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Global Attractivity of Matrix Difference Equations Using Enriched Jungck Contractions

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Abstract. This paper intuitively gives a justifiable approach to enriched-type Jungck contraction. In [11][Common fixed points theorems for enriched Jungck contractions in Banach spaces. J. Fixed Point Theory Appl. 23(4) (2021) 76], we have noted that there is a technical error in the proof of the main theorem as well as the example and so we argue with a counter-example. Moreover, we correct and prove it under an enriched general class of Jungck contraction using weak compatibility, and an example is given to validate our result. In addition, we obtain the fixed points under generalized enriched Prešić type contraction. Finally, we apply our result in proving the existence and uniqueness of globally attractive equilibrium points for matrix difference equations.

1. Introduction and preliminaries

For each self-mapping f on a nonempty set X, a point $\nu \in X$ is called a fixed point of f if $f(\nu) = \nu$, and we denote the set of all fixed points of T by $Fix(T)$. The averaged mapping associated with T is defined by $T_\lambda(x) = (1-\lambda)I + \lambda T$, where I is the identity mapping and $\lambda \in (0,1]$. In 1922, Banach [1] developed the important result in metric space theory to ensure the existence and uniqueness of the fixed point of self-mapping on a metric space. The Banach contraction principle effectively states that in a complete metric space (X,d) every contraction mapping $f : X \to X$, that is, any mapping for which there exists $k \in [0,1)$ such that

$$d(fx, fy) \le kd(x,y), \ \forall\, x, y \in X$$

has a unique fixed point. Moreover, for every $\nu_0 \in X$, the fixed point of f can be approximated by the Picard iteration $\{\nu_n\}$, defined by $f(\nu_{n-1}) = \nu_n$, for all $n \in \mathbb{N}$. In the previous nine decades or so, highly remarkable literature has been developed ([31], [41], [42], [43], [44], [45]) using Banach

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contraction principle. Furthermore, some of its extensions have shown to be extremely helpful and adaptable in handling a wide range of nonlinear problems, including differential equations, integral equations, optimization problems, variational inequalities, economics and many others (see [8], [23], [24], [25], [26], [27], [39] and references therein). Berinde and Păcurar [5] recently improved the Banach contraction principle (in the case of Banach space) by proving the fixed point result of enriched contraction mappings using Picard iteration of the averaged mapping. This theorem is stated as follows:

Theorem 1.1. [5] *Let T be a self-mapping on a Banach space X. If the mapping T is an enriched contraction, that is, there exists $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\|, \ \forall \ x, y \in X.$$

Then T has a unique fixed point and there is $\lambda \in (0, 1]$ such that for each $x_0 \in X$, the Krasnoselskij iteration $\{x_n\}$ given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, n \geq 0,$$

converges to a unique fixed point of T.

The proof of Theorem 1.1 consists of three fundamental phases, which are as follows:

(i) to prove the existence of the fixed point of the averaged mapping;
(ii) to show the uniqueness of the fixed point of the averaged mapping;
(iii) to utilize the well known fact that $Fix(T) = Fix(T_\lambda)$.

Because of these factors, the study of enriched mappings has nowadays become one of the most important and active research fields in nonlinear analysis. Given the importance of the averaged mapping in the construction of enriched mappings, recently, Păcurar [12] provided an analog of the averaged mapping for the case of operators defined on product spaces.

Definition 1.1. [12] *Let $(X, +, \cdot)$ be a linear space, k be a positive integer, and $T : X^k \to X$ be an operator. For $\lambda_0, \lambda_1, \ldots, \lambda_k \geq 0$, with $\sum_{i=0}^k \lambda_i = 1$ and $\lambda_k \neq 0$, the operator $T_\lambda : X^k \to X$ defined by*

$$T_\lambda(u_0, u_1, \ldots, u_{k-1}) = \lambda_0 u_0 + \lambda_1 u_1 + \cdots + \lambda_{k-1}u_{k-1} + \lambda_k T(u_0, u_1, \ldots, u_{k-1}), \tag{1.1}$$

$u_0, u_1, \ldots, u_{k-1} \in X$, is called the averaged mapping corresponding to T.

Remark 1.1. *For $k = 1$, the equation (1.1) simply reduced to $T_\lambda(u_0) = \lambda_0 u_0 + \lambda_1 T(u_0), u_0 \in X$, where $\lambda_0 + \lambda_1 = 1$, which is the usual definition of averaged mapping.*

Remark 1.2. *In the case of averaged mapping, when $T : X^k \to X$, it is easy to prove that $\bar{u} \in X$ is a fixed point of T if and only if it is a fixed point of the corresponding T_λ, where $T_\lambda : X^k \to X$, for some $\lambda_i \geq 0$, with $\sum_{i=0}^k \lambda_i = 1$ and $\lambda_k \neq 0$.*

Let $\bar{u} \in X$ be such that $T_\lambda(\bar{u}, \bar{u}, \ldots, \bar{u}) = \bar{u}$, so that

$$\lambda_0\bar{u} + \lambda_1\bar{u} + \cdots + \lambda_{k-1}\bar{u} + \lambda_k T(\bar{u}, \bar{u}, \ldots, \bar{u}) = \bar{u}.$$

It follows that,

$$(1-\lambda_k)\bar{u}+\lambda_k T(\bar{u},\bar{u},\ldots,\bar{u})=\bar{u}.$$

Since $\lambda_k \neq 0$, we have $T(\bar{u},\bar{u},\ldots,\bar{u})=\bar{u}$. The converse is also immediate.

There are several approaches to studying the concept of enriched mappings. In [7], Berinde initiated one of the outstanding works in this area, where he introduced the class of enriched nonexpansive mappings in Hilbert spaces by Krasnoselskij averaged mapping. He also employed the Krasnoselskij iteration to approximate the fixed points of enriched nonexpansive mappings. Afterward, Berinde and Păcurar [9] gave the concept of enriched Kannan mappings, which encompass all Kannan mappings as well as certain nonexpansive mappings, and they investigated the set of fixed points and showed a convergence theorem for enriched Kannan mappings using Krasnoselskij iteration in Banach spaces. Then they extend these mappings to the class of enriched Bianchini mappings. In 2021, the same authors studied [10] the class of enriched Chatterjea contractions, and subsequently provided some fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces in [6]. Păcurar recently presented [12] a general class of enriched Prešić-type operators and proved the convergence of two separate iterative approaches to the fixed point by establishing some criteria under which enriched Prešič operators have a unique fixed point. She also provided a data dependency finding, which is used to demonstrate the global asymptotic stability of the equilibrium of the k-th order difference equation.

Difference equations are typically used to explain how particular circumstances change over time. For instance, if a population is made up of discrete generations, then the m-th generation $v(m)$ determines the size of the $(m+1)-$st generation $v(m+1)$ as a function of the m-th generation $v(m)$. The difference equation expresses this relationship

$$v(m+1)=f(v(m)).$$

The study of global random attractors was started by Rulle [47]. Later, In [48], global attraction was considered for a particular type of higher order nonlinear difference equations. For additional information, see ([49], [50]). In 2016, Abbas et al. [46] derived global attractivity for the following sequence $\{U_n\} \subset \sigma(N)$,

$$U_{n+k}=Q_1+\frac{1}{k}\sum_{i=0}^{k-1}A^*\xi(U_{n+i})A, \quad n=1,2,\ldots,$$

where $\sigma(N)$ is the set of all $N \times N$ Hermitian positive definite matrices, Q is $N \times N$ Hermitian positive semidefinite matrix, A is an $N \times N$ nonsingular matrix, A^* is the conjugate transpose of A, $\psi : \sigma(N) \to \sigma(N)$. Now we shall see some basic definitions and concepts that will be needed in the sequel.

Definition 1.2. [51] *Let X be a non empty set, and $f, g : X \to X$ be two operators. An element $u \in X$ is said to be*

(i) *a coincidence point of f and g if*

$$f(u) = g(u).$$

The value $p = f(u) = g(u)$ is a coincidence value of f and g;

(ii) *a common coincidence point of f and g if*

$$f(u) = g(u) = u.$$

Definition 1.3. *Let $(X, \|.\|)$ be a normed linear space. The mappings $f, g : X \to X$ are said to be*

(i) *commuting* [2] *if*

$$fgx = gfx, \ \forall\, x \in X,$$

(ii) *weakly commuting* [19] *if*

$$\|fgx - gfx\| = \|fx - gx\|, \ \forall\, x \in X,$$

(iii) *compatible* [3] *if*

$$\lim_{n\to\infty} \|fgx_n - gfx_n\| = 0$$

whenever $\{x_n\} \in X$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, for some $t \in X$,

(iv) *weakly compatible* [4] *if they commute at their coincidence points. That is,*

$$f(g(u)) = g(f(u)) = u,$$

where u is any coincidence point of f and g.

Remark 1.3. [33] *Let $f : X^k \to X$, k a positive integer, we define its associate operator $F : X \to X$ by*

$$F(u) = f(u, u, \ldots, u), u \in X. \tag{1.2}$$

Clearly, $u \in X$ is a fixed point of f if and only if it is a fixed point of its associate operator F.

Definition 1.4. [33] *Let X be a nonempty set, k a positive integer, and $f : X^k \to X$, $g : X \to X$ be two operators, where F is an associate operator of f.*

(i) *An element $u \in X$ is said to be a coincidence point of f and g if it is a coincidence point of F and g.*
(ii) *The value $p = f(u) = g(u)$ is a coincidence value of f and g if it is a coincidence point of F and g.*
(iii) *An element $u \in X$ is said to be a common coincidence point of f and g if it is a common coincidence point of F and g.*

Definition 1.5. [33] *Let X be a nonempty set, k a positive integer, and $f : X^k \to X$, $g : X \to X$ be two operators. Then f and g are said to be weakly compatible if F and g are weakly compatible.*

Remark 1.4. *The following relationship between commutativity and compatibility was discovered in* [4], [38], [40];

(i) *commuting mappings $\Rightarrow$ weakly commuting mappings $\Rightarrow$ compatible mappings.*
(ii) *However, compatible mappings are not necessarily weakly commuting.*
(iii) *Also, compatible mappings $\Rightarrow$ weakly compatible, but weakly compatible mappings need not be compatible.*

For further information on these concepts, see ([19], [20], [22], [28], [29], [30], [32], [34], [35], [36], [37], [38], [40]).

In 1936, Jungck [3] used a constructive approach of sequence to prove a common fixed point theorem for commuting mappings in metric spaces. This work served as a foundation for other popular fixed point theorems that use commuting and noncommuting mappings that meet contractive requirements. In [16] and [17], Prešić proposed a new sort of generalization. Later on, Ćirić and Prešić [13] proved the following result.

Theorem 1.2. [13] *Let (X, d) be a complete metric space, k a positive integer, and $f : X^k \to X$ be a mapping satisfying*

$$\begin{aligned} d(f(u_1, u_2, \dots, u_k), f(u_2, u_3, \dots, u_{k+1})) \\ \leq \alpha \max\{d(u_i, u_{i+1})\ ;\ \ 1 \leq i \leq k\}, \end{aligned}$$

where $\alpha \in (0, 1)$ and $u_1, u_2, \dots, u_{k+1} \in X$. Then

(a) *f has a fixed point $\bar{u} \in X$;*

(b) *for any $u_1, u_2, u_3, \dots, u_k \in X$ and for $n \in \mathbb{N}$,*

$$u_{n+k} = f(u_n, u_{n+1}, \dots, u_{n+k-1}),$$

then the sequence $\{u_n\}_{n=1}^{\infty}$ is convergence to $\bar{u}$ and

$$\lim u_n = f(\lim u_n, \lim u_n, \lim u_n, \dots, \lim u_n);$$

(b) *in addition, if we suppose that*

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y),$$

for all $x, y \in X$, with $x \neq y$, then $\bar{u}$ is the unique fixed point of f.

2. Erratum

In [11], Marchiş extended Jungck's [3] work and proved common fixed point theorem for a pair of single-valued enriched Banach contraction which satisfies weak commutativity condition in Banach spaces. The theorem is given below.

Theorem 2.1. [11] *Let f and g be self-mappings on a Banach space X. If the mappings f and g are enriched Jungck contractions, that is, there exists $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that*

$$\|b(x - y) + fx - fy\| \leq \theta\|gx - gy\|,\ \forall\ x, y \in X, \tag{2.1}$$

and the following conditions are satisfied:

(i) *f_λ and g are weakly commuting mappings, where*

$$f_\lambda(x) = (1 - \lambda)x_n + \lambda f x_n, n \geq 0;$$

(ii) *$f_\lambda(X) \subset g(X)$, and both f_λ and g are continuous.*

Then $Fix(f) = Fix(g) = \{a\}$. Moreover, there exists $\lambda \in (0,1]$ such that the iterative method $\{gx_{n+1}\}_{n=0}^{\infty}$ converges to a.

Remark 2.1. (i) But indeed, in the proof of Theorem 2.1, page-5, line 31 in [11], it is given that,

$$\|f_\lambda f_\lambda x_n - f_\lambda a_1\| \leq c \cdot \|g f_\lambda x_n - g a_1\| < \|g f_\lambda x_n - g a_1\|,$$

allowing $n \to \infty$ in the above inequality,

$$\|f_\lambda a - f_\lambda a_1\| < \|ga - ga_1\|,$$

from this, we cannot conclude that $f_\lambda a = f_\lambda a_1$ using the given hypothesis. However, the author obtained incorrectly that $f_\lambda a = f_\lambda a_1$ and used this key fact throughout the proof to arrive at the conclusion.

(ii) Also, on page-7, line 17 in Example 3.3 of Theorem 2.1 in [11], it is proven that,

$$\left|(b-1)(x-y)\right| \leq \theta \cdot \left|\frac{(x-y)(4x+4y-1)}{5}\right|.$$

But if we take $x = 0$, $y = 1/4$ and $b = 5/7$ in the above inequality, we get

$$\frac{1}{14} \leq 0,$$

which is absurd.

Now, we present a counter-example which satisfies all the requirements of Theorem 2.1 but not its conclusion.

Example 2.1. *Consider the Banach space $X = \mathbb{R}$ with the usual norm. Define two functions $f, g : X \to X$ by $f(x) = \frac{1}{5}$, $g(x) = x + 1$, for all $x \in X$. Also take $b = \frac{1}{4}, \lambda = \frac{2}{3}$ and $\theta = \frac{1}{2}$.*

By given assumptions, $f_\lambda(x) = \frac{5x+12}{15}$, for all $x \in X$. First, we verify that f and g satisfy the enriched Junck contraction. Indeed, we have

$$\begin{aligned} \left|b(x-y) + \frac{1}{5} - \frac{1}{5}\right| &\leq \theta|x+1-y-1| \\ &= \theta|x-y|, \\ \text{implies, } \quad |b(x-y)| &\leq \theta|x-y| \\ b &\leq \theta, \text{for all } \ x \neq y \in X. \end{aligned}$$

Hence equation (2.1) is true, and so f and g satisfied enriched Junck's contraction. Next, we prove that f_λ and g satisfy the weakly commuting condition.

$$|f_\lambda(g(x)) - g(f_\lambda(x))| = \left|f_\lambda(1+x) - g\left(\frac{5x+1}{15}\right)\right|$$

$$\begin{aligned} &= \left|\frac{5(1+x)+2}{15} - \left(\frac{5x+2}{15}+1\right)\right| \\ &= \left|\frac{7+5x}{15} - \left(\frac{5x+17}{15}\right)\right| \\ &= \left|\frac{7+5x-5x-17}{15}\right| \\ &= \frac{10}{15}. \end{aligned}$$

Thus, $|f_\lambda(g(x)) - g(f_\lambda(x))| = \dfrac{2}{3}$ for all $x \in X$.
Now,

$$\begin{aligned} |f_\lambda(x) - g(x)| &= \left|\left(\frac{5x+2}{15}\right) - 1 - x\right| \\ &= \left|\frac{5x+2-15x-15}{15}\right| \\ &= \left|\frac{-10x-13}{15}\right| \\ &= \frac{10x+13}{15}. \end{aligned}$$

Hence, $|f_\lambda(x) - g(x)| = \dfrac{10x+13}{15}$ for all $x \in X$. Therefore, it is clear that

$$\frac{10}{15} \leq \frac{10x+13}{15}, \text{ for all } x \in X.$$

So, f_λ and g are weakly commuting mappings. Now, we note that all the conditions of Theorem 2.1 are satisfied, however, g does not have any fixed point, while f has a fixed point $\dfrac{1}{5}$.
Hence, Theorem 2.1 and further results use Theorem 2.1, they all turn out to be invalid.

Remark 2.2.

To address this issue, we use the notion of weak compatibility instead of weak commuting, which beneficially helps us prove Theorem 2.1 with the same conclusions but with a different proof technique. Undoubtedly, the support of an averaged mapping is also needed.

3. Main results

In this section, we first obtain a fixed point result for generalized enriched Prešić-type contractions.

Theorem 3.1. *Let $(X, \|.\|)$ be a Banach space, k a positive integer, and $f : X^k \to X$ be a mapping for which there exist $b_i \geq 0, i = 0, 1, \ldots, k-1$ and $(\alpha - \sum_{i=0}^{k-1} b_i) < 1$ such that*

$$\left\|\sum_{i=1}^{k} b_i(u_i - u_{i+1}) + f(u_0, u_1, \ldots, u_{k-1}) - f(u_1, u_2, \ldots, u_k)\right\|$$

$$\leq \alpha \max\{\|fu_i - fu_{i+1}\|\ ;\ \ 1 \leq i \leq k\}, \tag{3.1}$$

where $u_1, u_2, \ldots, u_{k+1} \in X$. *Then*

(a) f *has a fixed point* $\bar{u} \in X$;

(b) *for any* $u_1, u_2, u_3, \ldots, u_k \in X$ *and for* $n \in \mathbb{N}$,

$$u_{n+k} = f(u_n, u_{n+1}, \ldots, u_{n+k-1}),$$

then the sequence $\{u_n\}_{n=1}^{\infty}$ *is convergent to* $\bar{u}$ *and*

$$\lim u_n = f(\lim u_n, \lim u_n, \lim u_n, \ldots, \lim u_n);$$

(c) *moreover, if we assume that*

$$\|f(x, x, \ldots, x) - f(y, y, \ldots, y)\| < \|x - y\|,$$

for all $x, y \in X$, *with* $x \neq y$, *then* $\bar{u}$ *is the unique fixed point of* f.

Proof. For any $u_1, u_2, \ldots, u_k \in X$, define a sequence $\{u_n\}$ as follows;

$$u_{n+k} = f(u_n, u_{n+1}, \ldots, u_{n+k-1}),\ n = 1, 2, \ldots$$

Denote

$$S_n = \left\| \sum_{i=0}^{n-1} b_i(u_i - u_{i+1}) + u_n - u_{n+1} \right\|.$$

We shall prove by induction that for each $n \in \mathbb{N}$,

$$S_n \leq M\mu^n. \tag{3.2}$$

Where $M = \max\left\{\frac{S_1}{\mu^1}, \frac{S_2}{\mu^2}, \ldots, \frac{S_k}{\mu^k}\right\}$ and $\mu = \alpha^{1/k}$. By the definition of M, it is easy to see that (3.2) is true for $n = 1, 2, 3, \ldots, k$. Thus by induction hypothesis, we have,

$$S_n \leq M\mu^n, S_{n+1} \leq M\mu^{n+1}, \ldots, S_{n+k-1} \leq M\mu^{n+k-1}.$$

Now, we prove that $S_n \leq M\mu^{n+k}$.

$$\begin{aligned} S_{n+k} &= \left\| \sum_{i=n}^{n+k-1} b_i(u_i - u_{i+1}) + u_{n+k} - u_{n+k+1} \right\| \\ &= \left\| \sum_{i=n}^{n+k-1} b_i(u_i - u_{i+1}) + f(u_n, u_{n+1}, \ldots, u_{n+k-1}) - f(u_{n+1}, u_{n+2}, \ldots, u_{n+k}) \right\| \\ &\leq \alpha \max\{\|u_n - u_{n+1}\|, \|u_{n+1} - u_{n+2}\|, \ldots, \|u_{n+k-1} - u_{n+k}\|\}. \end{aligned} \tag{3.3}$$

We note that,

$$\begin{aligned} \|u_n - u_{n+1}\| &\leq \left\| \sum_{i=0}^{n-1} b_i(u_i - u_{i+1}) + u_n - u_{n+1} \right\|, \\ \|u_{n+1} - u_{n+2}\| &\leq \left\| \sum_{i=1}^{n} b_i(u_i - u_{i+1}) + u_{n+1} - u_{n+2} \right\|, \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\|u_{n+k-1}-u_{n+k}\| \leq \left\|\sum_{i=n-1}^{n+k-2} b_i(u_i-u_{i+1})+u_{n+k-1}-u_{n+k}\right\|.$$

Now, (3.3) can be written as

$$\begin{aligned}
S_{n+k} &\leq \alpha \max\Bigg\{\left\|\sum_{i=0}^{n-1} b_i(u_i-u_{i+1})+u_n-u_{n+1}\right\|, \left\|\sum_{i=1}^{n} b_i(u_i-u_{i+1})+u_{n+1}-u_{n+2}\right\|, \dots, \\
&\quad \left\|\sum_{i=n-1}^{n+k-2} b_i(u_i-u_{i+1})+u_{n+k-1}-u_{n+k}\right\|\Bigg\} \\
&\leq \alpha \max\{M\mu^n, M\mu^{n+1}, \dots, M\mu^{n+k-1}\} \\
&= \alpha M\mu^n \\
&= M\mu^{n+k}.
\end{aligned}$$

Hence, we have $S_{n+k} \leq M\mu^{n+k}$. Next we prove that $\{u_n\}$ is Cauchy.

$$\begin{aligned}
\|u_n-u_{n+p}\| &\leq \|u_n-u_{n+1}\|+\|u_{n+1}-u_{n+2}\|+\cdots+\|u_{n+p-1}-u_{n+p}\| \\
&\leq \left\|\sum_{i=0}^{n-1} b_i(u_i-u_{i+1})+u_n-u_{n+1}\right\|+\left\|\sum_{i=1}^{n} b_i(u_i-u_{i+1})+u_{n+1}-u_{n+2}\right\|+\dots \\
&\quad +\left\|\sum_{i=n-1}^{n+p-2} b_i(u_i-u_{i+1})+u_{n+p-1}-u_{n+p}\right\| \\
&\leq M\mu^n+M\mu^{n+1}+\cdots+M\mu^{n+p-1} \\
&= M\mu^n(1+\mu+\mu^2+\cdots+\mu^{p-1}) \\
&= M\mu^n/(1-\mu).
\end{aligned}$$

Therefore, we conclude that $\{u_n\}$ is Cauchy. Since $(X, \|.\|)$ is Banach space, there exists $\bar{u}$ in X such that $\lim_{n\to\infty} u_n = \bar{u}$. Then for each integer n, we have that

$$\begin{aligned}
\|\bar{u}-f(\bar{u},\bar{u},\dots,\bar{u})\| &\leq \|\bar{u}-u_{n+k}\|+\|u_{n+k}-f(\bar{u},\bar{u},\dots,\bar{u})\| \\
&= \|\bar{u}-u_{n+k}\|+\|f(u_n,u_{n+1},\dots,u_{n+k-1})-f(\bar{u},\bar{u},\dots,\bar{u})\| \\
&\leq \|\bar{u}-u_{n+k}\|+\|f(\bar{u},\bar{u},\dots,\bar{u})-f(\bar{u},\bar{u},\dots,u_n)\| \\
&\quad +\|f(\bar{u},\bar{u},\dots,u_n)-f(\bar{u},\bar{u},\dots,u_n,u_{n+1})\|+\dots\dots \\
&\quad +\|f(\bar{u},u_n,u_{n+1},\dots,u_{n+k-2})-f(u_n,u_{n+1},\dots,u_{n+k-1})\| \\
&\leq \|\bar{u}-u_{n+k}\|+\left\|\sum_{i=0}^{n-1} b_i(u_i-u_{i+1})+f(\bar{u},\bar{u},\dots,\bar{u})-f(\bar{u},,\dots,u_n)\right\| \\
&\quad +\left\|\sum_{i=1}^{n} b_i(u_i-u_{i+1})+f(\bar{u},\bar{u},\dots,u_n)-f(\bar{u},\bar{u},\dots,u_n,u_{n+1})\right\|+\dots
\end{aligned}$$

$$
\begin{aligned}
& + \left\| \sum_{i=n-1}^{n+p-2} b_i(u_i - u_{i+1}) + f(\bar{u}, u_n, u_{n+1}, \ldots, u_{n+k-2}) \right. \\
& \left. - f(u_n, u_{n+1}, \ldots, u_{n+k-1}) \right\| \\
\leq \; & \|\bar{u} - u_{n+k}\| + \alpha\|\bar{u} - u_n\| + \alpha \max\{\|\bar{u} - u_n\|, \|u_n - u_{n+1}\|\} \\
& + \cdots + \alpha \max\{\|\bar{u} - u_n\|, \|u_n - u_{n+1}\|, \ldots, \|u_{n+k-2} - u_{n+k-1}\|\}.
\end{aligned}
$$

It is now immediate that

$$\|\bar{u} - f(\bar{u}, \bar{u}, \ldots, \bar{u})\| \leq 0 \text{ as } n \to \infty.$$

Hence, we conclude that $f(\bar{u}, \bar{u}, \ldots, \bar{u}) = \bar{u}$. Thus, $\lim u_n = f(\lim u_n, \lim u_n, \ldots, \lim u_n)$. Now, we show that the fixed point of f is unique. Suppose to the contrary that $\bar{v}$ and $\bar{u}$ are two distinct fixed points of f in X. Then $f(\bar{v}, \bar{v}, \ldots, \bar{v}) = \bar{v}$ and $f(\bar{u}, \bar{u}, \ldots, \bar{u}) = \bar{u}$. We have that $\|\bar{u} - \bar{v}\| = \|f(\bar{u}, \bar{u}, \ldots, \bar{u}) - f(\bar{v}, \bar{v}, \ldots, \bar{v})\| < \|\bar{u} - \bar{v}\|$, a contradiction. Hence, $\bar{u}$ is the unique fixed point of f. □

Next, we prove our main result which is the correct version of Theorem 2.1 in product spaces using the concept of weak compatibility.

Theorem 3.2. *Let $(X, \|.\|)$ be a Banach space and k a positive integer. Assume $f : X^k \to X$ and $g : X \to X$ are two mappings for which there exist $b_i \geq 0, i = 0, 1, \ldots, k-1$ and $(\alpha - \sum_{i=0}^{k-1} b_i) < 1$ such that*

$$
\begin{aligned}
& \left\| \sum_{i=0}^{k-1} b_i(u_i - u_{i+1}) + f(u_0, u_1, \ldots, u_{k-1}) - f(u_1, u_2, \ldots, u_k) \right\| \\
& \qquad \leq \alpha \max\{\|gu_i - gu_{i+1}\| \,;\; 0 \leq i \leq k-1\},
\end{aligned} \tag{3.4}
$$

where f_λ is an averaged mapping corresponding to f. Then the following assertions holds;

(i) *If $f_\lambda(X^k) \subset Y \subseteq g(X)$, then f_λ and g have a unique coincidence point.*
(ii) *If $f_\lambda(x)$ and $g(x)$ are weakly compatible, then f and g have a unique common coincidence point.*
(iii) *The sequence $\{z_n\}$ defined by $\{z_n\} = f_\lambda(u_n, u_n, \ldots, u_n) = g(u_{n+1}), n \in \mathbb{N}$ converges to the common fixed point of f and g.*

Proof. Let $u_0 \in X$. Then $f_\lambda(u_0, u_0, \ldots, u_0) \in f_\lambda(X^k) \subset g(X)$, thus there exists $u_1 \in X$ such that

$$f_\lambda(u_0, u_0, \ldots, u_0) = g(u_1).$$

Further, $f_\lambda(u_1, u_1, \ldots, u_1) \in f_\lambda(X^k) \subset g(X)$, so there exists $u_2 \in X$ such that

$$f_\lambda(u_1, u_1, \ldots, u_1) = g(u_2).$$

In this way, we construct a sequence $\{g(u_n)\}_{n\geq 0}$, $u_0 \in X$ and

$$f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_{n-1}) = g(u_n), n \geq 1.$$

Now, let us denote $\hat{\zeta} = \sum_{i=0}^{k-1} b_i > 0$, and $\lambda_k = \dfrac{1}{\hat{\zeta}+1}$, k a positive integer. Take

$$\lambda_i = \lambda_k b_i, i = 0, 1, 2, \ldots, k-1. \tag{3.5}$$

On the other hand,

$$\sum_{i=0}^{k} \lambda_i = \sum_{i=0}^{k-1} \lambda_i + \lambda_k = \sum_{i=0}^{k-1} \lambda_k b_i + \lambda_k = \lambda_k(\hat{\zeta}+1).$$

Using (3.5) in (3.4), we obtain

$$\left\| \sum_{i=0}^{k-1} \frac{\lambda_i}{\lambda_k}(u_i - u_{i+1}) + f(u_0, u_1, \ldots, u_{k-1}) - f(u_1, u_2, \ldots, u_k) \right\| \\ \leq \alpha \max\{\|gu_i - gu_{i+1}\| \,;\ \ 0 \leq i \leq k-1\},$$

multiplying $\lambda_k > 0$, we have

$$\left\| \sum_{i=0}^{k-1} \lambda_i(u_i - u_{i+1}) + \lambda_k f(u_0, u_1, \ldots, u_{k-1}) - \lambda_k f(u_1, u_2, \ldots, u_k) \right\| \\ \leq \alpha\lambda_k \max\{\|gu_i - gu_{i+1}\| \,;\ \ 0 \leq i \leq k-1\},$$

implies,

$$\left\| \sum_{i=0}^{k-1} \lambda_i u_i - \sum_{i=0}^{k-1} \lambda_i u_{i+1} + \lambda_k f(u_0, u_1, \ldots, u_{k-1}) - \lambda_k f(u_1, u_2, \ldots, u_k) \right\| \\ \leq \varkappa \max\{\|gu_i - gu_{i+1}\| \,;\ \ 0 \leq i \leq k-1\},$$

where $\alpha\lambda_k = \varkappa$, for $i = 0, 1, 2, \ldots, k-1$. Now, by using the definition of averaged mapping, we deduce that

$$\|f_\lambda(u_0, u_1, \ldots, u_{k-1}) - f_\lambda(u_1, u_2, \ldots, u_k)\| \leq \varkappa \max\{\|gu_i - gu_{i+1}\| \,;\ \ 0 \leq i \leq k-1\}. \tag{3.6}$$

For any $n \geq 1$, we have

$$\begin{aligned} \|gu_n - gu_{n+1}\| &= \|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_{n-1}) - f_\lambda(u_n, u_n, \ldots, u_n)\| \\ &\leq \|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_{n-1}) - f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n)\| \\ &\quad + \|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n) - f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n, u_n)\| \\ &\quad + \|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n, u_n) - f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n, u_n, u_n)\| \\ &\quad + \cdots + \|f_\lambda(u_{n-1}, u_n, \ldots, u_n) - f_\lambda(u_n, u_n, \ldots, u_n)\|. \end{aligned} \tag{3.7}$$

Now by (3.6), we have the following observation,

$$\|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_{n-1}) - f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n)\| \\ \leq \varkappa \max\{0, 0, , \ldots, 0, \|gu_{n-1} - gu_n\|\},$$

$$\|f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n) - f_\lambda(u_{n-1}, u_{n-1}, \ldots, u_n, u_n)\| \\ \leq \varkappa \max\{0, 0, , \ldots, \|gu_{n-1} - gu_n\|, 0\},$$

$$\dots,$$

$$\|f_\lambda(u_{n-1},u_n,\dots,u_n)-f_\lambda(u_n,u_n,\dots,u_n,u_n)\|$$
$$\leq \varkappa\max\{\|gu_{n-1}-gu_n\|,0,0,,\dots,0\}.$$

Substituting all the inequalities above in (3.7), we get

$$\begin{aligned}\|gu_n-gu_{n+1}\| &\leq \varkappa(\|gu_{n-1}-gu_n\|+\|gu_{n-1}-gu_n\|+\cdots+\|gu_{n-1}-gu_n\|)\\ &= \varkappa k(\|gu_{n-1}-gu_n\|).\end{aligned}$$

Take $\beta=\varkappa k$ (choose suitable α such that $\varkappa k<1$). Now by induction, we have

$$\|gu_n-gu_{n+1}\|\leq\beta^n(\|gu_0-gu_1\|), n\geq 0.$$

Hence, $\{gu_n\}$ is a Cauchy sequence. Thus, by our assumption there exists $\bar{u}$ in Y such that $\lim_{n\to\infty} g(u_n)=\bar{u}$, and since $Y\subseteq g(X)$, there exists $p\in X$ such that $g(p)=\bar{u}=\lim_{n\to\infty} g(u_n)$. Now, we shall prove that $g(p)=f(p,p,\dots,p)=\bar{u}$.

$$\begin{aligned}\|gu_n-f_\lambda(p,p,\dots,p)\| &= \|f_\lambda(u_{n-1},u_{n-1},\dots,u_{n-1})-f_\lambda(p,p,\dots,p)\|\\ &\leq \|f_\lambda(u_{n-1},u_{n-1},\dots,u_{n-1})-f_\lambda(u_{n-1},u_{n-1},\dots,p)\|\\ &\quad+\|f_\lambda(u_{n-1},u_{n-1},\dots,p)-f_\lambda(u_{n-1},u_{n-1},\dots,p,p)\|\\ &\quad+\|f_\lambda(u_{n-1},u_{n-1},\dots,p,p)-f_\lambda(u_{n-1},u_{n-1},\dots,p,p,p)\|\\ &\quad+\cdots+\|f_\lambda(u_{n-1},p,\dots,p)-f_\lambda(p,p,\dots,p)\|.\end{aligned}$$

By the similar observation as above, we have

$$\begin{aligned}\|gu_n-f_\lambda(p,p,\dots,p)\| &\leq \beta\|gu_{n-1}-gp\|\\ &\leq \beta\{\|gu_{n-1}-gu_n\|+|gu_n-gp\|\}.\end{aligned}$$

By induction, we get

$$\|gu_n-f_\lambda(p,p,\dots,p)\|\leq\beta^n\{\|gu_0-gu_1\|+|gu_n-gp\|\}.$$

Since $\beta<1$, we deduce that $\|gu_n-f_\lambda(p,p,\dots,p)\|\leq 0$ as $n\to\infty$. Hence we have $gu_n=f_\lambda(p,p,\dots,p)=gp=\bar{u}$. Now we claim that p is the only coincidence point for f_λ and g. Assume that there is some q in X such that $gq=f_\lambda(q,q,\dots,q)\neq\bar{u}$.

$$\begin{aligned}\|gp-gq\| &= \|f_\lambda(p,p,\dots,p)-g(q,q,\dots,q)\|\\ &\leq \||f_\lambda(p,p,\dots,p)-f_\lambda(p,p,\dots,p,q)\|\\ &\quad+\|f_\lambda(p,p,\dots,p,q)-f_\lambda(p,p,\dots,q,q)\|\\ &\quad+\|f_\lambda(p,p,\dots,q,q)-f_\lambda(p,p,\dots,q,q,q)\|\\ &\quad+\cdots+\|f_\lambda(p,q,\dots,q,q)-g(q,q,\dots,q)\|.\end{aligned}$$

It follows that, $\|gp-gq\|\leq\beta|gp-gq\|$, which is a contradiction. Hence p is the only coincidence point for f_λ and g. Now, we show that $\bar{u}$ is the unique common coincidence point of f_λ and g.

Since, $\bar{u}$ is the coincidence value of f_λ and g, and p is the coincidence point of f_λ and g, we have $f_\lambda(p, p, \ldots, p, p) = g(p) = \bar{u}$. Since F is the associated operator of f_λ and by weak compatibility, we get $F(\bar{u}) = F(g(p)) = g(F(p)) = g(\bar{u})$. That is $F(\bar{u}) = g(\bar{u})$, which shows $\bar{u}$ is the point of coincidence of both F and g. Therefore by definition, $\bar{u}$ is the point of coincidence for f_λ and g, but we proved that p is the only coincidence point of f_λ and g, which shows $\bar{u} = p$. Hence, $\bar{u}$ is the unique common coincidence point of f_λ and g. Moreover, by remark 1.2, we conclude that $\bar{u}$ is the unique common coincidence point of f and g. As, we proved $\bar{u} = \lim_{n\to\infty} g(u_{n+1})$, the sequence $\{z_n\} = f_\lambda(u_n, u_n, \ldots, u_n) = g(u_{n+1})$, converges to $\bar{u}$. □

Remark 3.1.

(i) In the particular case when $b = 0$ and $k = 1$, Theorem 3.2 reduces to Jungck's Theorem [3].

(ii) Similarly, if we take $b = 0, k = 1$ and g as identity mapping in Theorem 3.2, we get the Banach contraction principle. Meanwhile, when we take $k = 1$ and g as identity mapping in Theorem 3.2, we obtain the enriched Banach contraction principle.

Now, we see an example that validates Theorem 3.2.

Example 3.1. *Consider the Banach space $X = \mathbb{R}$ with the usual norm. For $k = 2$, define the mappings $f : X^2 \to X$ and $g : X \to X$ by*

$$f(x,y) = \begin{cases} \dfrac{x-y}{12} + \dfrac{1}{2}, & x+y < 2, \\ \dfrac{x-y}{10} + \dfrac{1}{2}, & x+y \geq 2, \end{cases}$$

and

$$g(x) = \frac{x}{2} + \frac{1}{4},$$

for all $x, y \in X$.

We first prove that,

$$\left| \sum_{i=0}^{1} b_i(u_i - u_{i+1}) + f(u_0, u_1) - f(u_1, u_2) \right| \leq \alpha \max\{|gu_0 - gu_1|, |gu_1 - gu_2|\}, \tag{3.8}$$

for any $u_0, u_1, u_2 \in \mathbb{R}$, by taking $\alpha = \frac{3}{4}, b_0 = 0$ and $b_1 = \frac{1}{20}$. Since $b_0 = 0$, the left-hand side of (3.8) becomes

$$\left| \frac{1}{20}(u_1 - u_2) + f(u_0, u_1) - f(u_1, u_2) \right|.$$

Now, the following 4 cases arise.

Case 1: $u_0 + u_1 < 2$ and $u_1 + u_2 < 2$.

Now,

$$f(u_0, u_1) = \frac{(u_0 - u_1)}{12} + \frac{1}{2}, \qquad f(u_1, u_2) = \frac{(u_1 - u_2)}{12} + \frac{1}{2}.$$

Thus,

$$f(u_0,u_1)-f(u_1,u_2)=\frac{u_0-u_2}{12}.$$

Now, by applying the properties of modulus,

$$\left|\frac{1}{20}(u_1-u_2)+f(u_0,u_1)-f(u_1,u_2)\right| \leq \frac{1}{12}|u_0-u_1|+\frac{2}{15}|u_1-u_2|.$$

On the other hand, $|gu_0-gu_1|=\frac{1}{2}|u_0-u_1|, \quad |gu_1-gu_2|=\frac{1}{2}|u_1-u_2|$.
Thus,

$$\alpha \max(|gu_0-gu_1|,|gu_1-gu_2|)=\frac{3}{8}\max(|u_0-u_1|,|u_1-u_2|).$$

Let $M=\max(|u_0-u_1|,|u_1-u_2|)$, then $|u_0-u_1|\leq M$ and $|u_1-u_2|\leq M$.
Therefore,

$$\frac{1}{12}M+\frac{2}{15}M\leq\frac{3}{8}M$$

is true.
Case 2: $u_0+u_1<2$ and $u_1+u_2\geq 2$.
Now,

$$f(u_0,u_1)=\frac{u_0-u_1}{12}+\frac{1}{2}, \qquad f(u_1,u_2)=\frac{u_1-u_2}{10}+\frac{1}{2}.$$

Thus,

$$f(u_0,u_1)-f(u_1,u_2)=\frac{(u_0-u_1)}{12}-\frac{(u_1-u_2)}{10}.$$

Hence,

$$\left|\frac{1}{20}(u_1-u_2)+f(u_0,u_1)-f(u_1,u_2)\right|\leq\frac{1}{12}|u_0-u_1|+\frac{1}{20}|u_1-u_2|.$$

Similarly, by taking $M=\max(|u_0-u_1|,|u_1-u_2|)$, the inequality

$$\frac{1}{12}M+\frac{1}{20}M\leq\frac{3}{8}M$$

holds.
Also, it is easy to see that, for **Case-3:** $u_0+u_1\geq 2$ and $u_1+u_2<2$, the inequality (3.8) holds.

Case 4: $u_0+u_1\geq 2$ and $u_1+u_2\geq 2$.

$$f(u_0,u_1)=\frac{u_0-u_1}{10}+\frac{1}{2}, \qquad f(u_1,u_2)=\frac{u_1-u_2}{10}+\frac{1}{2}.$$

Thus

$$f(u_0,u_1)-f(u_1,u_2)=\frac{u_0-u_2}{10}.$$

Hence

$$\left|\frac{1}{20}(u_1-u_2)+\frac{u_0-u_2}{10}\right|\leq\frac{1}{10}|u_0-u_1|+\frac{3}{20}|u_1-u_2|.$$

So that in this case as well, the inequality

$$\frac{1}{10}M+\frac{3}{20}M\leq\frac{3}{8}M$$

holds. Thus, in all the cases, we see that

$$\left|\sum_{i=0}^{1} b_i(u_i - u_{i+1}) + f(u_0, u_1) - f(u_1, u_2)\right| \le \alpha \max\{|gu_0 - gu_1|, |gu_1 - gu_2|\},$$

for all u_0, u_1 and $u_2 \in X$. We now consider the averaged mapping

$$f_\lambda(u_0, u_1) = \lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 f(u_0, u_1),$$

where $\lambda_0 = 0$, $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{2}{3}$.
Now, $f_\lambda\left(\frac{1}{2}, \frac{1}{2}\right) = 0 + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{2}$ and $g\left(\frac{1}{2}\right) = \frac{1}{2}$. Thus,

$$f_\lambda\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} = g\left(\frac{1}{2}\right).$$

Hence, clearly, $\frac{1}{2}$ is the unique coincidence point of f_λ and g. Moreover, we notice that f_λ and g are weakly compatible. Thus, by applying Theorem 3.2, f and g have a unique common coincidence point $\frac{1}{2}$.

4. Application to pair of matrix difference equations

Now, we shall study the global attractivity of the sequence $\{U_{n+k}\} \subset \sigma(N)^k$ defined by

$$U_{n+k} = Q_1 + \sum_{i=0}^{k-1} A^* \xi(U_{n+i}) A, \quad n = 1, 2, \ldots, \tag{4.1}$$

and $\{U_{n+1}\} \subset \sigma(N)$ by

$$U_{n+1} = Q_2 + A^* \psi(U_n) A, \quad n = 1, 2, \ldots \tag{4.2}$$

where $\sigma(N)$ is the set of all $N \times N$ Hermitian positive definite matrices, Q_1, Q_2 are $N \times N$ Hermitian positive semidefinite matrices, A is an $N \times N$ nonsingular matrix, A^* is the conjugate transpose of A, $\xi : \sigma(N)^k \to \sigma(N)$ and $\psi : \sigma(N) \to \sigma(N)$. First, we recall some basic definitions and important results.

Definition 4.1. *Let X be a non empty set, k a positive integer. Assume $f : X^k \to X$ is a mapping. For given $u_1, u_2, \ldots, u_k \in X$, define the sequence $\{u_n\}$ by*

$$u_{n+k} = f(u_n, u_{n+1}, \ldots, u_{n+k-1}), \quad n = 0, 1, 2, \ldots, \tag{4.3}$$

The point $\bar{u}$ is said to be equilibrium if it satisfies the condition:

$$\bar{u} = f(\bar{u}, \bar{u}, \ldots, \bar{u}).$$

Moreover, on the metric space (X, d), the equilibrium point $\bar{u}$ of the equation (4.3) *is said to be global attractor if for all $u_1, u_2, \ldots, u_k \in X$ we have*

$$d(u_n, \bar{u}) \to 0 \ \ as \ \ n \to \infty.$$

For $N \geq 2, \sigma(N)$ denotes the open convex cone of all $N \times N$ Hermitian positive definite matrices, endowed with the Thompson metric defined by

$$d(C, D) = \max\{\ln G(C/D), \ln G(D/C)\}, \ \forall \ C, D \in \sigma(N),$$

where $G(C/D) = \inf\{\tau > 0 : C \leq \tau D\} = \tau^+(D^{-1/2}CD^{-1/2})$ is the maximum eigenvalue of $D^{-1/2}CD^{-1/2}$. Here $R \leq S(S-R \geq 0)$ means that $S-R$ is positive semidefinite and $R < S(S-R > 0)$ means that $S-R$ is positive definite. Moreover, with respect to the Thompson metric d, $(\sigma(N), d)$ is complete [15] and

$$d(C, D) = \| \ln C^{-1/2}DC^{-1/2}\|,$$

where $\|.\|$ denotes the spectral norm. From [18], we observe that the Thompson metric exists on any open normal convex cones of real Banach spaces. We now see some of the important properties of Thompson metric:

(i) For any nonsingular matrix P and $C, D \in \sigma(N)$,

$$d(C, D) = d(C^{-1}, D^{-1}) = d(P^*CP, P^*DP).$$

(ii) $d(R^l, S^l) \leq ld(R, S)$, for all $R, S \in \sigma(N)$ and $l \in [0, 1]$.

(iii) For any nonsingular matrix P and $R, S \in \sigma(N)$,

$$d(P^*RP, P^*SP) \leq |l|d(R, S), l \in [1, -1].$$

(iv) For any $C, D, E, F \in \sigma(N)$,

$$d(C + D, E + F) \leq \max\{d(C, E), d(D, F)\}.$$

Assume that the mappings $\xi : \sigma(N)^k \to \sigma(N)$ and $\psi : \sigma(N) \to \sigma(N)$ satisfy (3.4).

Theorem 4.1. *The equations* (4.1) *and* (4.2) *have a unique common equilibrium point $\bar{\rho} \in \sigma(N)$. Furthermore, $\bar{\rho}$ is a global attractor.*

Proof. Define the mappings $f : \sigma(N)^k \to \sigma(N)$ by

$$f(\rho_1, \rho_2, \ldots, \rho_k) = Q_1 + \frac{1}{k}[A^*\xi(\rho_1, \rho_2, \ldots, \rho_k)A],$$

for all $\rho_1, \rho_2, \ldots, \rho_k \in \sigma(N)$, and $g : \sigma(N) \to \sigma(N)$ by

$$g(\rho) = Q_2 + A^*\psi(\rho)A, \text{ for any } \rho \in \sigma(N).$$

For $\rho_0, \rho_1, \rho_2, \ldots, \rho_k \in \sigma(N)$,

$d(f(\rho_0, \rho_1, \rho_2, \ldots, \rho_{k-1}), f(\rho_1, \rho_2, \ldots, \rho_k))$

$$\begin{aligned} &= d\left(Q_1 + \frac{1}{k}[A^*\xi(\rho_0, \rho_1, \ldots, \rho_{k-1})A], Q_1 + \frac{1}{k}[A^*\xi(\rho_1, \rho_2, \ldots, \rho_k)A]\right) \\ &= d\left(\left(\frac{A}{\sqrt{k}}\right)^*\xi(\rho_0, \rho_1, \ldots, \rho_{k-1})\left(\frac{A}{\sqrt{k}}\right), \left(\frac{A}{\sqrt{k}}\right)^*\xi(\rho_1, \rho_2, \ldots, \rho_k)\left(\frac{A}{\sqrt{k}}\right)\right). \end{aligned} \tag{4.4}$$

Denote $H = \dfrac{A}{\sqrt{k}}$. Then H is also nonsingular. Using property (i) we have

$$d(H^*\xi(\rho_0, \rho_1, \dots, \rho_{k-1})H, H^*\xi(\rho_1, \rho_2, \dots, \rho_k)H) = d(\xi(\rho_0, \rho_1, \dots, \rho_{k-1}), \xi(\rho_1, \rho_2, \dots, \rho_k.)$$

Thus, (4.4) becomes

$$d(f(\rho_0, \rho_1, \rho_2, \dots, \rho_{k-1}), f(\rho_1, \rho_2, \dots, \rho_k)) = d(\xi(\rho_0, \rho_1, \rho_2, \dots, \rho_{k-1}), \xi(\rho_1, \rho_2, \dots, \rho_k)).$$

Similarly, we obtain

$$d(g\rho_i, g\rho_{i+1}) = d(\psi\rho i, \psi\rho_{i+1}).$$

Hence,

$$\left\| \sum_{i=0}^{k-1} b_i(\rho_i - \rho_{i+1}) + f(\rho_0, \rho_1, \dots, \rho_{k-1}) - f(\rho_1, \rho_2, \dots, \rho_k) \right\| \\ \leq \alpha \max\{\|g\rho_i - g\rho_{i+1}\| \; ; \; 0 \leq i \leq k-1\}.$$

Now, utilizing Theorem 3.2, we conclude that both (4.1) and (4.2) have a unique common global attractor equilibrium point $\bar{\rho} \in \sigma(N)$. □

For a positive integer k, define the sequence $\{U_{n+k}\} \subset \sigma(N)^k$ by

$$U_{n+k} = Q_1 + \sum_{i=0}^{k-1} A^* U_{n+i}^z A, \;\; n = 1, 2, \dots, \tag{4.5}$$

and $\{U_{n+1}\} \subset \sigma(N)$ by

$$U_{n+1} = Q_2 + A^* U_n^z A, \;\; n = 1, 2, \dots, \tag{4.6}$$

where $U_1, U_2, \dots U_k \in \sigma(N)$, Q_1, Q_2 are $N \times N$ Hermitian positive semidefinite matrices, and $|z| \in [0, 1)$.

Corollary 4.1. *Both the equations* (4.5) *and* (4.6) *have a unique common equilibrium point* $\bar{\rho} \in \sigma(N)$. *Moreover,* $\bar{\rho}$ *is a global attractor.*

Our final objective is to emphasize the Theorem 4.1 by giving an example with a numerical approximation of the convergence of iterated sequence.

Example 4.1. *Take* $N = 3$ *and* $k = 2$. *Assume* $\xi : \sigma(3)^2 \to \sigma(3)$, $\psi : \sigma(3) \to \sigma(3)$ *and defined by* $\xi(U_1, U_2) = U_1^{1/2}$, $\psi(U) = U^{1/2}$, *where* $U, U_1, U_2 \in \sigma(3)$. *Consider the recursive sequence as follows:*

$$U_{n+2} = Q_1 + \frac{1}{2}[A^* U_n^{1/2} A + A^* U_{n+1}^{1/2} A], \;\; n = 1, 2, \dots \tag{4.7}$$

$$U_{n+1} = Q_2 + A^* U_n^{1/2} A, \;\; n = 1, 2, \dots \tag{4.8}$$

where $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$,

$$Q_2 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}, U_1 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix} \text{ and } U_2 = \begin{pmatrix} 7 & 3 & 2 \\ 3 & 7 & 3 \\ 2 & 3 & 11 \end{pmatrix}.$$

Then after 42 iterations, we get the unique equlibrium point

$$\bar{U} \approx U_{42} = \begin{pmatrix} 61.50 & 50.01 & -41 \\ 50.01 & 45.65 & -31.93 \\ -41 & -31.93 & 38.93 \end{pmatrix}.$$

5. Conclusion

We found an obvious error in the main theorem of [11], thus, in this paper, we proved it under suitable mapping called weak compatibility. Moreover, using Krasnoselskij averaged mappings, we obtained a common coincidence point for a pair of mappings. In addition, we have showed a unique fixed point for generalized enriched Prešić type contraction. A couple of examples are given, one is to validate our main result and another as a counter-example to Theorem 2.1. In the end, we used our findings to prove the existence and uniqueness of an equilibrium point in matrix difference equations.

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