$ is self-adjoint and $\Upsilon_\xi(\mathfrak{R}_\xi) = \mathfrak{R}_\xi$. Assume that $\mathcal{T}$ is a linear operator on span$\{\mathfrak{R}_\xi\}_{\xi\in\mathfrak{S}}$ such that it has an extension to a bounded linear operator $\tilde{\mathcal{T}} : \mathcal{H} \to \mathcal{H}$ with $\tilde{\mathcal{T}} \Upsilon^*_\xi = \Upsilon^*_{\xi+1}$. Let there exist $n_0 \in \mathfrak{S}$ such that $\Upsilon_{n_0} : \mathcal{H} \to \mathfrak{R}_{n_0}$ is surjective.*

*If $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ is represented via $\mathcal{T}$, then $\{\Upsilon_\xi\}_{\xi\in\mathfrak{S}}$ is linearly dependent.*

*Proof.* Suppose that $\{(\mathfrak{R}_\xi, \Gamma_\xi)\}_{\xi\in\mathfrak{S}}$ be the dual $g-$fusion frame of $\{(\mathfrak{R}_\xi, \Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$.

Thus every $f \in \mathcal{H}$,

$$f = \sum_{\xi\in\mathfrak{S}} \Upsilon^*_\xi \Gamma_\xi(f).$$

So $\tilde{\mathcal{T}}(f) = \sum_{\xi\in\mathfrak{S}} \Upsilon^*_{\xi+1} \Gamma_\xi(f)$. This implies that

$$\mathcal{T}(\Upsilon_j(f)) = \sum_{\xi\in\mathfrak{S}} \Upsilon^*_{\xi+1} \Gamma_\xi \Upsilon_j(f).$$

For every $f \in \mathcal{H}$ we have

$$\Upsilon_{j+1}(f) = \sum_{\xi\in\mathfrak{S}} \Upsilon^*_{\xi+1} \Gamma_\xi \Upsilon_j(f). \tag{3.1}$$

Choose $\theta, \theta' \in \mathfrak{S}$ such that $\theta \neq \theta'$ and $n_0 \notin \{\theta-1, \theta, \theta'-1, \theta'\}$. Let $\{\tilde{Y}_\xi\}_{\xi\in\mathfrak{S}}$ denote the sequence consisting of the same elements as $Y_\xi$, but with $Y_\theta$ and $Y_{\theta'}$ interchanged. Clearly, that

$$\widetilde{\Gamma}_\xi = \begin{cases} \Gamma_\xi, & \xi \neq \theta, \theta', \\ \Gamma_{\theta'}, & \xi = \theta, \\ \Gamma_\theta, & \xi = \theta', \end{cases}$$

is the canonical dual $g$−fusion frame $\{(\mathfrak{R}_\xi, \widetilde{Y}_\xi)\}_{\xi\in\mathfrak{S}}$. From 3.1 we obtain

$$Y_{j_0+1} = \sum_{\xi\in\mathfrak{S}} Y_{\xi+1} Y_\xi S^{-1} Y_{j_0} \tag{3.2}$$

Hence

$$\begin{aligned} Y_{j_0+1} &= Y_\theta Y_{\theta-1} S^{-1} Y_{j_0} + Y_{\theta+1} Y_\theta S^{-1} Y_{j_0} + Y_{\theta'} Y_{\theta'-1} S^{-1} Y_{j_0} + Y_{\theta'+1} Y_{\theta'} S^{-1} Y_{j_0} \\ &+ \sum_{\xi\notin\{\theta-1,\theta,\theta'-1,\theta'\}} Y_{\xi+1} Y_\xi S^{-1} Y_{j_0} \end{aligned}$$

Then we have

$$\begin{aligned} Y_{j_0+1} &= \widetilde{Y}_{j_0+1} \\ &= \sum_{\xi\in\mathfrak{S}} \widetilde{Y}_{\xi+1} \widetilde{\Gamma}_\xi \widetilde{Y}_{j_0}. \end{aligned}$$

So,

$$\begin{aligned} Y_{j_0+1} &= Y_{\theta'} Y_{\theta-1} S^{-1} Y_{j_0} + Y_{\theta+1} Y_{\theta'} S^{-1} Y_{j_0} + Y_\theta Y_{\theta'-1} S^{-1} Y_{j_0} + Y_{\theta'+1} Y_\theta S^{-1} Y_{j_0} \\ &+ \sum_{\xi\notin\{\theta-1,\theta,\theta'-1,\theta'\}} Y_{\xi+1} Y_\xi S^{-1} Y_{j_0}. \end{aligned}$$

Applying 3.2, we obtain

$$\begin{aligned} Y_\theta Y_{\theta-1} S^{-1} Y_{j_0} &+ Y_{\theta+1} Y_\theta S^{-1} Y_{j_0} + Y_{\theta'} Y_{\theta'-1} S^{-1} Y_{j_0} + Y_{\theta'+1} Y_{\theta'} S^{-1} Y_{j_0} \\ &= Y_{\theta'} Y_{\theta-1} S^{-1} Y_{j_0} + Y_{\theta+1} Y_{\theta'} S^{-1} Y_{j_0} + Y_\theta Y_{\theta'-1} S^{-1} Y_{j_0} + Y_{\theta'+1} Y_\theta S^{-1} Y_{j_0}. \end{aligned}$$

Now, we assume that $\theta > \theta'$ and $\theta' = \theta + q$, for some $q \in \mathbb{N}$. So

$$\begin{aligned} 0 &= Y_\theta Y_{\theta-1} S^{-1} Y_{j_0} + Y_{\theta+1} Y_\theta S^{-1} Y_{j_0} + Y_{\theta+q} Y_{\theta+(q-1)} S^{-1} Y_{j_0} + Y_{\theta+(q+1)} Y_{\theta+q} S^{-1} Y_{j_0} \\ &- Y_{\theta+q} Y_{\theta-1} S^{-1} Y_{j_0} - Y_{\theta+1} Y_{\theta+q} S^{-1} Y_{j_0} - Y_\theta Y_{\theta+(q-1)} S^{-1} Y_{j_0} - Y_{\theta+(q+1)} Y_\theta S^{-1} Y_{j_0}. \end{aligned}$$

Hence

$$\begin{aligned} Y_{2\theta-1} S^{-1} Y_{j_0} &+ Y_{2\theta+1} S^{-1} Y_{j_0} + Y_{2(\theta+q)-1} S^{-1} Y_{j_0} \\ &+ Y_{2(\theta+q)+1} S^{-1} Y_{j_0} - 2Y_{2\theta+(q-1)} S^{-1} Y_{j_0} \\ &- 2Y_{2\theta+(q+1)} S^{-1} Y_{j_0} = 0. \end{aligned}$$

We choose $j_0 = n_0$, where $Y_{n_0}$ is surjective.

Since $\Upsilon_{j_0}$ and $S^{-1}$ are surjective, we have

$$\begin{aligned} \Upsilon_{2\theta-1} &+ \Upsilon_{2\theta+1} + \Upsilon_{2(\theta+q)-1} \\ &+ \Upsilon_{2(\theta+q)+1} - 2\Upsilon_{2\theta+(q-1)} \\ &- 2\Upsilon_{2\theta+(q+1)} = 0, \end{aligned}$$

so $\Upsilon_\xi$ is linearly dependent. □

**Theorem 3.1.** *Consider the sequence $\{\widehat{\Upsilon}_\xi\}_{\xi\in\mathfrak{S}}$ within $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, where for each $\xi\in\mathfrak{S}$, $\widehat{\Upsilon}_\xi(\mathcal{H})\subseteq\mathfrak{R}_\xi$. Let $\{(\mathfrak{R}_\xi,\Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$ represent a $g$–fusion frame via a linear operator $\mathcal{T}$ acting on span$\{\mathfrak{R}_\xi\}_{\xi\in\mathfrak{S}}$. Assume the existence of constants $\eta,\beta\in[0,1)$, fulfilling the inequality*

$$\left\|\sum_{\xi=1}^{n}\alpha_\xi\Big(\Upsilon_\xi-\widehat{\Upsilon}_\xi\Big)(f)\right\|\leq\eta\left\|\sum_{\xi=1}^{n}\alpha_\xi\Upsilon_\xi(f)\right\|+\beta\left\|\sum_{\xi=1}^{n}\alpha_\xi\widehat{\Upsilon}_\xi(f)\right\| \tag{3.3}$$

*for any $f\in\mathcal{H}$ and all finite complex sequences $\{\alpha_\xi\}_{\xi=1}^{n}$. Under these conditions, the subsequent statements are valid:*

*(i)*

$$\left(\frac{1-\eta}{1+\beta}\sqrt{A}\right)^2\langle f,f\rangle\leq\sum_{\xi\in\mathfrak{S}}\langle\widehat{\Upsilon}_\xi(f),\Upsilon_\xi(f)\rangle\leq\left(\frac{1+\eta}{1-\beta}\sqrt{B}\right)^2\langle f,f\rangle,$$

*where $A$ and $B$ are frame bounds of $\{(\mathfrak{R}_\xi,\Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$.*

*(ii) If the span of $\Upsilon_\xi$ is infinite-dimensional and $\{\widehat{\Upsilon}_\xi\}_{\xi\in\mathfrak{S}}$ forms an infinite set, then $\{\widehat{\Upsilon}_\xi\}_{\xi\in\mathfrak{S}}$ exhibits linear independence.*

*Proof.* Select $l\in\mathfrak{S}$ and define $\alpha_\xi=\varepsilon_{l\xi}$ in 3.3, where

$$\varepsilon_{l\xi}=\begin{cases}0 & \text{if } l\neq\xi,\\ 1 & \text{if } l=\xi.\end{cases}$$

Consequently, for any $f\in\mathcal{H}$

$$\|\Upsilon_l(f)-\widehat{\Upsilon}_l(f)\|\leq\eta\|\Upsilon_l(f)\|+\beta\|\widehat{\Upsilon}_l(f)\|.$$

Then

$$(1-\beta)\|\widehat{\Upsilon}_l(f)\|\leq(1+\eta)\|\Upsilon_l(f)\|.$$

Thus

$$\langle\widehat{\Upsilon}_l(f),\Upsilon_l(f)\rangle\leq\left(\frac{1+\eta}{1-\beta}\right)^2\langle\Upsilon_l(f),\Upsilon_l(f)\rangle.$$

This implies

$$\sum_{i\in\mathfrak{S}}\langle\widehat{\Upsilon}_l(f),\Upsilon_l(f)\rangle\leq\left(\frac{1+\eta}{1-\beta}\right)^2\sum_{i\in\mathfrak{S}}\langle\Upsilon_l(f),\Upsilon_l(f)\rangle\leq\left(\frac{1+\eta}{1-\beta}\right)^2B\langle f,f\rangle,$$

where $B$ is the upper frame bound of $\{(\mathfrak{R}_\xi,\Upsilon_\xi)\}_{\xi\in\mathfrak{S}}$.

Similarly, for all $i \in \Im$ and $f \in \mathcal{H}$,

$$\langle \widehat{\Upsilon}_l(f), \Upsilon_l(f) \rangle \geq \left( \frac{1-\eta}{1+\beta} \right)^2 \langle \Upsilon_l f, \Upsilon_l f \rangle.$$

Hence

$$\sum_{i \in \Im} \langle \widehat{\Upsilon}_l(f), \Upsilon_l(f) \rangle \geq \left( \frac{1-\eta}{1+\beta} \right)^2 \sum_{i \in \Im} \langle \Upsilon_l f, \Upsilon_l f \rangle.$$

In conclusion,

$$\sum_{i \in \Im} \langle \widehat{\Upsilon}_l(f), \Upsilon_l(f) \rangle \geq \left( \frac{1-\eta}{1+\beta} \right)^2 A \langle f, f \rangle.$$

That is,

$$\left( \frac{1-\eta}{1+\beta} \sqrt{A} \right)^2 \langle f, f \rangle \leq \sum_{\xi \in \Im} \langle \widehat{\Upsilon}_\xi(f), \Upsilon_\xi(f) \rangle \leq \left( \frac{1+\eta}{1-\beta} \sqrt{B} \right)^2 \langle f, f \rangle,$$

Let $\{\alpha_\xi\}_{\xi=1}^n$ be a finite sequence in $\mathbb{C}$ with $\sum_{\xi=1}^n \alpha_\xi \widehat{\Upsilon}_\xi = 0$. Given $\eta \in [0, 1)$, by 3.3

$$\sum_{\xi=1}^n \alpha_\xi \Upsilon_\xi = 0.$$

As per Lemma 3.1, $\alpha_\xi = 0$ for all $k = 1, \dots, n$. Thus, $\{\widehat{\Upsilon}_\xi\}_{\xi \in \Im}$ is linearly independent. □

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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