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Analysis of Existence, Uniqueness, Stability and Controllability in Pantograph with Caputo–Hadamard Volterra–Fredholm Fractional Integro–Differential Equations

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Abstract. This study addresses the analytical verification that a solution exists and that it is uniquely determined Boundary value problems involving the Caputo–Hadamard fractional operator that contain nonlinear Volterra–Fredholm type integrals and pantograph-type arguments under nonlocal boundary conditions, by making use of strategies that involve constructing upper and lower solutions. By converting the fractional differential equations into an equivalent integral form, a nonlinear operator is defined in a Banach space. Existence of a solution is shown through a fixed point theorem (FPT) argument, and uniqueness is obtained by applying the Banach fixed point theorem under suitable assumptions. The stability of the system is examined in the Ulam–Hyers sense, and controllability is verified using an appropriate fixed point framework. A illustrating example is provided to exhibit the practical relevance of the theoretical results.

1. Introduction

Nonlinear differential equations play a central role in the modelling of diverse real-world phenomena. Many natural and engineered systems exhibit hereditary, memory, or long-range interaction effects, and classical integer-order models often fail to capture their full complexity. This has motivated the extensive use of non-integer order derivatives and fractional differential equations (FDEs) as accurate tools for describing such processes [1–5]. Fractional models, with or without delay, arise naturally in numerous branches of science and engineering, including mechanics, viscoelasticity, physics, control theory, chemical processes, biological systems, economics, and dynamical systems theory [8–11]. Considerable attention has therefore been devoted to the

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qualitative analysis of linear and nonlinear FDEs, particularly their pantograph $(0 < \gamma < 1)$ with existence, uniqueness, stability, and long-term behaviour [6,13].

Among various fractional operators, the Hadamard fractional derivative is distinguished by its logarithmic kernel, which distinguishes it from the Riemann–Liouville and Caputo derivatives [7]. This operator is especially suitable for systems evolving on multiplicative time scales or with logarithmic growth behaviour. Significant research progress has been made in the theory of Hadamard FDEs, and many analytical properties, solution techniques, and qualitative results have been established [**?**,13,14].

Parallel to these developments, FIDEs (Fractional Integro-Differential Equations) have emerged as powerful modelling tools for systems involving combined local and nonlocal memory effects [17]. Such equations provide realistic descriptions of phenomena in acoustics, porous media, electrochemistry, polymer science, signal processing, chaotic dynamics, electro -magnetics, astrophysics, medicine, rheology, and several other applied disciplines [6,11]. Consequently, examination of Volterra, Fredholm, and Volterra–Fredholm fractional integro-differential equations (VFFIDEs) has attracted increasing interest [13,14]. These models can capture complex behaviours that are difficult to describe using classical formulations, thereby leading to a growing body of research on their analytic and qualitative properties [7,14].

In [8] The work addresses a family of nonlinear differential equations with several Caputo FIDEs domain boundary restrictions.

$$\mathfrak{D}^{\kappa}[\mathfrak{D}^{\nu}\mathfrak{u}(\rho) - \mathfrak{g}(\rho,\mathfrak{u}(\rho))] = \mathfrak{f}(\rho,\mathfrak{u}(\rho)), \quad \rho \in [0,T],$$

$$u(0) = 0, \quad (D^{fl}u)(T) = \check{}\mathcal{I}^{æ}u(T), \quad \mathring{}, æ < 1, 0 < \check{}.$$

In [9]Addressed for the existence of solutions to Caputo–Hadamard neutral FDEs with Dirichlet boundary conditions are established.

$$\mathfrak{D}^{\kappa}[\mathfrak{D}^{\nu}\mathfrak{u}(\rho) - \mathfrak{g}(\rho,\mathfrak{u}(\rho))] = \mathfrak{f}(\rho,\mathfrak{u}(\rho)), \quad < \kappa, \nu < g,$$

$$u(1) = 0, \quad u(T) = 0, \quad \rho \in [1,T].$$

Ntouyas [11] Addressed the existence of solutions for a FDEs governed by a fractional integral type boundary condition.

$${}^{\mathfrak{c}}\mathfrak{D}^{\nu}_{+}\mathfrak{u}(\rho) = \mathfrak{f}(\rho,\mathfrak{u}(\rho)), \quad g < \nu \leq ,$$

$$u(0) = 0, \quad u(1) = \check{}\mathcal{I}^{æ}u(\jmath), \quad \check{} \in \mathbb{R},\ 0 < æ < 1,\ 0 < \jmath < 1.$$

Akiladevi et al. [10] Analyzed the conditions ensuring both Well-posedness of solutions in the nonlinear neutral fractional boundary value problem

$$\mathfrak{D}^{\nu}_{+}[\mathfrak{u}(\rho) - \mathfrak{g}(\rho,\mathfrak{u}(\rho))] = \mathfrak{f}(\rho,\mathfrak{u}(\rho)), \quad < \nu \leq g,$$

$$\mathfrak{u}() = \kappa\mathfrak{I}^{\rho}\mathfrak{u}(\eta), \quad \kappa \in \mathbb{R},\ < \eta < g, < \rho < g, .$$

A. A. Hamoud, [14] Examined the well-posedness of results for a Caputo–Hadamard fractional VFIDEs under nonlocal boundary conditions.

$$^{\mathfrak{H}}\mathfrak{D}^{\nu}_{g+}\mathfrak{u}(\rho) = \mathfrak{f}(\rho, \mathfrak{u}(\rho)) + \int_{g}^{\rho} \mathfrak{k}(\rho, \mathfrak{r}, \mathfrak{u}(\mathfrak{r}))\, \mathfrak{dr} + \int_{g}^{\mathfrak{T}} \mathfrak{h}(\rho, \mathfrak{r}, \mathfrak{u}(\mathfrak{r}))\, \mathfrak{dr}, \qquad \rho \in [g, \mathfrak{T}],$$

$$u(1) = u_0 + {}^{-}\int_{1}^{T} u(r)\, dr,$$

A. A. Hamoud [12] Examined the Well-posedness of solutions for nonlinear Caputo fractional neutral VFFIDEs subject to a fractional integral boundary condition.

$$^{\mathfrak{C}}\mathfrak{D}^{\nu}_{+}[\mathfrak{u}(\rho) - \mathfrak{g}(\rho, \mathfrak{u}(\rho))] = \mathfrak{f}(\rho, \mathfrak{u}(\rho), \mathfrak{K}\mathfrak{u}(\rho), \mathfrak{Q}\mathfrak{u}(\rho)), \qquad < \nu \leq g,$$

$$\mathfrak{u}() = \kappa \mathfrak{I}^{\rho}\mathfrak{u}(\eta), \qquad < \eta < g, \ < \rho < g.$$

Inspired by the preceding studies, we address a more comprehensive class Caputo–Hadamard FIDEs, referred to as Caputo–Hadamard VFFIDEs with pantograph of the form

$$^{CH}D^{\kappa}_{1+} = \psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\gamma\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big), \qquad \rho \in \mathfrak{J} := [1, T] \tag{1.1}$$

with boundary condition,

$$\mathrm{u}(1) = \mathfrak{A}, \qquad \mathrm{u}(\mathrm{T}) = \mathfrak{B}. \tag{1.2}$$

Where, $^{CH}D^{\kappa}_{1+}$ is the Caputo Hadamard Fractional derivative κ, the function $\psi : \mathfrak{J} \times \mathfrak{R}^4 \rightarrow \mathfrak{R}$ is continuous. $\mathfrak{A}, \mathfrak{B} \in \mathfrak{R}$, $\gamma \in (0,1)$, $\mathfrak{K}(\rho) = \int_1^{\rho} k(\rho,s)\mathrm{u}(s)\, ds$, $\mathfrak{Q}\mathrm{u}(\rho) = \int_1^{T} q(\rho,s)\mathrm{u}(s)\, ds$. This study investigates the conditions under which a system of pantograph with VFFIDEs, defined via the Caputo–Hadamard derivative, admits existence, uniqueness, stability, and controllability of solutions an area that has received limited attention in the literature. The analysis combines several rigorous mathematical tools, including Schauder's FPT, the Banach's FPT, and Ulam–Hyers stability theory, to establish a comprehensive theoretical framework for systems subject to nonlocal boundary conditions. The paper proceeds as follows: part 2 presents the essential preliminaries and key lemmas; part 3 develops the main results regarding existence and uniqueness; part 4 addresses the stability analysis; part 5 examines the controllability properties; part 6 provides a illustrating example to demonstrate the applicability of the theoretical findings; and part 7 concludes the study, highlighting potential directions for future research.

2. Preliminaries

This part outlines the preliminary concepts that will be utilized throughout the study.

Definition 2.1. [6] Let $\hat{h}$ be a suitable function. Then, the H-FIE of order κ is defined as:

$$\left({}^{H}I^{\kappa}_{l+}\hat{h}\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{l}^{\theta} \left(\log\left(\frac{\theta}{r}\right)\right)^{\kappa-1} \frac{\hat{h}(r)}{r} dr$$

Definition 2.2. [6] The CH-FDE of order $m - 1 < \kappa \leq m$ for a function $\hat{h}$ is given by:

$$\left({}^{CH}D_{1+}^{\kappa}\hat{h}\right)(\theta) = \theta\frac{d}{d\theta}\left({}^{H}I_{1+}^{m-\kappa}\hat{h}\right)(\theta) = \frac{1}{\Gamma(m-\kappa)}\,\theta\frac{d}{d\theta}\int_{1}^{\theta}\left(\log\left(\frac{\theta}{r}\right)\right)^{m-\kappa-1}\frac{\hat{h}(r)}{r}dr$$

Definition 2.3. [7] Alternatively, the CH-FDE of order $m-1<\kappa\leq m$ is also defined by:

$$\left({}^{CH}D_{1+}^{\kappa}\hat{h}\right)(\theta) = \frac{1}{\Gamma(m-\kappa)}\int_{1}^{\theta}\left(\log\left(\frac{\theta}{r}\right)\right)^{m-\kappa-1}\left(r\frac{d}{dr}\right)^{m}\hat{h}(r)\frac{dr}{r}$$

Lemma 2.1. [7] Let $m-1<\kappa\leq m$. Then:

$${}^{H}I_{1+}^{\kappa}\left({}^{CH}D_{1+}^{\kappa}\hat{h}(\theta)\right) = \hat{h}(\theta) - \sum_{i=0}^{m-1}\kappa_i(\log\theta)^i, \quad \kappa_i\in\mathfrak{R}, i=0,1,...,m-1$$

Theorem 2.1. [4] [Banach's FPT] Let $(\mathfrak{X},\mathrm{d})$ be a complete metric space, and let $\mathfrak{T}:\mathfrak{X}\to\mathfrak{X}$ be a contraction mapping. Then $\mathfrak{T}$ admits a single fixed point $\check{x}\in\mathfrak{X}$ satisfying

$$\mathfrak{T}(\check{x}) = \check{x}.$$

Theorem 2.2. [4][Arzelà–Ascoli Theorem] Let $[\mathrm{a},\mathrm{b}]$ be a compact interval. If a sequence of functions defined on $[\mathrm{a},\mathrm{b}]$ is both uniformly bounded and equicontinuous then one can extract a subsequence that converges uniformly on $[\mathrm{a},\mathrm{b}]$.

Theorem 2.3. [4] [Schauder's FPT] Suppose $\mathfrak{X}$ represents a Banach space and let B subset of $\mathfrak{X}$ be a nonempty, closed, bounded, and convex subset. If $\aleph: B\to B$ is a continuous operator so that the set $\aleph(B)$ is compactly contained in $\mathfrak{X}$, then $\aleph$ has at least single fixed point in B.

3. Well-Posedness of Solutions

This part is intended to prove the well- posedness of reults to Eq. 1.1 together with the criteria in 1.2. To facilitate the discussion, we first state the lemma below.

Lemma 3.1. For $0<\kappa\leq 1$, suppose that $\mathfrak{H}$ is a continuous on C([1, T], taking values in $\mathbb{R}$) is a given function with continuity. If u belongs to $\mathrm{C}(\mathfrak{J},\mathfrak{R})$, then u fulfills the problem

$${}^{CH}D_{1+}^{\kappa} = \mathfrak{H}(\rho), \qquad \rho\in J:=[1,T] \tag{3.1}$$

with boundary condition,

$$\mathrm{u}(1) = \mathfrak{A}, \qquad \mathrm{u}(\mathrm{T}) = \mathfrak{B} \tag{3.2}$$

iff, the function u solves the corresponding integral equation.
(Here, $\psi\Big(\rho,\mathrm{u}(\rho),\mathrm{u}(\lambda\rho),\mathfrak{K}\mathrm{u}(\rho),\mathfrak{Q}\mathrm{u}(\rho)\Big)=\mathfrak{H}(\rho)$.)

$$\mathrm{u}(\rho) = \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\left[\mathfrak{B}-\mathfrak{A}-\frac{1}{\Gamma\kappa}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\right] + \frac{1}{\Gamma\kappa}\int_{1}^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds, \quad \rho\in\mathfrak{J}. \tag{3.3}$$

Proof: Applying the Hadamard fractional derivatives ${}^{H}I^{\kappa}_{1+}$ to right and left side of the given fractional differential equation, we obtain:

$$ {}^{H}I^{\kappa}_{1+}\,{}^{CH}D^{\kappa}_{1+}\mathrm{u}(\rho) = {}^{H}I^{\kappa}_{1+}\mathfrak{H}(\rho), \quad \rho \in [1, T]. \tag{3.4} $$

$$ \mathrm{u}(\rho) - c_0 - c_1 log(\rho) = ({}^{H}I^{\kappa}_{1+}\mathfrak{H}(\rho), \quad c_0, c_1 \in \mathfrak{R} $$

Now, we using property

$$ \mathrm{u}(\rho) = c_0 + c_1 log(\rho) + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds \tag{3.5} $$

The boundary condition $\mathrm{u}(1) = \mathfrak{A}$, implies $c_0 = \mathfrak{A}$, and the second boundary condition $\mathrm{u}(T) = \mathfrak{B}$, gives

$$ \mathrm{u}(\rho) = \mathfrak{A} + c_1 log(\rho) + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds $$

Hence,

$$ c_1 = \frac{1}{\log T}\left(\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\right) \tag{3.6} $$

substitute c_0, c_1, 3.5 in ,we get

$$ \mathrm{u}(\rho) = \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\left[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{T}\left(\log\frac{T}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\right] + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds \tag{3.7} $$

On the contrary, when u satisfies 3.1, then applying ${}^{CH}I^{\kappa}_{1+}$ to each side of 3.1, we obtain

$$ \begin{aligned} {}^{CH}D^{\kappa}_{1+}u(\rho) =& {}^{CH}I^{\kappa}_{1+}\left(\mathfrak{A} + \frac{\log(\rho)}{\log(T)}\left[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{T}\left(\log\frac{T}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\right]\right. \\ &\left. + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\right) \\ =&\mathfrak{H}(\mathrm{s}) \\ =&\psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda\mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big) \end{aligned} $$

To rewrite 3.3 in a form suitable for applying theorem 2.3, we introduce the mapping $\aleph : \mathrm{B} \to \mathrm{B}$ by

$$ \begin{aligned} (\aleph\mathrm{u})(\rho) = \mathfrak{A} &+ \frac{\log(\rho)}{\log(T)}\left[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{T}\left(\log\frac{T}{s}\right)^{\kappa-1}\psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda\mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\right] \\ &+ \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)^{\kappa-1}\psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda\mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s} \end{aligned} \tag{3.8} $$

A function u is a unchanged point of the operator $\aleph$ if it that satisfies the identity $\aleph\mathrm{u} = \mathrm{u}$. To ensure the existence of such a fixed point, we formulate the assumptions that follow.

A1 Consider $\mathrm{u}^{\star}, \mathrm{u}_{\star} \in \mathrm{B}$ such that $a \le \mathrm{u}_{\star}(\rho) \le \mathrm{u}^{\star}(\rho) \le b$ and

$$ \begin{aligned} &{}^{CH}D^{\kappa}_{1+}\mathrm{u}_{\star}(\rho) - \psi\Big(\rho, \mathrm{u}_{\star}(\rho), \mathrm{u}_{\star}(\lambda\rho), \mathfrak{K}\mathrm{u}_{\star}(\rho), \mathfrak{Q}\mathrm{u}_{\star}(\rho)\Big) \ge 0, \\ &{}^{CH}D^{\kappa}_{1+}\mathrm{u}^{\star}(\rho) - \psi\Big(\rho, \mathrm{u}^{\star}(\rho), \mathrm{u}^{\star}(\lambda\rho), \mathfrak{K}\mathrm{u}^{\star}(\rho), \mathfrak{Q}\mathrm{u}^{\star}(\rho)\Big) \le 0, \qquad \forall\rho \in \mathfrak{J}. \end{aligned} $$

A2 There exist non-negative constants $L_\psi, L_\lambda, L_{\mathfrak{K}}, L_{\mathfrak{Q}} > 0$ so that for all $\rho \in \mathfrak{J} := [1,T]$ and for all $\mathrm{u}, \breve{\mathrm{u}} \in \mathfrak{R}$, the nonlinear function $\psi : \mathfrak{J} \times \mathfrak{R}^4 \to \mathbb{R}$ satisfies

$$\begin{aligned}&\left|\psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big) - \psi\Big(\rho, \breve{\mathrm{u}}(\rho), \breve{\mathrm{u}}(\lambda\rho), \mathfrak{K}\breve{\mathrm{u}}(\rho), \mathfrak{Q}\breve{\mathrm{u}}(\rho)\Big)\right| \\ &\leq L_\psi|\mathrm{u} - \breve{\mathrm{u}}| + L_\lambda|\mathrm{u} - \breve{\mathrm{u}}| + L_{\mathfrak{K}}|\mathrm{u} - \breve{\mathrm{u}}| + L_{\mathfrak{Q}}|\mathrm{u} - \breve{\mathrm{u}}|.\end{aligned}$$

(i.e.,)

$$|\psi(\rho, \mathrm{u}) - \psi(\rho, \breve{\mathrm{u}})| \leq L_\psi|\mathrm{u} - \breve{\mathrm{u}}|,$$

$$|\psi(\mathrm{u}(\lambda\rho)) - \psi(\breve{\mathrm{u}}(\lambda\rho))| \leq L_\lambda|\mathrm{u} - \breve{\mathrm{u}}|,$$

$$|\mathfrak{K}(\rho, s, \mathrm{u}) - \mathfrak{K}(\rho, s, \breve{\mathrm{u}})| \leq L_{\mathfrak{K}}|\mathrm{u} - \breve{\mathrm{u}}|,$$

$$|\mathfrak{Q}(\rho, s, \mathrm{u}) - \mathfrak{Q}(\rho, s, \breve{\mathrm{u}})| \leq L_{\mathfrak{Q}}|\mathrm{u} - \breve{\mathrm{u}}|.$$

The function $\mathrm{u}_\star$ and $\mathrm{u}^\star$ are referred to as the lower and upper results for the system 1.1-1.2 respectively. The approach is initially built upon 2.3.

Theorem 3.1. If condition (A1),(A2) hold, then there exists at least single non- negative results for the system 1.1-1.2.

Proof: Consider

$$\Xi = \Big\{\mathrm{u} \in \mathrm{B} \;\Big|\; \mathrm{u}_\star(\rho) \leq \mathrm{u}(\rho) \leq \mathrm{u}^\star(\rho),\ \rho \in \mathfrak{J}\Big\}.$$

provided with the norm $\|\mathrm{u}\| = \max_{\rho\in\mathfrak{J}} |\mathrm{u}\rho|$, thus, we deduce $\mathrm{A}\|\mathrm{u}\| \leq \mathrm{b}$. Therefore, Ξ is identified as a convex,closed and subset with bounded elements in the Banach space $\mathrm{C}([1,T])$. In addition, since $\psi, \lambda\mathfrak{K}, \mathfrak{Q}$ are continuous the operator $\aleph$ in 4 is likewise continuous on Ξ. Now, if $\mathrm{u} \in \Xi$,**(A3)** there exist non- negative constants $\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_{\mathfrak{K}}, \mathfrak{M}_{\mathfrak{Q}}$ such that

$$\begin{aligned}\max\{\psi(\rho, \mathrm{u}(\rho)) : \rho \in \mathfrak{J},\ \mathrm{u}(\rho) \leq b\} &\leq \mathfrak{M}_\psi, \\ \max\{\psi(\mathrm{u}(\lambda\rho)) : \rho \in \mathfrak{J},\ \mathrm{u}(\lambda\rho) \leq b\} &\leq \mathfrak{M}_\lambda, \\ \max\{\mathfrak{K}(\rho, s, \mathrm{u}(s)) : \rho, s \in \mathfrak{J},\ \mathrm{u}(s) \leq b\} &\leq \mathfrak{M}_{\mathfrak{K}}, \\ \max\{\mathfrak{Q}(\rho, s, \mathrm{u}(s)) : \rho, s \in \mathfrak{J},\ \mathrm{u}(s) \leq b\} &\leq \mathfrak{M}_{\mathfrak{Q}}.\end{aligned}$$

Then

$$\begin{aligned}|(\aleph\mathrm{u})(\rho)| \leq \Big|\mathfrak{A} &+ \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda\mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\Big] \\ &+ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda\mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\Big|\end{aligned} \tag{3.9}$$

$$\begin{aligned}|(\aleph\mathrm{u})(\rho)| &\leq |\mathfrak{A}| + \Big|\frac{\log(\rho)}{\log(T)}\Big|\Big[|\mathfrak{B} - \mathfrak{A}| + \frac{(\log T)^\kappa}{\Gamma(\kappa+1)}|(\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_{\mathfrak{K}}, \mathfrak{M}_{\mathfrak{Q}})|\Big] + \frac{(\log\rho)^\kappa}{\Gamma(\kappa+1)}|(\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_{\mathfrak{K}}, \mathfrak{M}_{\mathfrak{Q}})| \\ &\leq 2|\mathfrak{A}| + |\mathfrak{B}| + \frac{2(\log T)^\kappa}{\Gamma(\kappa+1)}|(\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_{\mathfrak{K}}, \mathfrak{M}_{\mathfrak{Q}})|.\end{aligned}$$

Hence, $\aleph(\Xi)$ is bounded uniformly. Next, we show that the functions are equicontinuous of $\aleph(\Xi)$. For each $u \in \Xi$. Then for $\rho_1, \rho_2 \in \mathfrak{J}$ with $\rho_1 < \rho_2$, we have

$$
\begin{aligned}
&|(\aleph \mathrm{u})(\rho_2) - (\aleph \mathrm{u})(\rho_1)| \\
\leq &|\mathfrak{A}| + \Big|\frac{\log(\rho_2) - \log(\rho_1)}{\log(T)}\Big| \\
&\Big|\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}\Big]\Big| \\
&+ \frac{1}{\Gamma\kappa}\Big|\int_1^{\rho_2} \Big(\log\frac{\rho_2}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s} \\
&- \int_1^{\rho_1} \Big(\log\frac{\rho_1}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}\Big| \\
&+ \frac{1}{\Gamma\kappa}\int_{\rho_2}^{\rho_1} \Big(\log\frac{\rho_2}{s}\Big)^{\kappa-1} |\psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)|\frac{ds}{s} \\
&\leq |\mathfrak{A}| + \frac{1}{\log T}\log\Big(\frac{\rho_2}{\rho_1}\Big)|[\mathfrak{B} - \mathfrak{A}]| + \frac{2(\mathfrak{M}_\psi + \mathfrak{M}_\lambda + \mathfrak{M}_\mathfrak{K} + \mathfrak{M}_\mathfrak{Q})}{\Gamma(\kappa+1)}\Big[\log\Big(\frac{\rho_2}{\rho_1}\Big)\Big]^\kappa
\end{aligned}
\tag{3.10}
$$

$\to 0$ as $\rho_1 \to \rho_2$. The convergence does not depend on u in Ξ, which ensure that $\aleph(\Xi)$ is equicontinuous.Theorem 2.2 ensures that $\aleph : \Xi \to B$ is compact. The only requirement to apply theorem 2.3 is to show that $\aleph(\Xi) \subset \Xi$. For any $\mathrm{u} \in \Xi$, then $\mathrm{u}_\star(\rho) \leq \mathrm{u}(\rho) \leq \mathrm{u}^\star(\rho)$ and by (A1), we get

$$
\begin{aligned}
(\aleph \mathrm{u})(\rho) &= \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}\Big] \\
&+ \frac{1}{\Gamma\kappa}\int_1^{\rho} \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s} \\
&\leq \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \mathrm{U}\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}u(\mathrm{s}), \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}\Big] \\
&+ \frac{1}{\Gamma\kappa}\int_1^{\rho} \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \mathrm{U}\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s} \\
&\leq \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \mathrm{U}\Big(\mathrm{s}, \mathrm{u}^\star(\mathrm{s}), \mathrm{u}^\star(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}^\star(\mathrm{s}), \mathfrak{Q}\mathrm{u}^\star(\mathrm{s})\Big)\frac{ds}{s}\Big] \\
&+ \frac{1}{\Gamma\kappa}\int_1^{\mathrm{s}} \Big(\log\frac{\mathrm{s}}{s}\Big)^{\kappa-1} \mathrm{U}\Big(\mathrm{s}, \mathrm{u}^\star(\mathrm{s}), \mathrm{u}^\star(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}^\star(\mathrm{s}), \mathfrak{Q}\mathrm{u}^\star(\mathrm{s})\Big)\frac{ds}{s} \\
&\leq \mathrm{u}^\star(\rho)
\end{aligned}
$$

and

$$
\begin{aligned}
(\aleph \mathrm{u})(\rho) &= \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}\Big] \\
&+ \frac{1}{\Gamma\kappa}\int_1^{\rho} \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u(s)}, u(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u(s)}, \mathfrak{Q}\mathrm{u(s)}\Big)\frac{ds}{s}
\end{aligned}
$$

$$\leq \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \mathrm{L}\Big(\mathrm{s}, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big)\frac{ds}{s}\Big]$$
$$+ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \mathrm{L}\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}$$
$$\leq \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \mathrm{L}\Big(\mathrm{s}, \mathrm{u}_\star(\mathrm{s}), \mathrm{u}_\star(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}_\star(\mathrm{s}), \mathfrak{Q}\mathrm{u}_\star(\mathrm{s})\Big)\frac{ds}{s}\Big]$$
$$+ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \mathrm{L}\Big(\mathrm{s}, \mathrm{u}_\star(\mathrm{s}), \mathrm{u}_\star(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}_\star(\mathrm{s}), \mathfrak{Q}\mathrm{u}_\star(\mathrm{s})\Big)\frac{ds}{s}$$
$$\leq \mathrm{u}_\star(\rho).$$

Hence, $\mathrm{u}^\star(\rho) \leq (\aleph \mathrm{u})(\rho) \leq \mathrm{u}^\star(\rho), \rho \in \mathfrak{J}$. *i.e.*, $\aleph(\Xi) \subset \Xi$ By the theorem 2.3, the mapping $\aleph$ possesses ensures the exists of a fixed point $\mathrm{u} \in \Xi$.

∴ It follows that the system 1.1–1.2 possesses a non-negative solution, concluding the argument. We now proceed to a further result that relies on Theorem 2.1.

Theorem 3.2. Provided that assumptions $(A1), (A2)$ hold,

$$\Delta = \frac{2(L_\psi + L_\lambda + L_{\mathfrak{K}} + L_{\mathfrak{Q}})(\log T)^\kappa}{\Gamma(\kappa+1)} < 1. \tag{3.11}$$

which guarantees the existence of a single unique solution to the system 1.1 -1.2 .

Proof:

According to theorem 3.1, the system 1.1-1.2 yields at least single non-negative solution. Thus, it remains to verify that the operator introduced in 4 acts as a contractive mapping in Ξ. In particular, for every $\mathrm{u}, \mathrm{v} \in \Xi$, we have

$$|(\aleph \mathrm{u})(\rho) - (\aleph \breve{\mathrm{u}})(\rho)|$$
$$\leq \int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big) - \psi\Big(\mathrm{s}, (\breve{\mathrm{u}}\mathrm{s}), \breve{\mathrm{u}}(\lambda \mathrm{s}), \mathfrak{K}(\breve{\mathrm{u}}\mathrm{s}), \mathfrak{Q}(\breve{\mathrm{u}}\mathrm{s})\Big)|\frac{ds}{s}$$
$$+ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big) - \psi\Big(\mathrm{s}, (\breve{\mathrm{u}}\mathrm{s}), (\breve{\mathrm{u}}\lambda \mathrm{s}), \mathfrak{K}\breve{\mathrm{u}}(\mathrm{s}), \mathfrak{Q}(\breve{\mathrm{u}}\mathrm{s})\Big)\frac{ds}{s}$$
$$\leq \frac{2(L_\psi + L_\lambda + L_{\mathfrak{K}} + L_{\mathfrak{Q}})(\log T)^\kappa}{\Gamma(\kappa+1)}\|\mathrm{u} - \mathrm{v}\|.$$

Hence,

$$\|\aleph \mathrm{u} - \aleph \breve{\mathrm{u}}\| \leq \Delta\|\mathrm{u} - \mathrm{v}\|.$$

Therefore, using 3.11, we conclude that the operator $\aleph$ satisfies the contraction property. Thus, based on the theorem 2.1, consequently, the system 1.1-1.2 has a unique non-negative results.

4. Analysis of Stability Properties

In this part, we examine the structural stability characteristics of the system 1.1 which is initiated by analyzing the following inequality.

$$|{}^{CH}D^\kappa_{1+} - \psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big)| \leq \varepsilon, \qquad \rho \in \mathfrak{J} \tag{4.1}$$

$$|^{CH}D^{\kappa}_{1+} - \psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}(\mathrm{u}\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big)| \leq \varepsilon\varphi(\rho), \qquad \rho \in \mathfrak{J} \tag{4.2}$$

for our analysis, the essential definitions are given below.

Definition 4.1. [13] The system1.1 possesses Ulam–Hyers type stability if one can find a non-negative constant $\mathfrak{C}_f > 0$ so that, for every $\varepsilon > 0$ and for each results $\mathrm{z} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of 4.1 corresponds to a solution $\mathrm{u} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of 1.1 satisfying.

$$|\mathrm{z}(\rho) - \mathrm{u}(\rho)| \leq \mathfrak{C}_f \varepsilon.$$

Definition 4.2. [13] The system1.1 exhibits Ulam–Hyers stability if one can find function $\mathfrak{F} \in \mathrm{C}(\mathfrak{R}^+, \mathfrak{R}^+)$ with the property $\mathfrak{F}(0) = 0$, so that for all $\varepsilon > 0$, each solution $z \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of 4.1 corresponds to a solution $\mathrm{u}, \mathrm{v} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of 1.1 for which the inequality

$$|z(\rho) - \mathrm{u}(\rho)| \leq \mathfrak{F}\varepsilon$$

holds.

Definition 4.3. [13] The problem1.1possesses Ulam–Hyers–Rassias stability corresponding to φ if there exits a constant $\mathfrak{C}_{f,\varphi} > 0$ such that, for every results $z \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of 4.2, one can find a solution $\mathrm{u}, \check{\mathrm{u}} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ of1.1 satisfying

$$|z(\rho) - \mathrm{u}(\rho)| \leq \varepsilon\, \mathfrak{C}_{f,\varphi}\, \varphi(\rho).$$

Theorem 4.1. Assuming the assumptions stated in Theorem 3.2 are satisfied, the results corresponding to 1.1 possesses the property of Ulam–Hyers type stability.

Proof: Let $\varepsilon < 0$, $\mathrm{z} \in \mathrm{u}$ be a function that complies with the inequality 4.1

$$|^{CH}D^{\kappa}_{1+} - \psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big)| \leq \varepsilon, \qquad \forall\rho \in \mathfrak{J} := [1, T] \tag{4.3}$$

and $\mathrm{u} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R})$ be the well defined results of the following Caputo–Hadamard VFFIDEs with pantograph-type conditions

$$^{CH}D^{\kappa}_{1+} = \psi\Big(\rho, \mathrm{u}(\rho), \mathrm{u}(\lambda\rho), \mathfrak{K}\mathrm{u}(\rho), \mathfrak{Q}\mathrm{u}(\rho)\Big), \qquad \rho \in \mathfrak{J} := [1, T]$$

with boundary condition,

$$\mathrm{u}(1) = \mathfrak{A}, \qquad \mathrm{u}(\mathrm{T}) = \mathfrak{B}$$

By using lemma 3.1,we obtain

$$\mathrm{u}(\rho) = \mathrm{A_u} + \frac{1}{\Gamma(\kappa)} \int_1^{\rho} \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}$$

where,

$$\begin{aligned}\mathrm{A_u} = \mathfrak{A} + \frac{\log(\rho)}{\log(\rho)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\Big] \\ + \frac{1}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\end{aligned}$$

conversely, provided that $u(T)$ equals $z(T)$ and $u(1) = z(1)$, then $A_u = A_z$.
Indeed,
$|A_u - A_z| = 0$
It follows that , $A_u = A_z$.
Accordingly, we arrive at

$$u(\rho) = A_z + \frac{1}{\Gamma(\kappa)} \int_1^{\rho} \left(\log \frac{\rho}{s}\right)^{\kappa-1} \psi\Big(s, u(s), u(\lambda s), \mathfrak{K}u(s), \mathfrak{Q}u(s)\Big)\frac{ds}{s}$$

An integration of the inequality 4.3 leads to

$$\Big|z(\rho) - A_u - \frac{1}{\Gamma(\kappa)} \int_1^{\rho} \left(\log \frac{\rho}{s}\right)^{\kappa-1} \psi\Big(s, z(s), z(\lambda s), \mathfrak{K}z(s), \mathfrak{Q}z(s)\Big)\frac{ds}{s}\Big| = \frac{\varepsilon}{1-\Lambda}$$

we have

$$\begin{aligned}
&|z(\rho) - u(\rho)| \\
&\leq \Big|z(\rho) - A_u - \frac{1}{\Gamma(\kappa)} \int_1^{\rho} \left(\log \frac{\rho}{s}\right)^{\kappa-1} \psi\Big(s, z(s), z(\lambda s), \mathfrak{K}z(s), \mathfrak{Q}z(s)\Big)\frac{ds}{s}\Big| \\
&+ \Big|\frac{1}{\Gamma(\kappa)} \int_1^{\rho} \left(\log \frac{\rho}{s}\right)^{\kappa-1} \Big[\psi\Big(s, z(s), z(\lambda s), \mathfrak{K}z(s), \mathfrak{Q}z(s)\Big) - \psi\Big(s, u(s), u(\lambda s), \mathfrak{K}u(s), \mathfrak{Q}u(s)\Big)\Big]\frac{ds}{s}\Big| \\
&\leq \frac{\varepsilon}{1-\Delta} + \frac{2\mathfrak{L}}{\Gamma\kappa} \int_1^{\rho} \left(\log \frac{\rho}{s}\right)^{\kappa-1} |z(s) - u(s)|\frac{ds}{s}
\end{aligned}$$

Here, $\mathfrak{L} = L_{\psi}|u - \breve{u}| + L_{\lambda}|u - \breve{u}| + L_{\mathfrak{K}}|u - \breve{u}| + L_{\mathfrak{Q}}|u - \breve{u}|$.
Where $E_{\check{},1}$ Mittag-leffler function defined by

$$E_{\check{},1}z = \sum_{k=0}^{\infty} \frac{z}{\Gamma(k\kappa + 1)}, \qquad z \in C \tag{4.4}$$

by apply 4.4, we obtain

$$\begin{aligned}
|z(\rho) - u(\rho)| &\leq \frac{E_{\check{}}, 1(2\mathfrak{L}(logT)^{\kappa})}{1-\Lambda}.\varepsilon \\
&= \mathfrak{C}_f \varepsilon
\end{aligned}$$

Therefore, equation 1.1 satisfies is Ulam- Hyers stability.
Remark 1. By taking $\mathfrak{F}(\varepsilon) = \mathfrak{C}_f \varepsilon$. As a result,the problem 1.1 satisfies the Ulam–Hyers type generalized stability.

Theorem 4.2. Consider the assumptions of 3.2 along with assumption (A4) hold.
In particular, there is an non-decreasing function $\varphi \in (\mathfrak{J}, \mathfrak{R}^+)$ and a constant $\lambda_{\varphi} > 0$ such that, for every $\rho \in J$, the inequality

$$^{CH}D^{\kappa}_{1+}\varphi(\rho) \leq \lambda_{\varphi}\, \varphi(\rho)$$

is satisfied. Under these assumptions, problem 1.1 exhibits generalized Ulam- Hyers-Rassias stability.

Proof:
Define $\varepsilon < 0$ and $z \in C(\mathfrak{J}, \mathfrak{R})$ be function which is satisfies the inequality:

$$|{}^{CH}D_{1+}^{\kappa} - \psi\Big(\rho, u(\rho), u(\lambda\rho), \mathfrak{K}u(\rho), \mathfrak{Q}u(\rho)\Big)| \leq \varepsilon, \qquad \rho \in \mathfrak{J} := [1, T] \tag{4.5}$$

and let $z \in C(\mathfrak{J}, \mathfrak{R})$ be the exactly one solution of the following Caputo-Hadamard type pantograph with conditions.

$$u(1) = \mathfrak{A}, \qquad u(T) = \mathfrak{B}$$

Applying Lemma 3.1, we arrive at

$$u(\rho) = A_u + \frac{1}{\Gamma(\kappa)} \int_1^{\rho} \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(s, u(s), u(\lambda s), \mathfrak{K}u(s), \mathfrak{Q}u(s)\Big)\frac{ds}{s}$$

where,

$$\begin{aligned} A_u = \mathfrak{A} &+ \frac{\log(\rho)}{\log(\rho)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\psi\Big(s, u(s), u(\lambda s), \mathfrak{K}u(s), \mathfrak{Q}u(s)\Big)\frac{ds}{s}\Big] \\ &+ \frac{1}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\psi\Big(s, u(s), u(\lambda s), \mathfrak{K}u(s), \mathfrak{Q}u(s)\Big)\frac{ds}{s} \end{aligned}$$

By integration of the inequality 4.5,we get

$$\Big|z(\rho) - A_u - \frac{1}{\Gamma(\kappa)}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\psi\Big(s, z(s), z(\lambda s), \mathfrak{K}z(s), \mathfrak{Q}z(s)\Big)\frac{ds}{s}\Big| = \varepsilon\, \mathfrak{C}_{f,\varphi}\,\varphi(\rho)$$

Conversely, we have

$$\begin{aligned} |z(\rho) - u(\rho)| \leq &\Big|z(\rho) - A_u - \frac{1}{\Gamma(\kappa)}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\psi\Big(s, z(s), z(\lambda s), \mathfrak{K}z(s), \mathfrak{Q}z(s)\Big)\frac{ds}{s}\Big| \\ &+ \frac{2\mathfrak{L}}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}|z(s) - u(s)|\frac{ds}{s} \\ \leq &\varepsilon\, \mathfrak{C}_{f,\varphi}\,\varphi(\rho) + \frac{2\mathfrak{L}}{\Gamma\kappa}\int_1^{\rho}\Big(\log\frac{\rho}{s}\Big)^{\kappa-1}|z(s) - u(s)|\frac{ds}{s} \end{aligned}$$

By applying 4.4, we get

$$|z(\rho) - u(\rho)| \leq \varepsilon\, \mathfrak{C}_{f,\varphi}\,\varphi(\rho) E_{\check{}}, 1(2\mathfrak{L}(logT)^{\kappa}). \qquad \rho \in [1, T]. \tag{4.6}$$

Thus, the equation 1.1 is Ulam–Hyers type generalized stability

5. Controllability

In this part, the for nonlinear system described by equation 1.1, we adopt the definition in [15,17], [16] controllability:

$${}^{CH}D_{1+}^{\kappa} = \psi\Big(\rho, u(\rho), u(\lambda\rho), \mathfrak{K}u(\rho), \mathfrak{Q}u(\rho)\Big) + (Bv)(\rho), \qquad \rho \in \mathfrak{J} := [1, T] \tag{5.1}$$

with boundary condition,

$$u(1) = \mathfrak{A}, \qquad u(T) = \mathfrak{B} \tag{5.2}$$

referring to equation 1.1-1.2. Moreover, the state variable u(.) belongs to the Banach space $\mathfrak{R}$, at the same time the control input v(.) belongs to the space $\mathfrak{L}^2(\mathfrak{J}, \mathrm{U})$, which is a Space of admissible controls, forming a Banach space with U as the associated Banach space. The operator B is linear and bounded, acting from U in to $\mathfrak{R}$. A mild solution exists for the equation 5.1-5.2, which can be expressed as:

$$\begin{aligned}(\mathrm{u})(\rho) = \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s}\Big] \\ + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\frac{ds}{s} + \\ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \psi\Big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\Big)\mathrm{B}\mathrm{v}\frac{ds}{s}\end{aligned} \tag{5.3}$$

Definition 5.1. The considered system 5.1-5.2 is controllable within the $\mathfrak{J}$ if, for any initial values $\mathfrak{A}, \mathfrak{B}$ in the Banach space $\mathfrak{R}$, there exist a control input $(v)\rho \in \mathfrak{L}\,(\mathfrak{J}, (U))$ such that the corresponding mild solution $\mathrm{u}(\rho)$ fulfilled the conditions 5.2. To establish the conttrollability results for the caputo-Hadamard fractional system 5.2, before presenting the theorem, the following assumptions are stated **A3**. **A5** The bounded linear mapping $W\ \mathfrak{L}(\mathfrak{J}, \mathrm{U}) \to \mathfrak{R}$ be determined by

$$W_v = \frac{1}{\Gamma(\kappa+1)}\int_1^\rho \Big(\log\frac{\rho}{r}\Big)^{\kappa} (\mathrm{Bv})(s)\, ds,$$

and its corresponding inverse W^{-1} is well - defined on the space obtained as a quotient of $\mathrm{L}^2(\mathfrak{J}, \mathrm{U})/$ ker. Furthermore, there is a constants $M_1, M_2 > 0$ satisfying the following conditions

$$\|\mathrm{B}\| \le M_1, \qquad \|W^{-1}\| \le M_2.$$

$M_1, M_2 > 0$ *so that* $\|\mathrm{B}\| \le M_1$ *and* $\|W^{-1}\| \le M_2$.

Theorem 5.1. If the assumptions (A1)-(A3), (A5),the system characterized by equation 5.1 the system is exhibiting full controllability within $\mathfrak{J}$

Proof:
Given the conditions outlined in hypothesis (A3), it is feasible to establish control based on the attributes of any arbitrary function $\mathrm{u}(\rho)$. Consider the collection of functions $\mathrm{Y}_{\gamma 1}$, which includes all functions that are continuous u represented on the interval $\mathfrak{J}$ that take real values, satisfy the restriction $\|u\|_\infty \le \gamma_1$.

$$\Omega(\rho) = W^{-1}\Bigg[\mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \frac{\mathfrak{H}s}{s}ds\Bigg]. \tag{5.4}$$

The goal is to demonstrate that the operator Ψ maps Y_{γ_1} into itself. Once this is established, the operator can be defined.

$$\Psi\mathrm{u}(\rho) = \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1} \frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1} \frac{\mathfrak{H}s}{s}ds + (\mathrm{B}\Omega)(s) \tag{5.5}$$

It holds that $\Psi \mathrm{u}(T) = \mathfrak{B}$, demonstrating that the system defined by equation 5.1- 5.2 is controllable on [1,T]. A fixed point for exists, with the control function $\mathrm{v}(\rho)$ defined as in equation 5.5, serving as the generalized solution to the control problem. Using the control input $\Omega(\rho)$, the following expression is derived:

$$\begin{aligned}\Psi \mathrm{u}(\rho) =& \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds + \\ &(\mathrm{B})W^{-1}\Bigg[\mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Bigg]\end{aligned} \tag{5.6}$$

It can be concluded that the operator Ψ is continuous. For any function $\mathrm{u} \in \mathrm{Y}_{\gamma_1}$ and for all $\rho \in [1, T]$, the following relationship can be derived from equation 5.3:

$$\|\Omega(\rho)\| \leq \|W^{-1}\|\Bigg\|\Bigg[\mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Bigg]\Bigg\|.$$

$$\begin{aligned}\|\Omega(\rho)\| =& M_1\Bigg[\|\mathfrak{A}\| + \Bigg\|\frac{\log(\rho)}{\log(T)}\Bigg\|\Big[\|\mathfrak{B}\| - \|\mathfrak{A}\| - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\|\psi\big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\big)\|ds\Big] + \\ &\frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\|\psi\big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\big)\|ds\Bigg]\end{aligned} \tag{5.7}$$

utilizing equations 5.8 and 5.7,we get

$$\begin{aligned}\Psi \mathrm{u}(\rho) =& \mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds + \\ &M_1M_2\Bigg[\mathfrak{A} + \frac{\log(\rho)}{\log(T)}\Big[\mathfrak{B} - \mathfrak{A} - \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Big] + \frac{1}{-}\Gamma\kappa\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\frac{\mathfrak{H}s}{s}ds\Bigg] \\ &\leq (1 + M_1, M_2)\Big(2\|\mathfrak{A}\| + \|\mathfrak{B}\| + \frac{2(\log T)^\kappa}{\Gamma(\kappa+1)}\|(\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_{\mathfrak{K}}, \mathfrak{M}_{\mathfrak{Q}})\|\Big). \\ &\leq \mathfrak{N}\end{aligned} \tag{5.8}$$

Next, it will be demonstrated that the operator $\Psi : \mathrm{Y}_{\gamma_1} \to \mathrm{Y}_{\gamma_1}$satisfies all the conditions of Theorem 2.3. The following proof, divided into several parts, show that the operator Ψ transforms the set $\mathrm{Y}_{\gamma_1} = \{\mathrm{u} \in \mathrm{C}(\mathfrak{J}, \mathfrak{R}) : \|\mathrm{u}\|_\infty \leq \gamma_1\}$ into itself.

Step 1: The continuity of Ψ can be established as follows: Consider a sequence u_n such that $\mathrm{u}_\mathrm{n} \to \mathrm{u}$ in $\mathrm{Y}_{\gamma 1}$

$$\begin{aligned}\|(\Psi \mathrm{u}_\mathrm{n})(\rho) - (\Psi \mathrm{u})(\rho)\| \leq & \frac{1}{\Gamma\kappa}\int_1^T \Big(\log\frac{T}{s}\Big)^{\kappa-1}\|\psi\big(\mathrm{s}, \mathrm{u}_\mathrm{n}(\mathrm{s}), \mathrm{u}_\mathrm{n}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}_\mathrm{n}(\mathrm{s}), \mathfrak{Q}\mathrm{u}_\mathrm{n}(\mathrm{s})\big) - \\ &\psi\big(\mathrm{s}, \mathrm{u}(\mathrm{s}), \mathrm{u}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}(\mathrm{s}), \mathfrak{Q}\mathrm{u}(\mathrm{s})\big)\|ds \\ &+ \frac{1}{\Gamma\kappa}\int_1^\rho \Big(\log\frac{\rho}{s}\Big)^{\kappa-1}\|\psi\big(\mathrm{s}, \mathrm{u}_\mathrm{n}(\mathrm{s}), \mathrm{u}_\mathrm{n}(\lambda \mathrm{s}), \mathfrak{K}\mathrm{u}_\mathrm{n}(\mathrm{s}), \mathfrak{Q}\mathrm{u}_\mathrm{n}(\mathrm{s})\big) -\end{aligned}$$

$$\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)\|ds+$$
$$(B)W^{-1}\Bigg[\frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{s}\right)^{\kappa-1}\|\psi\Big(s,u_n(s),u_n(\lambda s),\mathfrak{K}u_n(s),\mathfrak{Q}u_n(s)\Big)-$$
$$\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)\|ds$$
$$+\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{s}\right)^{\kappa-1}\|\psi\Big(s,u_n(s),u_n(\lambda s),\mathfrak{K}u_n(s),\mathfrak{Q}u_n(s)\Big)-$$
$$\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)\|ds\Bigg]$$

Due to the continuity of $\mathfrak{K},\mathfrak{Q}$, it can be inferred that:

$$\|(\Psi u_n)(\rho)-(\Psi u)(\rho)\|\to 0 \qquad as \quad n\to\infty.$$

Thus,Ψ exhibits continuity over the set $Y_{\gamma 1}$.

Step 2: Since $\Psi Y_{(}\gamma_1)\subset Y_{\gamma_1}$, it follows that the set $\Psi(Y_{\gamma_1})$ is uniformly bounded, which implies boundedness.

Step 3: To illustrate the equicontinuity of $\Psi_{(}Y_{\gamma_1})$, let's consider ρ_1 and ρ_2 from the set with bounded elements [1,T] in $C(\mathfrak{J},\mathfrak{R})$, along with u from Y_{γ_1}, where $\rho_1<\rho_2$. In this scenario, it holds that:

$$\|(\Psi u)(\rho_2)-(\Psi u)(\rho_1)\|$$
$$=\Bigg\|\mathfrak{A}+\frac{\log(\rho_2-\rho_1)}{\log(T)}\Big[\mathfrak{B}-\mathfrak{A}-\frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds\Big]+$$
$$\frac{1}{\Gamma\kappa}\int_1^{\rho_2}\left(\log\frac{\rho_2}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds-$$
$$\frac{1}{\Gamma\kappa}\int_1^{\rho_2}\left(\log\frac{\rho_1}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds$$
$$+\frac{1}{\Gamma\kappa}\int_{\rho_1}^{\rho_2}\left(\log\frac{\rho_2}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds+$$
$$(B)W^{-1}\Bigg[\mathfrak{A}+\frac{\log(\rho_2-\rho_1)}{\log(T)}\Big[\mathfrak{B}-\mathfrak{A}-$$
$$\frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds\Big]+$$
$$\frac{1}{\Gamma\kappa}\int_1^{\rho-2}\left(\log\frac{\rho_2}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds$$
$$-\frac{1}{\Gamma\kappa}\int_1^{\rho_1}\left(\log\frac{\rho_1}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds$$
$$+\frac{1}{\Gamma\kappa}\int_{\rho_1}^{\rho_2}\left(\log\frac{\rho_2}{s}\right)^{\kappa-1}\psi\Big(s,u(s),u(\lambda s),\mathfrak{K}u(s),\mathfrak{Q}u(s)\Big)ds\Bigg]\Bigg\|$$

$$\leq (1+\mathrm{BW}^{-1})\Bigg(|\mathfrak{A}| + \frac{1}{\log T}\log\Bigg(\frac{\rho_2}{\rho_1}\Bigg)|[\mathfrak{B}-\mathfrak{A}]| + \frac{2(\mathfrak{M}_\psi + \mathfrak{M}_\lambda + \mathfrak{M}_\mathfrak{K} + \mathfrak{M}_\mathfrak{Q})}{\Gamma(\kappa+1)}\left[\log\left(\frac{\rho_2}{\rho_1}\right)\right]^\kappa\Bigg)$$

By considering the results from above steps and applying Theorem 2.3, it is established that Ψ is both continuous and compact. As ρ_2 approaches ρ_1, the expression on the right tends to zero, represented as $\|(\Psi\mathrm{u})(\rho_2) - (\Psi\mathrm{u})(\rho_1)\| \to 0$. Applying Theorem 2.3 ensures that a fixed point exits u, which solves the problem define by equations 5.1 and 5.2. Consequently, the system described by 5.1 and 5.2 is controllable within the interval $\mathfrak{J} = [1, T]$.

6. Examples

Example 6.1. We examine the Caputo–Hadamard Volterra-Fredholm fractional integro-differential equation subject to the imposed boundary criteria

$$^{CH}D^{1/3}_{1+}\mathrm{u}(t) = \frac{1}{2}\left\{\frac{\sin(\mathrm{u}(t))}{6} + \frac{1}{20}\cos\left(\frac{1}{4}\mathrm{u}(t)\right) + \frac{1}{36}\int_1^t e^{-s/9}\mathrm{u}(s)\,ds + \frac{1}{36}\int_1^5 (4-e^{-s/3})(\mathrm{u}s)\,ds\right\}, \quad t \in [1,5] \tag{6.1}$$

with boundary conditions:

$$\mathrm{u}(1) = 0, \quad \mathrm{u}(5) = 1, \tag{6.2}$$

Where, $\mathrm{T} = 5$, $\kappa = 1/3$, $\psi(t, u(t)) = \frac{\sin(u(t))}{6}$, $\mathfrak{K}(t, s, u(s)) = e^{-s/9}u(s)$, $\mathfrak{Q}(t, s, u(s)) = (4 - e^{-s/3})u(s)$. here t= ρ *and*

$$\Big|\psi\Big(\rho, u(\rho), u(\lambda\rho), \mathfrak{K}u(\rho), \mathfrak{Q}u(\rho)\Big) - \psi\Big(\rho, v(\rho), v(\lambda\rho), \mathfrak{K}v(\rho), \mathfrak{Q}v(\rho)\Big)\Big|$$
$$\leq 0.0833|u-v| + 0.00625|u-v| + 0.040125|u-v| + 0.02205|u-v|.$$

(i.e.,)

$$|\psi(\rho, u) - \psi(\rho, \breve{u})| \leq 0.0833|u - \breve{u}|,$$

$$|\psi(u(\lambda\rho)) - \psi(\breve{u}(\lambda\rho))| \leq 0.00625|u - \breve{u}|,$$

$$|\mathfrak{K}(\rho, s, u) - \mathfrak{K}(\rho, s, \breve{u})| \leq 0.040125|u - \breve{u}|,$$

$$|\mathfrak{Q}(\rho, s, u) - \mathfrak{Q}(\rho, s, \breve{u})| \leq 0.02205|u - \breve{u}|.$$

we get the value of

$$\Delta = \frac{2(L_\psi + L_\lambda + L_\mathfrak{K} + L_\mathfrak{Q})(\log T)^\kappa}{\Gamma(\kappa+1)} = 0.151725 < 1. \tag{6.3}$$

then, by Theorem 3.2, the problem has a exactly one solution.

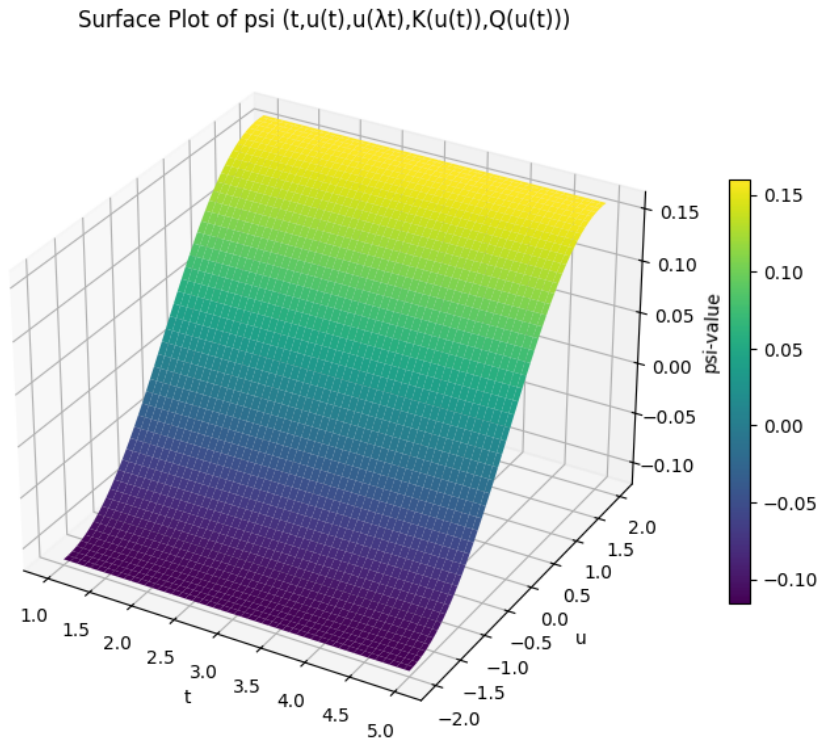

FIGURE 1. Graphical represented the nonlinear functions $\Psi(t, u(t), u(\lambda t), K(u(t)), Q(u(t))), over \quad \mathfrak{J}[1,5]$ illustrating existence and uniqueness behavior. for example 6.1

Surface plot of the operator $\Psi(t, u(t), u(\lambda t), K(u(t)), Q(u(t)))$ illustrating the smooth and well-behaved variation needed to ensure the exactly one of the result to the system of fractional integro–differential.

Now, the constant $\mathfrak{C}_f$ used in the Ulam Stability analysis is computed:

$$\mathfrak{C}_f = \frac{\mathrm{E}_{\check{\ }}, 1(2\mathfrak{L}(logT)^{\kappa})}{1-\Delta}.\varepsilon = \frac{1.5656}{0.602} = 2.60.\varepsilon$$

With $\varepsilon = 0.5$, we obtain: $\mathfrak{C}_f \varepsilon = 2.60.0.3 = 0.78 < 1$

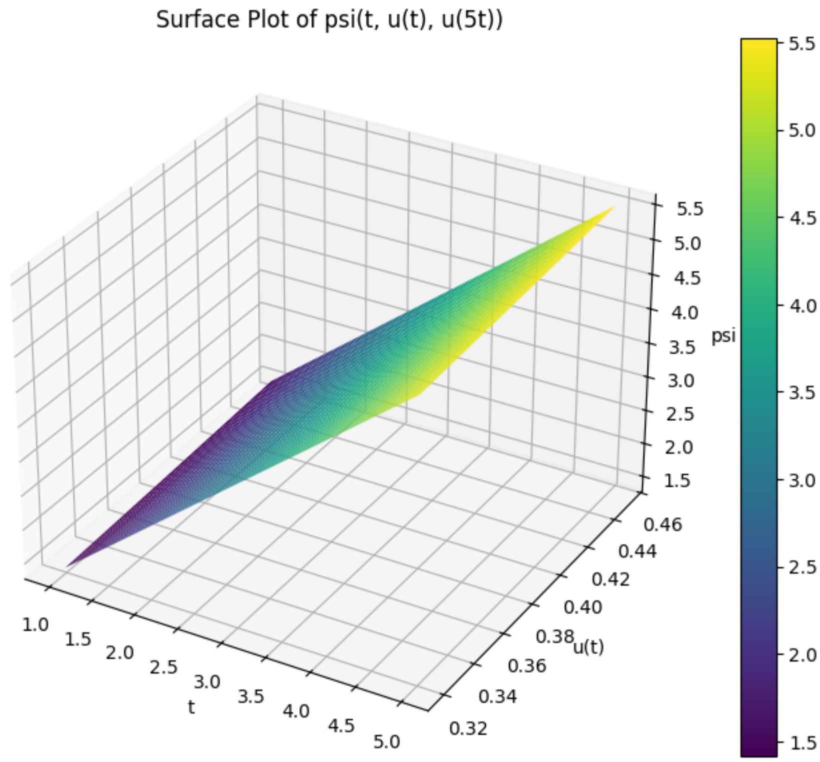

Figure 2. Graphical represented the nonlinear functions $\Psi(t, \mathrm{u}(t), \mathrm{u}(\lambda t), K(u(t)), Q(u(t)))$, over $\Im[1,5]$ illustrating its smooth variation and stable behavior. for example 6.1

The surface plot illustrates the variation of the operator $\Psi(t, u(t), u(5t))$ associated with the given fractional integro-differential system. The horizontal axes represent the time variable $t \in [1,5]$ and the solution values $u(t)$, while the vertical axis displays the computed Ψ-values. The smooth upward surface indicates how Ψ increases simultaneously with t and $u(t)$, reflecting the combined effect of the nonlinear sine–cosine terms, the exponential component, and the delayed argument $u(5t)$. This visualization helps to understand the operator's behavior and supports the analysis of the system's stability and controllability characteristics.

Example 6.2. We examine the Caputo–Hadamard Volterra-Fredholm fractional integro-differential equation subject to the imposed boundary criteria

$$
\begin{aligned}
&{}^{CH}D_{1+}^{1/7}u(t)\\
&=\frac{1}{2}\left\{0.01\,\sin(u(t)) + 0.01\,\cos(u(\frac{1}{2}t)) + \frac{1}{36}\int_1^t e^{-s/9}u(s)\,ds + \frac{1}{36}\int_1^5 (4-e^{-s/3})u(s)\,ds + (Bu)(t)\right\},\\
&t \in [1,5],
\end{aligned}
\tag{6.4}
$$

with boundary condition

$$u(1) = 0, \quad u(5) = 1.$$

The parameters associated with the theoretical framework of Theorem 5.1 are selected as follows:

$$M_1 = 0.1, \qquad M_2 = 0.5,$$
$$\|\mathfrak{A}\| = 0.01, \qquad \|\mathfrak{B}\| = 0.01,$$
$$T = 5, \qquad \kappa = \frac{1}{7}, \qquad \lambda = \frac{1}{2},$$
$$M_\psi = 0.01, \qquad M_\lambda = 0.01,$$
$$M_K = 0.0028, \qquad M_Q = 0.0028.$$

The controllability condition is verified by calculating the value of

$$\begin{aligned}\mathfrak{N} &= (1 + M_1, M_2)\Big(2\|\mathfrak{A}\| + \|\mathfrak{B}\| + \frac{2(\log T)^\kappa}{\Gamma(\kappa+1)}\|(\mathfrak{M}_\psi, \mathfrak{M}_\lambda, \mathfrak{M}_\mathfrak{K}, \mathfrak{M}_\mathfrak{Q})\|\Big). \\ &= (1 + 0.1(0.5))(2(0.01) + 0.01 + \frac{2(1.067)}{0.935}(0.0584)) \\ &= (1 + 1.05)(2(0.01) + 0.01 + 2.28(0.0256)) \\ &= 0.0926 < 1\end{aligned} \tag{6.5}$$

Since, $\mathfrak{N} < 1$, all the requirements of Theorem 5.1 are verified. Hence, it follows that a controllable element corresponding to the given system exists on the interval $\mathfrak{J}$.

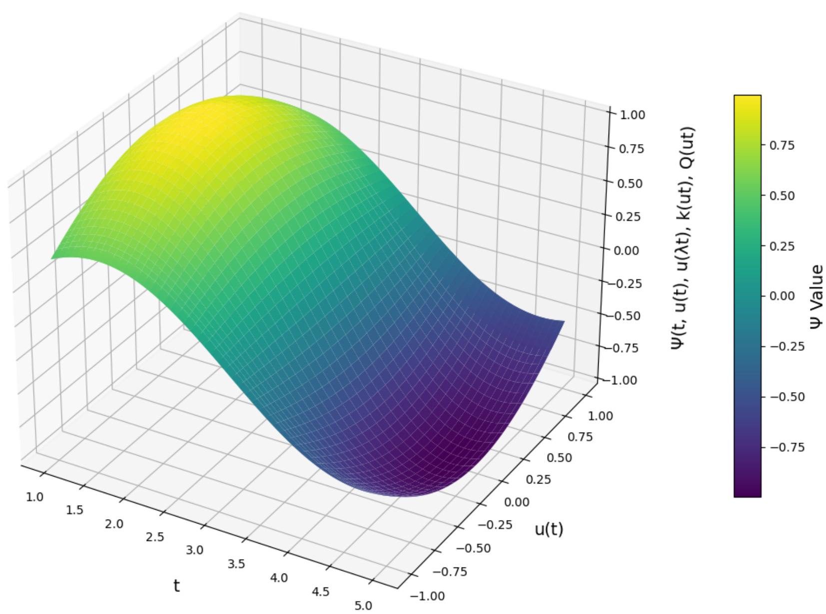

FIGURE 3. Graphical representation of the solution $u(t)$ over time and initial condition for example 6.2

The graphical representation behavior of the nonlinear operator $\Psi(t, u(t))$ corresponding to the Caputo–Hadamard (FIDEs)under consideration. The horizontal axes represent the time variable

$t \in [1,5]$ and the solution values $u(t)$, while the vertical axis displays the computed values of $\Psi(t, u(t))$. This graphical representation shows how the delayed term $u(\lambda t)$, the integral kernels, and the control operator $(Bu)(t)$ collectively influence the system's dynamic behavior.

7. Conclusion

The principal objectives of this study have been accomplished. Through the application of Banach's contraction principle along with the fixed point theorem of Schauder's, we established rigorous well- posedness results for non-negative results of the nonlinear Caputo–Hadamard VFFIDEs. Additionally, the system's Ulam–Hyers and Ulam–Hyers–Rassias stability properties, as well as its controllability, were demonstrated under appropriate assumptions. A illustrating example was provided to confirm the applicability of the theoretical analysis. These findings collectively validate the effectiveness and completeness of the proposed framework.

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