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# Controlled Fuzzy 2-Metric Spaces: A Soft Computing Framework with Dynamic Applications

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**Abstract.** In this article, we introduce the concept of a controlled fuzzy 2-metric space, formulated by incorporating three control functions that flexibly regulate the fuzzy distance relationships among triplets of points. This structure provides a flexible analytical tool for modeling systems influenced by uncertainty, interdependence, and approximate reasoning. We establish several fundamental properties of this structure and derive fixed-point results. To demonstrate its practical relevance, we apply the proposed framework to a dynamic market-equilibrium problem, in which agents' interactions are governed by fuzzy relations and control-dependent adjustments. The study also discusses implications for soft computing and decision-making systems, highlighting the framework's potential in modeling adaptive and uncertain environments.

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## 1. Introduction

In recent decades, the concept of fuzzy metric spaces has emerged as a powerful extension of classical metric spaces, offering a flexible framework for measuring uncertainty and proximity. Its foundation was laid by Kramosil and Michalek [1] in 1975 and later redefined by George and Veeramani [2] through continuous $t$-norms, forming the foundation for subsequent work in fuzzy topology and fixed point theory. These foundational models laid the groundwork for further developments in fuzzy topology, fixed-point theory, and its applications.

In parallel with these advances, the theory of 2-metric spaces [3], a higher-order generalization of metric spaces, was investigated, in which the distance function depends on three variables rather than two. Inspired by this multi-variable structure, Sharma [4] introduced fuzzy 2-metric spaces, which extend fuzzy metrics by including three elements providing a richer structure for handling multidimensional relationships. Subsequently, Sezen [5] introduced controlled fuzzy metric spaces, where control functions enhance modeling flexibility for systems involving additional parameters such as decision networks or complex data structures.

Recent advances have demonstrated the utility of these frameworks. Moussaoui *et al.* [6], Thangaraj *et al.* [7], and Tiwari *et al.* [8] developed fixed point results in controlled fuzzy metric settings, while Samuel *et al.* [9] and Ishtiaq *et al.* [10, 11] extended such results to generalized controlled metric spaces with applications to fractional integrals, market equilibrium, and fractional differential equations. Further studies [12–15] on fuzzy double, triple, n-controlled, and 2-metric spaces provide the theoretical and application-oriented motivation for the present study, which develops fixed-point theory in controlled fuzzy metric frameworks. This article contributes to this line of research by introducing a new type of distance function called a controlled fuzzy 2-metric space. It builds on foundational studies in the fuzzy metric space setting by integrating a higher-order metric structure that accounts for three-point interactions, in addition to pairwise distances. This framework is motivated by the need to model complex systems where triple interactions and uncertainty coexist, such as multi-agent networks, distributed decision-making, and optimization under uncertainty. By incorporating control functions, the framework regulates the influence of each point, allowing the metric to capture dynamic inter-dependencies, consensus evolution, and uncertainty tolerance more effectively than traditional fuzzy metrics. Mathematically, it integrates a continuous $t$-norm, a fuzzy set, and control functions to ensure reflexivity, symmetry, and continuity, and coincides with the classical fuzzy 2-metric whenever the control functions are constant.

To demonstrate its utility, we establish fixed-point results and apply the framework to dynamic market equilibrium and multi-agent simulations. In such settings, participants make decisions based on limited information and uncertain behaviors, and their choices often depend on one another. Such complex interactions cannot be properly described using classical metrics that measure only exact, fixed distances. In contrast, a fuzzy 2-metric measures the degree of closeness among three interacting agents rather than just pairs, thereby accommodating collective behaviors and shared uncertainties in evolving markets. This integration bridges abstract fuzzy theory with soft computing, distributed control, and dynamic equilibrium modeling.

The article is organized as follows. In Section 2, we review preliminaries and essential concepts in fuzzy metric spaces. Section 3 introduces the controlled fuzzy 2-metric space, along with illustrative examples highlighting the role of control functions. Section 4 presents a fixed-point theorem in this new framework, including its potential applications for modeling dynamic market equilibrium. Lastly, Section 5 provides simulation examples to demonstrate the effectiveness of the proposed structure.

## 2. Preliminaries

In this section, we present some definitions and results that are necessary for the main results of this article.

**Definition 2.1.** *[16] A t-norm '$\star$' is a binary operation on $[0,1]$ that satisfies the following conditions:*

*(i)* $\gamma \star \nu = \nu \star \gamma$, $\forall \gamma, \nu \in [0,1]$;

*(ii)* $(\nu_1 \star \gamma) \star \nu_2 = \nu_1 \star (\gamma \star \nu_2)$, $\forall \gamma, \nu_1, \nu_2 \in [0,1]$;

*(iii)* $\nu \star 1 = \nu$, $\forall \nu \in [0,1]$;

*(iv)* $\nu_1 \star \gamma \leq \nu_2 \star \alpha$ *whenever* $\nu_1 \leq \nu_2$ *and* $\gamma \leq \alpha$, $\forall \nu_1, \nu_2, \gamma, \alpha \in [0,1]$.

*If $\star$ is continuous, then it is called a continuous t-norm.*

Some well-known examples of $t$-norms (see [16]) are: for $\alpha, \nu \in [0,1]$, $\alpha \star \nu =$

(i) $\min\{\alpha, \nu\}$(Standard intersection); (ii) $\alpha \cdot \nu$ (Algebraic product); (iii) $\max\{0, \alpha + \nu - 1\}$ (Bounded difference).

**Definition 2.2.** *[2] A George-Veeramani (GV)-type fuzzy metric space is a 3-tuple $(\mho, \mathcal{M}, \star)$, where $\star$ is a continuous t-norm and the function $\mathcal{M} : \mho^2 \times (0,\infty) \to [0,1]$ satisfies the following axioms:*

*($\mathcal{M}$1)* $\mathcal{M}(\kappa, \varsigma, v_1) > 0$;

*($\mathcal{M}$2)* $\mathcal{M}(\kappa, \varsigma, v_1) = 1$, $\forall v_1 > 0$ *if and only if* $\kappa = \varsigma$;

*($\mathcal{M}$3)* $\mathcal{M}(\kappa, \varsigma, v_1) = \mathcal{M}(\varsigma, \kappa, v_1)$;

*($\mathcal{M}$4)* $\mathcal{M}(\kappa, \varsigma, v_1) \star \mathcal{M}(\varsigma, z, v_2) \leq \mathcal{M}(\kappa, z, v_1 + v_2)$ *where $\star$ is a continuous t-norm;*

*($\mathcal{M}$5)* $\mathcal{M}(\kappa, \varsigma, .) : (0,\infty) \to [0,1]$ *is continuous;*

*for all $\kappa, \varsigma, z \in \mho$ and $v_1, v_2 > 0$.*

**Lemma 2.1.** *[2] GV-type fuzzy metric $\mathcal{M}(\kappa, \varsigma, v)$ is non-decreasing with respect to $v > 0$, for all $\kappa, \varsigma \in \mho$.*

**Remark 2.1.** *[2] In GV-type fuzzy metric space $(\mho, \mathcal{M}, \star)$, the following holds:*

*(i) If for some $q \in (0,1)$ and $v \in (0,\infty)$, $\mathcal{M}(\kappa, \varsigma, v) > 1 - q$, for $\kappa, \varsigma \in \mho$, then there exists $v_0 \in (0, v)$ such that $\mathcal{M}(\kappa, \varsigma, v) > 1 - q$.*

*(ii) For any $r_1 > r_2$ in $(0,1)$, there exists $q \in (0,1)$ such that $r_1 \star q \geq r_2$ holds.*

*(iii) For any $q_1 \in (0,1)$, there exists $q_2 \in (0,1)$ such that $q_2 \star q_2 \geq q_1$ holds.*

We begin by recalling the notion of a 2-metric and fuzzy 2-metric space, which serves as the underlying structure for our subsequent analysis.

**Definition 2.3.** *[3] A pair $(\mho, d)$ is called a 2-metric space if $d$ is a real valued function on $\mho^3$ that satisfies the following axioms:*

$(d1)$ *given distinct elements $\kappa, \varsigma$ of $\mho$, there is an element $z$ of $\mho$ such that $d(\kappa, \varsigma, z) \neq 0$;*

$(d2)$ *$d(\kappa, \varsigma, z) = 0$, when at least two of $\kappa, \varsigma, z$ are equal;*

$(d3)$ *$d(\kappa, \varsigma, z) = d(z, \varsigma, \kappa) = d(z, \kappa, \varsigma)$, for every $\kappa, \varsigma, z \in \mho$;*

$(d4)$ *$d(\kappa, \varsigma, z) \leq d(w, \varsigma, z) + d(\kappa, w, z) + d(\kappa, \varsigma, w)$, for every $\kappa, \varsigma, z, w \in \mho$.*

**Definition 2.4.** *[4] A fuzzy 2-metric space is a 3-tuple $(\mho, \mathcal{M}, \star)$, where $\star$ is a continuous t-norm and the function $\mathcal{M} : \mho^3 \times (0, \infty) \to [0, 1]$ satisfies following axioms:*

$(F\mathcal{M}1)$ *$\mathcal{M}(\kappa, \varsigma, z, 0) = 0$;*

$(F\mathcal{M}2)$ *$\mathcal{M}(\kappa, \varsigma, z, v_1) = 1$, $\forall v_1 > 0$ if at least two of $\{\kappa, \varsigma, z\} \subset \mho$ are equal;*

$(F\mathcal{M}3)$ *$\mathcal{M}(\kappa, \varsigma, z, v_1) = \mathcal{M}(\kappa, z, \varsigma, v_1) = \mathcal{M}(\varsigma, z, \kappa, v_1)$;*

$(F\mathcal{M}4)$ *$\mathcal{M}(\kappa, \varsigma, z, v_1 + v_2 + v_3) \geq \mathcal{M}(\kappa, \varsigma, u, v_1) \star \mathcal{M}(\kappa, u, z, v_2) \star M(u, \varsigma, z, v_3)$;*

$(F\mathcal{M}5)$ *$\mathcal{M}(\kappa, \varsigma, z, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;*

*for all $\kappa, \varsigma, z, u \in \mho$ and $v_1, v_2, v_3 > 0$.*

The following definition of a controlled fuzzy metric space was given by Sezen [5].

**Definition 2.5.** *[5] A controlled fuzzy metric space is a 3-tuple $(\mho, \mathcal{M}_\lambda, \star)$, where $\star$ is a continuous t-norm and the function $\mathcal{M}_\lambda : \mho^2 \times (0, \infty) \to [0, 1]$ satisfies following axioms:*

*($\mathcal{M}_\lambda$1)* $\mathcal{M}_\lambda(\kappa, \nu, 0) = 0$;

*($\mathcal{M}_\lambda$2)* $\mathcal{M}_\lambda(\kappa, \nu, v_1) = 1 \iff \kappa = \nu$;

*($\mathcal{M}_\lambda$3)* $\mathcal{M}_\lambda(\kappa, \nu, v_1) = \mathcal{M}_\lambda(\nu, \kappa, v_1)$;

*($\mathcal{M}_\lambda$4)* $\mathcal{M}_\lambda(\kappa, z, v_1 + v_2) \geq \mathcal{M}_\lambda(\kappa, \nu, \frac{v_1}{\lambda(\kappa, \nu)}) \star \mathcal{M}_\lambda(\nu, z, \frac{v_2}{\lambda(\nu, z)})$;

*for all $\kappa, \nu, z \in \mho$ and $v_1, v_2 > 0$.*

## 3. Introduction to Controlled Fuzzy 2-Metric Space

In this section, we lay the foundations of controlled fuzzy 2-metric spaces, illustrated by examples that connect the abstract definitions with a concrete understanding. We then define notions of convergence and Cauchy sequences and establish related foundational results.

**Definition 3.1.** *Let $\mho$ be a nonempty set, $f, g, h : \mho^3 \to [1, \infty)$ be given functions and $\star$ be a continuous t-norm. Assume that the fuzzy set $\mathfrak{M} : \mho^3 \times (0, \infty) \to [0, 1]$ satisfies the following conditions:*

*($\mathfrak{M}$1)* $\mathfrak{M}(\kappa, v, \varsigma, \rho) > 0$, $\forall \kappa, v, \varsigma, \rho$, $\forall \rho > 0$;

*($\mathfrak{M}$2)* $\mathfrak{M}(\kappa, v, \varsigma, \rho) = 1$, $\forall \rho > 0$ *if and only if at least two among $\kappa, v, \varsigma \in \mho$ are identical;*

*($\mathfrak{M}$3) for all $\kappa, v \in \mho$ with $\kappa \neq v$, there exists $\varsigma \in \mho$ such that $\mathfrak{M}(\kappa, v, \varsigma, \rho) < 1$ for all $\rho > 0$;*

*($\mathfrak{M}$4) for every $\rho > 0$, $\mathfrak{M}(\kappa, v, \varsigma, \rho)$ is invariant under any permutation of $(\kappa, v, \varsigma) \in \mho^3$;*

*($\mathfrak{M}$5) for all $\kappa, v, \varsigma, a \in \mho$ and $\rho_\iota > 0$, $\iota = 1,2,3$,*

$$\mathfrak{M}\left(\kappa, v, \varsigma, \sum_{\iota=1}^{3} \rho_\iota\right) \geq \mathfrak{M}\left(\kappa, v, a, \frac{\rho_1}{f(\kappa, v, a)}\right) \star \mathfrak{M}\left(v, \varsigma, a, \frac{\rho_2}{g(v, \varsigma, a)}\right) \star \mathfrak{M}\left(\kappa, \varsigma, a, \frac{\rho_3}{h(\kappa, \varsigma, a)}\right);$$

*($\mathfrak{M}$6) for all $\kappa, v, \varsigma \in \mho$, the mapping $\mathfrak{M}(\kappa, v, \varsigma, .) \to [0,1]$ is continuous on $(0, \infty)$.*

*A triple $(\mho, \mathfrak{M}, \star)$ satisfying the above conditions is called a controlled fuzzy 2-metric space with respect to the control functions $f, g$ & $h$.*

**Remark 3.1.** *If the three functions $f, g$ and $h$ of the definition be fixed with the constant value 1, then we can deduce fuzzy 2-metric space of Sharma [4].*

Now we illustrate this concept with a series of examples to highlight the distinctive nature of the approached definition.

**Example 3.1.** *We consider the 2-metric space $(\mathbb{R}, d)$ [3, Example 2.2] where $d(\varsigma, a, \eta) = |(\varsigma - a)(a - \eta)(\eta - \varsigma)|$, $\forall \varsigma, a, \eta \in \mathbb{R}$ and define a function $\mathfrak{M} : \mathbb{R}^3 \times (0, \infty) \to [0,1]$ by $\mathfrak{M}(\kappa, a, \eta, \rho) = \frac{\rho}{\rho + D(\kappa, a, \eta)}$, $\forall \kappa, a, \eta \in \mathbb{R}$, $\forall \rho > 0$ where $D : \mathbb{R}^3 \to [0, \infty)$ is defined by $D(\kappa, a, \eta) = d^2(\kappa, a, \eta)$, $\forall \kappa, a, \eta \in \mathbb{R}$.*

*Since $d$ is a 2-metric on $\mathbb{R}$, so $\mathfrak{M}$ satisfies ($\mathfrak{M}$1)-($\mathfrak{M}$4) trivially. To check ($\mathfrak{M}$5), we consider $\kappa, a, \varsigma, \eta \in \mathbb{R}$; $\rho > 0$ where $\rho = \sum_{\iota=1}^{3} \rho_\iota$ with $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ and '$\min$' as the t-norm $\star$.*

*Before going to the proof, we recall the following result:*

$$a^2 + b^2 \geq 2ab, \ \forall a, b \in \mathbb{R}. \tag{3.1}$$

*Since $d$ is a 2-metric on $\mathbb{R}$, so*

$$d(\kappa, a, \eta) \leq d(\kappa, a, \varsigma) + d(\kappa, \varsigma, \eta) + d(\varsigma, a, \eta), \ \forall \varsigma \in \mathbb{R} \tag{3.2}$$

*holds. Again using (3.2), we get*

$$\begin{aligned} D(\kappa, a, \eta) =& d^2(\kappa, a, \eta) \\ &\leq (d(\kappa, a, \varsigma) + d(\kappa, \varsigma, \eta) + d(a, \eta, \varsigma))^2 \\ &= (d(\kappa, a, \varsigma) + d(\kappa, \eta, \varsigma))^2 + d^2(a, \eta, \varsigma) + 2(d(\kappa, a, \varsigma) + d(\kappa, \eta, \varsigma))d(a, \eta, \varsigma) \\ &\leq 2[(d(\kappa, a, \varsigma) + d(\kappa, \eta, \varsigma))^2 + d^2(a, \eta, \varsigma)] \quad (\textit{using (3.1)}) \\ &= 2d^2(a, \eta, \varsigma) + 2\{d^2(\kappa, a, \varsigma) + d^2(\kappa, \eta, \varsigma) + 2d(\kappa, a, \varsigma)d(\kappa, \eta, \varsigma)\} \\ &\leq 2D(a, \eta, \varsigma) + 2(D(\kappa, a, \varsigma) + D(\kappa, \eta, \varsigma)) + 2(d^2(\kappa, a, \varsigma) + d^2(\kappa, \eta, \varsigma)) \\ &= 4D(\kappa, a, \varsigma) + 4D(\kappa, \eta, \varsigma) + 2D(a, \eta, \varsigma). \end{aligned}$$

*Therefore, $D$ satisfies*

$$D(\kappa, a, \eta) \leq 4D(\kappa, a, \varsigma) + 4D(\kappa, \eta, \varsigma) + 2D(a, \eta, \varsigma), \ \forall \kappa, a, \eta, \varsigma \in \mho. \tag{3.3}$$

*Now,*

$$\begin{aligned}&\mathfrak{M}\left(\kappa, a, \varsigma, \frac{\rho_1}{4}\right) \star \mathfrak{M}\left(\kappa, \eta, \varsigma, \frac{\rho_2}{4}\right) \star \mathfrak{M}\left(a, \eta, \varsigma, \frac{\rho_3}{2}\right)\\ &= \min\left\{\frac{\rho_1}{\rho_1 + 4D(\kappa, a, \varsigma)}, \frac{\rho_2}{\rho_2 + 4D(\kappa, \eta, \varsigma)}, \frac{\rho_3}{\rho_3 + 2D(a, \eta, \varsigma)}\right\}.\end{aligned}$$

*Suppose that,* $\min\left\{\mathfrak{M}(\kappa, a, \varsigma, \frac{\rho_1}{4}), \mathfrak{M}(\kappa, \eta, \varsigma, \frac{\rho_2}{4}), \mathfrak{M}(a, \eta, \varsigma, \frac{\rho_3}{4})\right\} = \mathfrak{M}(\kappa, a, \varsigma, \frac{\rho_1}{4})$*. Then we have*

$$\frac{\rho_1}{\rho_1 + 4D(\kappa, a, \varsigma)} \leq \frac{\rho_2}{\rho_2 + 4D(\kappa, \eta, \varsigma)} \quad or \quad \rho_1 D(\kappa, \eta, \varsigma) \leq \rho_2 D(\kappa, a, \varsigma) \tag{3.4}$$

*and*

$$\frac{\rho_1}{\rho_1 + 4D(\kappa, a, \varsigma)} \leq \frac{\rho_3}{\rho_3 + 2D(a, \eta, \varsigma)} \quad or \quad \rho_1 D(a, \eta, \varsigma) \leq 2\rho_3 D(\kappa, a, \varsigma). \tag{3.5}$$

*In this case,*

$$\begin{aligned}&\mathfrak{M}(\kappa, a, \eta, \rho) - \mathfrak{M}(\kappa, a, \varsigma, \frac{\rho_1}{4})\\ &= \frac{\rho}{\rho + D(\kappa, a, \eta)} - \frac{\rho_1}{\rho_1 + 4D(\kappa, a, \varsigma)}\\ &= \frac{\rho(\rho_1 + 4D(\kappa, a, \varsigma)) - \rho_1(\rho + D(\kappa, a, \eta))}{(\rho + D(\kappa, a, \eta))(\rho_1 + 4D(\kappa, a, \varsigma))}\\ &\geq \frac{4(\rho_1 + \rho_2 + \rho_3)D(\kappa, a, \varsigma) - \rho_1\{4D(\kappa, a, \varsigma) + 4D(\kappa, \eta, \varsigma) + 2D(a, \eta, \varsigma)\}}{(\rho + D(\kappa, a, \eta))(\rho_1 + 4D(\kappa, a, \varsigma))} \quad (using\ (3.3))\\ &\geq 0 \quad (using\ (3.4)\ \&\ (3.5)).\end{aligned}$$

*This implies,* $\mathfrak{M}(\kappa, a, \eta, \rho) \geq \min\left\{\mathfrak{M}\left(\kappa, a, \varsigma, \frac{\rho_1}{4}\right), \mathfrak{M}\left(\kappa, \eta, \varsigma, \frac{\rho_2}{4}\right), \mathfrak{M}\left(a, \eta, \varsigma, \frac{\rho_3}{4}\right)\right\}$*. The other cases also can be verified similarly. Hence,*

$$\mathfrak{M}(\kappa, a, \eta, \rho) \geq \mathfrak{M}\left(\kappa, a, \varsigma, \frac{\rho_1}{f(\kappa, a, \varsigma)}\right) \star \mathfrak{M}\left(\kappa, \eta, \varsigma, \frac{\rho_2}{g(\kappa, \eta, \varsigma)}\right) \star \mathfrak{M}\left(a, \eta, \varsigma, \frac{\rho_3}{h(a, \eta, \varsigma)}\right)$$

*holds for the constant control functions* $f = g = 4, h = 2$ *in* $\mathbb{R}$*. Thus* $(\mathfrak{M}5)$ *holds for* $\mathfrak{M}$*. Moreover,* $\mathfrak{M}$ *is satisfies* $(\mathfrak{M}6)$*. Therefore,* $(\mathbb{R}, \mathfrak{M}, \min)$ *is a controlled fuzzy 2-metric space.*

**Example 3.2.** *Let* $\mho = \mathbb{R}$ *and defined a function* $\mathfrak{M} : \mho^3 \times (0, \infty) \to [0, 1]$ *by*

$$\mathfrak{M}(\kappa, v, \varsigma, \rho) = \exp\left(-\frac{\sqrt{|(\kappa - v)(v - \varsigma)(\varsigma - \kappa)|}}{\rho}\right), \quad \forall \kappa, v, \varsigma \in \mho, \ \forall \rho > 0.$$

*From the definition of* $\mathfrak{M}$ *it is clear that* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}1)$*-*$(\mathfrak{M}4)$ *and* $(\mathfrak{M}6)$*.*
*For* $(\mathfrak{M}5)$*, we define three functions*

$$f(\kappa, v, \varsigma) = \sqrt{|(\kappa - v)(v - \varsigma)(\varsigma - \kappa)|} + 1, \quad g(\kappa, v, \varsigma) = (\kappa - v)^2(v - \varsigma)^2(\varsigma - \kappa)^2 + 2$$

$$\& \quad h(\kappa, v, \varsigma) = \sqrt{|(\kappa - v)(v - \varsigma)|}(\varsigma - \kappa)^2 + 1, \quad \forall \kappa, v, \varsigma \in \mho.$$

*Now, take $\kappa, v, z, \varsigma \in \mho$ and $\rho = \sum_{\iota=1}^{3} \rho_\iota$ with $\rho_1, \rho_2, \rho_3 > 0$. Then, we have*

$$\begin{aligned}
& f(\kappa,v,\varsigma)\ \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|} + g(v,z,\varsigma)\ \sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|} + h(\kappa,z,\varsigma)\ \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|} \\
\geq\ & \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|} + 2\sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|} + \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|} \\
\geq\ & \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)| + |(v-z)(z-\varsigma)(\varsigma-v)| + |(\kappa-z)(\kappa-\varsigma)(z-\varsigma)|} \\
\geq\ & \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa) + (v-z)(z-\varsigma)(\varsigma-v) + (\kappa-z)(\kappa-\varsigma)(z-\varsigma)|} \\
=\ & \sqrt{|-\kappa^2 v + \kappa v^2 - v^2 z + v z^2 + \kappa^2 z - \kappa z^2|} \\
=\ & \sqrt{|(\kappa-v)(v-z)(z-\kappa)|}
\end{aligned}$$

*which implies*

$$\begin{aligned}
& \frac{f(\kappa,v,\varsigma)\ \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|}}{\rho_1} + \frac{g(v,z,\varsigma)\ \sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|}}{\rho_2} + \\
& \qquad \frac{h(\kappa,z,\varsigma)\ \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|}}{\rho_3} \\
> & \frac{f(\kappa,v,\varsigma)\ \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|}}{\rho} + \frac{g(v,z,\varsigma)\ \sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|}}{\rho} + \\
& \qquad \frac{h(\kappa,z,\varsigma)\ \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|}}{\rho} \\
> & \frac{\sqrt{|(\kappa-v)(v-z)(z-\kappa)|}}{\rho}.
\end{aligned}$$

*Henceforth,*

$$\begin{aligned}
& -\frac{f(\kappa,v,\varsigma)\ \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|}}{\rho_1} - \frac{g(v,z,\varsigma)\ \sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|}}{\rho_2} - \\
& \qquad \frac{h(\kappa,z,\varsigma)\ \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|}}{\rho_3} \leq -\frac{\sqrt{|(\kappa-v)(v-z)(z-\kappa)|}}{\rho} \\
\text{or } & \exp\left(-\frac{f(\kappa,v,\varsigma)\ \sqrt{|(\kappa-v)(v-\varsigma)(\varsigma-\kappa)|}}{\rho_1}\right) \cdot \exp\left(-\frac{g(v,z,\varsigma)\ \sqrt{|(v-z)(z-\varsigma)(\varsigma-v)|}}{\rho_2}\right) \cdot \\
& \qquad \exp\left(\frac{h(\kappa,z,\varsigma)\ \sqrt{|(\kappa-z)(z-\varsigma)(\varsigma-\kappa)|}}{\rho_3}\right) \leq \exp\left(-\frac{\sqrt{|(\kappa-v)(v-z)(z-\kappa)|}}{\rho}\right) \\
\text{or } & \mathfrak{M}\left(\kappa, v, \varsigma, \frac{\rho_1}{f(\kappa,v,\varsigma)}\right) \cdot \mathfrak{M}\left(v, z, \varsigma, \frac{\rho_2}{g(v,z,\varsigma)}\right) \cdot \mathfrak{M}\left(\kappa, z, \varsigma, \frac{\rho_3}{h(\kappa,z,\varsigma)}\right) \leq \mathfrak{M}(\kappa, v, z, \rho).
\end{aligned}$$

*Thus $\mathfrak{M}$ satisfies ($\mathfrak{M}5$) with respect to the 'product' t-norm and the control functions $f, g, h$ defined above. Therefore $(\mathbb{R}, \mathfrak{M}, \cdot)$ is a controlled fuzzy 2-metric space.*

**Example 3.3.** *Let $\mho = A \cup B$ where $A = (0,3)$ and $B = [3,\infty)$. Then define a function $\mathfrak{M} : \mho^3 \times (0,\infty) \to [0,1]$ by*

$$\mathfrak{M}(\kappa, v, z, \rho) = \begin{cases} 1 & \text{if and only if atleast two of } \kappa, v, \& z \text{ are equal} \\ \exp(-\frac{1}{z\rho}) & \text{if } \kappa, v \in A \text{ and } z \in B \\ \exp(-\frac{1}{\max\{v,z\}\rho}) & \text{if } \kappa \in A \text{ and } v, z \in B \\ \exp(-\frac{1}{\rho}) & \text{otherwise} \end{cases}$$

*for all $\kappa, v, z \in \mho$ and for all $t > 0$. Now, $(\mathfrak{M}1)$-$(\mathfrak{M}4)$ holds for $\mathfrak{M}$. So, we verify the axiom $(\mathfrak{M}5)$ for the defined $\mathfrak{M}$. For, we consider a function $f : \mho^3 \to [1,\infty)$ by*

$$f(\varkappa, v, w) = \begin{cases} 1 & \text{if } \varkappa, v, w \in A \\ \max\{\varkappa, v, w\} & \text{otherwise.} \end{cases}$$

*for all $\varkappa, v, w \in \mho$ and take $\kappa, v, \varkappa, w \in \mho$, $\rho = \sum_{\iota=1}^{3} \rho_\iota$ where $\rho_\iota \in (0,\infty)$, $\iota = 1,2,3$. Then we have the following cases:*

***Case-I:*** *$\kappa, v, \varkappa \in A$. Therefore $\mathfrak{M}(\kappa, v, \varkappa, \rho) = \exp\left(\frac{-1}{\rho}\right)$.*

*(a) Let $w \in A$. Then*

$$\mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right)$$
$$= \prod_{\iota=1}^{3} \exp\left(-\frac{1}{\rho_\iota}\right) = \exp\left(-\sum_{\iota=1}^{3} \frac{1}{\rho_\iota}\right) < \exp\left(-\frac{3}{\rho}\right) < \mathfrak{M}(\kappa, v, \varkappa, \rho).$$

*(b) Let $w \in B$. Then*

$$\mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right)$$
$$= \exp\left(-\frac{w}{w\rho_1}\right) \cdot \exp\left(-\frac{w}{w\rho_2}\right) \cdot \exp\left(\frac{w}{w\rho_3}\right)$$
$$= \exp\left(-\sum_{\iota=1}^{3} \frac{1}{\rho_\iota}\right) < \mathfrak{M}(\kappa, v, \varkappa, \rho) \quad \text{(as similar as case I(a) ).}$$

***Case-II:*** *$\kappa, v \in A$; $\varkappa \in B$. Therefore $\mathfrak{M}(\kappa, v, \varkappa, \rho) = \exp\left(\frac{-1}{\varkappa\rho}\right)$.*

*(a) Let $w \in A$. Then*

$$M\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot M\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot M\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right)$$
$$= \exp\left(-\frac{1}{\rho_1}\right) \cdot \exp\left(-\frac{\varkappa}{\varkappa\rho_2}\right) \cdot \exp\left(-\frac{\varkappa}{\varkappa\rho_3}\right)$$
$$= \exp\left(-\sum_{\iota=1}^{3} \frac{1}{\rho_\iota}\right) < \exp\left(-\frac{1}{\rho}\right) < \exp\left(-\frac{1}{\varkappa\rho}\right) = M(\kappa, v, \varkappa, \rho).$$

*(b) Let $w \in B$. Then*

$$\begin{aligned}
&\mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right) \\
&= \exp\left(-\frac{w}{w\rho_1}\right) \cdot \exp\left(-\frac{\max\{\kappa, \varkappa, w\}}{\max\{\varkappa, w\}\rho_2}\right) \cdot \exp\left(-\frac{\max\{v, w, \varkappa\}}{\max\{w, \varkappa\}\rho_3}\right) \\
&= \prod_{\iota=1}^{3} \exp\left(-\frac{1}{\rho_\iota}\right) = \exp\left(-\sum_{\iota=1}^{3} \frac{1}{\rho_\iota}\right) < \mathfrak{M}(\kappa, v, \varkappa, \rho) \quad (\textit{as similar as case } II(a)).
\end{aligned}$$

***Case-III:*** *$\kappa \in A$; $v, \varkappa \in B$. Therefore, $\mathfrak{M}(\kappa, v, \varkappa, \rho) = \exp\left(\frac{-1}{\max\{v,\varkappa\}\rho}\right)$.*

*(a) Let $w \in A$. Then*

$$\begin{aligned}
&\mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right) \\
&= \exp\left(-\frac{\max\{\kappa, v, w\}}{y\rho_1}\right) \cdot \exp\left(-\frac{\max\{v, \varkappa, w\}}{\max\{v, \varkappa\}\rho_2}\right) \cdot \exp\left(-\frac{\max\{\kappa, \varkappa, w\}}{\varkappa\rho_3}\right) \\
&= \exp\left(-\sum_{\iota=1}^{3} \frac{1}{\rho_\iota}\right) < \exp\left(-\frac{1}{\rho}\right) < \exp\left(-\frac{1}{\max\{v, \varkappa\}\rho}\right) = \mathfrak{M}(\kappa, v, \varkappa, \rho).
\end{aligned}$$

*(b) Let $w \in B$. Then*

$$\begin{aligned}
&\mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \cdot \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_2}{f(\kappa, \varkappa, w)}\right) \cdot \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_3}{f(v, \varkappa, w)}\right) \\
&= \exp\left(-\frac{\max\{v, w\}}{\max\{v, w\}\rho_1}\right) \cdot \exp\left(-\frac{\max\{v, \varkappa, w\}}{\max\{\varkappa, w\}\rho_2}\right) \cdot \exp\left(-\frac{\max\{\kappa, \varkappa, w\}}{\rho_3}\right) \\
&< \prod_{\iota=1}^{3} \exp\left(-\frac{1}{\rho_\iota}\right) < \exp\left(-\frac{1}{\rho}\right) < \mathfrak{M}(\kappa, v, \varkappa, \rho) \quad (\textit{as similar as case III(a)}).
\end{aligned}$$

*Therefore, for all $\kappa, v, \varkappa, w \in \mho$ and $\rho_1, \rho_2, \rho_3 > 0$, $\mathfrak{M}$ satisfies $(\mathfrak{M}5)$ :*

$$\mathfrak{M}(\kappa, v, \varkappa, \rho) \geq \mathfrak{M}\left(\kappa, v, w, \frac{\rho_1}{f(\kappa, v, w)}\right) \star \mathfrak{M}\left(v, \varkappa, w, \frac{\rho_2}{f(v, \varkappa, w)}\right) \star \mathfrak{M}\left(\kappa, \varkappa, w, \frac{\rho_3}{f(\kappa, \varkappa, w)}\right)$$

*with respect to 'product' t-norm and defined control function $f$. As $(\mathfrak{M}6)$ also holds for $\mathfrak{M}$, henceforth $(\mho, \mathfrak{M}, \cdot)$ is a controlled fuzzy 2-metric space.*

After presenting the definition and initial examples, we now introduce the notion of convergent and Cauchy sequences, which will allow us to investigate the behavior of sequences in the presence of the imposed control functions.

**Definition 3.2.** *Let $(\mho, \mathfrak{M}, \star)$ be a controlled fuzzy 2-metric space and $\{\varsigma_n\} \subset \mho$ be a sequence.*

*(i) $\{\varsigma_n\}$ is said to be convergent to some $\varkappa \in \mho$ if for each $\rho > 0$ and $l \in (0, 1)$, there exists $N(\rho, l) \in \mathbb{N}$ such that $\mathfrak{M}(\varsigma_n, \varkappa, a, \rho) > 1 - l$, for all $n \geq N(\rho, l)$ and all $a \in \mho$.*

*(ii) $\{\varsigma_n\}$ is said to be a Cauchy sequence in $\mho$ if for each $\rho > 0$ and $l \in (0, 1)$, there exists $N(\rho, l) \in \mathbb{N}$ such that $\mathfrak{M}(\varsigma_n, \varsigma_m, a, \rho) > 1 - l$, for all $m, n \geq N(\rho, l)$ and all $a \in \mho$.*

**Proposition 3.1.** *Let $\{\varsigma_n\}$ be a sequence in a controlled fuzzy 2-metric space $(\mho, \mathfrak{M}, \star)$. Then*

*(1) $\{\varsigma_n\}$ converges to $\varkappa \iff \lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \varkappa, a, \rho) = 1$, $\forall \rho > 0$, $\forall a \in \mho$.*

*(2) $\{\varsigma_n\}$ is a Cauchy sequence in $\mho \iff \lim_{m,n\to\infty} \mathfrak{M}(\varsigma_n, \varsigma_m, a, \rho) = 1$, $\forall \rho > 0$, $\forall a \in \mho$ .*

*Proof.* The lines of proof for (ii) are as similar as (i). So we only prove (i).

For, consider a sequence $\{\varsigma_n\} \subset \mho$ converging to $\varkappa \in \mho$. Then by the definition, for each $\rho > 0$ and $r \in (0, 1)$, there exists $N(\rho, r) \in \mathbb{N}$ such that

$$\mathfrak{M}(\varsigma_n, \varkappa, a, \rho) > 1 - r, \quad \forall n \geq N(\rho, r),\ \forall a \in \mho.$$

Since $r \in (0, 1)$ is arbitrary, we can write $\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \varkappa, a, \rho) = 1$, $\forall \rho > 0$, $\forall a \in \mho$.

Conversely suppose that $\{\varsigma_n\}$ be a sequence in $\mho$ and $\varkappa \in \mho$ be such that

$$\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \varkappa, a, \rho) = 1, \quad \forall \rho > 0,\ \forall a \in \mho. \tag{3.6}$$

Let $r \in (0, 1)$ be given. Then relation (3.6) implies, for each $\rho > 0$ and given $r$, there exists $N(\rho, r) \in \mathbb{N}$ such that

$$\mathfrak{M}(\varsigma_n, \varkappa, a, \rho) > 1 - r, \quad \forall n \geq N(\rho, r),\ \forall\forall a \in \mho.$$

This concludes our proof. □

**Proposition 3.2.** *Let $(\mho, \mathfrak{M}, \star)$ be a controlled fuzzy 2-metric space with respect to the control functions $f, g$ and $h$. Assume that, for any $a \in \mho$*

$$\lim_{n\to\infty} f(a, b, b_n),\ \lim_{n\to\infty} g(a, b, b_n) \text{ and } \lim_{n\to\infty} h(a, b, b_n) \text{ exist finitely} \tag{3.7}$$

*whenever $\{b_n\}$ is a sequence in $\mho$ converging to $b \in \mho$. Then every convergent sequence in $(\mho, \mathfrak{M}, \star)$ has a unique limit.*

*Proof.* Suppose, $\{\varsigma_n\}$ is a convergent sequence in $\mho$ which converges to $\nu_1$ and $\nu_2$, where $\nu_1 \neq \nu_2$.

Then by $(\mathfrak{M}2)$, there exists $z \in \mho$ such that

$$\mathfrak{M}(\nu_1, \nu_2, z, \rho) \neq 1, \quad \forall \rho > 0. \tag{3.8}$$

Since $\{\varsigma_n\}$ converges to both $\nu_1$ and $\nu_2$,

$$\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \nu_1, a, \rho) = 1 = \lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \nu_2, a, \rho), \quad \forall \rho > 0,\ \forall a \in \mho.$$

Now, using the inequality $(\mathfrak{M}5)$, we can write

$$\mathfrak{M}(\nu_1, \nu_2, z, \rho_0) \geq \mathfrak{M}\left(\nu_1, \nu_2, \varsigma_n, \frac{\rho_0}{3f(\nu_1, \nu_2, \varsigma_n)}\right) \star \mathfrak{M}\left(\nu_2, z, \varsigma_n, \frac{\rho_0}{3g(\nu_2, z, \varsigma_n)}\right) \star \mathfrak{M}\left(\nu_1, z, \varsigma_n, \frac{\rho_0}{3h(\nu_1, z, \varsigma_n)}\right).$$

Using the limiting approach, we get

$$\begin{aligned}
&\mathfrak{M}(\nu_1, \nu_2, z, \rho_0) \\
&\geq \lim_{n\to\infty} \left[\mathfrak{M}\left(\nu_1, \nu_2, \varsigma_n, \frac{\rho_0}{3f(\nu_1, \nu_2, \varsigma_n)}\right) \star \mathfrak{M}\left(\nu_2, z, \varsigma_n, \frac{\rho_0}{3g(\nu_2, z, \varsigma_n)}\right) \star \mathfrak{M}\left(\nu_1, z, \varsigma_n, \frac{\rho_0}{3h(\nu_1, z, \varsigma_n)}\right)\right] \\
&= 1 \star 1 \star 1 = 1.
\end{aligned}$$

This leads to the conclusion that $\mathfrak{M}(\nu_1, \nu_2, z, \rho_0) = 1$ that contradicts our assumption (3.8). Hence, the proposition is established. □

**Remark 3.2.** *The conclusion of Proposition 3.2 may not hold in general without the assumption (3.7). To justify consider the sequence $\{\varsigma_n\} = \{n\}$, $n \geq 3$ in the controlled fuzzy 2-metric space of Example 3.3.*

*Now as $\{\varsigma_n\} \subset B$, so for any $\nu, \kappa \in \mho$, $f(\varsigma_n, \nu, \kappa) = \max\{n, \nu, \kappa\} = n$. Therefore $\lim_{n\to\infty} f(\varsigma_n, \nu, \kappa) = \infty$.*

*Then for any two arbitrary elements $a, \nu \in A$, we have*

$$\mathfrak{M}(\varsigma_n, a, \nu, \rho) = \mathfrak{M}(n, a, \nu, \rho) = \exp(-\frac{1}{n \cdot \rho}), \quad \forall \rho > 0, \forall n \geq 3$$

$$\Longrightarrow \lim_{n\to\infty} \mathfrak{M}(n, a, \nu, \rho) = \lim_{n\to\infty} \exp\{-\frac{1}{n \cdot \rho}\} = 1, \quad \forall \rho > 0.$$

*This shows that the sequence $\{\varsigma_n\}$ converges to every point of $A$. Consequently, $\{\varsigma_n\}$ has more than one limit point.*

**Proposition 3.3.** *Let $(\mho, \mathfrak{M}, \star)$ be a controlled fuzzy 2-metric space with respect to $f_i$, $i = 1, 2, 3$. Also, assume that for every $a \in \mho$,*

$$\lim_{n\to\infty} f_i(a, b, b_n) \text{ and } \lim_{m,n\to\infty} f_i(b, b_m, b_n) \tag{3.9}$$

*exist finitely whenever $\{b_n\}$ is a sequence in $\mho$ converging to $b \in \mho$. Then every convergent sequence is a Cauchy sequence in $(\mho, \mathfrak{M}, \star)$.*

*Proof.* Suppose $\{\varsigma_n\} \subset \mho$ is a convergent sequence that converges to $x \in \mho$ such that. Then

$$\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, x, a, \rho) = 1, \quad \forall \rho > 0, \forall a \in \mho.$$

Using $(\mathfrak{M}5)$, for all $m, n \in \mathbb{N}$ and all $a \in \mho$, we can write

$$\begin{aligned}&\mathfrak{M}(\varsigma_n, \varsigma_m, a, \rho)\\ \geq &\mathfrak{M}\left(\varsigma_n, \varsigma_m, x, \frac{\rho}{3f_1(\varsigma_n, \varsigma_m, x)}\right) \star \mathfrak{M}\left(\varsigma_m, a, x, \frac{\rho}{3f_2(\varsigma_m, a, x)}\right) \star \mathfrak{M}\left(\varsigma_n, a, x, \frac{\rho}{3f_3(\varsigma_n, a, x)}\right).\end{aligned}$$

Using the limiting approach, we get

$$\begin{aligned}\lim_{m,n\to\infty} \mathfrak{M}(\varsigma_n, \varsigma_m, a, \rho) \geq \lim_{m,n\to\infty} &\mathfrak{M}\left(\varsigma_n, \varsigma_m, x, \frac{\rho}{3f_1(\varsigma_n, \varsigma_m, x)}\right) \star \lim_{m\to\infty} \mathfrak{M}\left(\varsigma_m, a, x, \frac{\rho}{3f_2(\varsigma_m, a, x)}\right)\\ &\star \lim_{n\to\infty} \mathfrak{M}\left(\varsigma_n, a, x, \frac{\rho}{3f_3(\varsigma_n, a, x)}\right) = 1 \star 1 \star 1\end{aligned}$$

which leads to the conclusion

$$\lim_{m,n\to\infty} \mathfrak{M}(\varsigma_n, \varsigma_m, a, \rho) = 1, \quad \forall \rho > 0, \forall a \in \mho.$$

□

The concept of completeness plays a crucial role in fixed-point theory and is defined below in controlled fuzzy 2-metric spaces.

**Definition 3.3.** *A controlled fuzzy 2-metric space $(\mho, \mathfrak{M}, \star)$ is said to be complete if every Cauchy sequence in $\mho$ converges to somewhere in $\mho$.*

## 4. Fixed Point Theorem and Its Application

This section establishes a fixed-point theorem over a controlled fuzzy 2-metric space. We introduce a contractive condition involving the function $\mathfrak{M}$ and a parameter $k \in (0,1)$ which ensures the existence and uniqueness of fixed points for self-mappings. The theorem further incorporates certain limiting conditions on the control functions $f, g, h$, which play a crucial role in ensuring the convergence of sequences in the space.

4.1. **Fixed point theorem.** To facilitate our main result, we first present the following lemma, which ensures a specific behavior of the controlled fuzzy 2-metric function $\mathfrak{M}$ satisfying an inequality. Although the proof of this lemma is omitted, as it follows a similar reasoning to the corresponding result in [17].

**Lemma 4.1.** *In a controlled fuzzy 2-metric space $(\mho, \mathfrak{M}, \star)$, if $\mathfrak{M}(\varsigma, \nu, \rho) \geq \mathfrak{M}(\varsigma, \nu, \frac{\rho}{k})$ holds for all $\varsigma, \nu \in \mho$ and all $\rho > 0$ where $k \in (0,1)$, then $\varsigma = \nu$.*

Utilizing this fundamental result on the coincidence of points, we proceed to develop our main fixed-point theorem.

**Theorem 4.1.** *Let $(\mho, \mathfrak{M}, \star)$ be a complete controlled fuzzy 2-metric space such that $\lim_{\rho\to\infty} \mathfrak{M}(\varsigma, \nu, a, \rho) = 1$, $\forall \varsigma, \nu, a \in \mho$ and $\mathcal{S} : \mho \to \mho$ be a mapping satisfying:*

$$\mathfrak{M}(\mathcal{S}\varsigma, \mathcal{S}\nu, a, k\rho) \geq \mathfrak{M}(\varsigma, \nu, a, \rho), \quad \forall \varsigma, \nu, a \in \mho, \ \forall \rho > 0 \tag{4.1}$$

*where $k \in (0,1)$. Further assume, $f, g, h : \mho^3 \to [1, \infty)$ be three control functions such that for any sequence $\{a_n\} \subseteq \mho$ and for each $b \in \mho$,*

$$\lim_{m,n\to\infty} f(a_n, a_m, b), \lim_{m,n\to\infty} g(a_n, a_m, b), \lim_{m,n\to\infty} h(a_n, a_m, b) \text{ exists finitely.} \tag{4.2}$$

*Then $F(\mathcal{S})$, the set of fixed points of $\mathcal{S}$ contains exactly one point.*

*Proof.* Choose $\varsigma_0 \in \mho$ arbitrarily and define the iterative sequence $\varsigma_n = \mathcal{S}\varsigma_{n-1}$, $n \in \mathbb{N}$.

Applying (4.1), for any $a \in \mho$ and $\gamma > 0$, we get

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+1}, a, \gamma) = \mathfrak{M}(\mathcal{S}\varsigma_{n-1}, \mathcal{S}\varsigma_n, a, \gamma) \geq \mathfrak{M}\left(\varsigma_{n-1}, \varsigma_n, a, \frac{\gamma}{k}\right). \tag{4.3}$$

Repeated applying (4.3) n times gives

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+1}, a, \gamma) \geq \mathfrak{M}\left(\varsigma_0, \varsigma_1, a, \frac{\gamma}{k^n}\right). \tag{4.4}$$

Therefore, by induction, for each fixed $p \in \mathbb{N}$,

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+p}, a, \gamma) \geq \mathfrak{M}\left(\varsigma_0, \varsigma_p, a, \frac{\gamma}{k^n}\right). \tag{4.5}$$

By applying the iterative contraction repeatedly to (4.5), we observe that when the third argument is chosen as $\varsigma_{n+1}$

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1}, \gamma) \geq \mathfrak{M}(\varsigma_0, \varsigma_p, \varsigma_{n+1}, \frac{\gamma}{k^n}) = \mathfrak{M}(\varsigma_p, \varsigma_{n+1}, \varsigma_0, \gamma') \tag{4.6}$$

where $\gamma' = \frac{\gamma}{k^n}$ (say).

Let $p, n \in \mathbb{N}$. By successive applications of the contractive condition (4.1), the fourth variable is scaled at each step by the factor $\frac{1}{k}$, while the indices decrease accordingly.

If $p > n+1$, then after $(n+1)$ iterations we obtain

$$\mathfrak{M}(\varsigma_p, \varsigma_{n+1}, \varsigma_0, \gamma') \geq \mathfrak{M}\left(\varsigma_{p-n-1}, \varsigma_0, \varsigma_0, \frac{\gamma'}{k^{n+1}}\right) = \mathfrak{M}\left(\varsigma_{p-n-1}, \varsigma_0, \varsigma_0, \frac{\gamma}{k^{2n+1}}\right) = 1.$$

Similarly, if $p < n+1$, an analogous backward $p$ iteration yields

$$\mathfrak{M}(\varsigma_p, \varsigma_{n+1}, \varsigma_0, \gamma') \geq \mathfrak{M}\left(\varsigma_{n+1-p}, \varsigma_0, \varsigma_0, \frac{\gamma'}{k^{p}}\right) = \mathfrak{M}\left(\varsigma_{n+1-p}, \varsigma_0, \varsigma_0, \frac{\gamma}{k^{n+p}}\right) = 1.$$

In both of the above cases, for each $p, n \in \mathbb{N}$, we have

$$\mathfrak{M}(\varsigma_p, \varsigma_{n+1}, \varsigma_0, \frac{\gamma}{k^n}) = 1$$

and hence (4.6) yields

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1}, \gamma) = 1. \tag{4.7}$$

Using the inequality ($\mathfrak{M}$5), for each $p \in \mathbb{N}$, $\gamma > 0$, and $a \in \mho$, we have

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+p}, a, \gamma) \geq \mathfrak{M}\left(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1}, \frac{\gamma}{3f(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1})}\right) \star \mathfrak{M}\left(\varsigma_{n+p}, a, \varsigma_{n+1}, \frac{\gamma}{3h(\varsigma_{n+p}, a, \varsigma_{n+1})}\right) \star \mathfrak{M}\left(a, \varsigma_n, \varsigma_{n+1}, \frac{\gamma}{3h(a, \varsigma_n, \varsigma_{n+1})}\right).$$

Applying the earlier estimates (4.4) and (4.5), we further obtain

$$\mathfrak{M}(\varsigma_n, \varsigma_{n+p}, a, \gamma) \geq \mathfrak{M}\left(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1}, \frac{\gamma}{3f(\varsigma_n, \varsigma_{n+p}, \varsigma_{n+1})}\right) \star \mathfrak{M}\left(\varsigma_1, \varsigma_p, a, \frac{\gamma}{3k^n g(\varsigma_{n+1}, a, \varsigma_{n+p})}\right) \star \mathfrak{M}\left(\varsigma_0, \varsigma_1, a, \frac{\gamma}{3k^n h(\varsigma_n, \varsigma_{n+1}, a)}\right).$$

On the combined application of the condition $\lim_{\rho\to\infty} \mathfrak{M}(\nu, y, a, \rho) = 1$, (4.2) and (4.7), for each fixed $p \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, \varsigma_{n+p}, a, \gamma) \geq 1 \star 1 \star 1 = 1, \quad \forall \gamma > 0.$$

This yields that $\{\varsigma_n\}$ is a Cauchy sequence in $\mho$. Since $\mho$ is a complete controlled fuzzy 2-metric space, so there exists $x \in \mho$ such that

$$\lim_{n\to\infty} \mathfrak{M}(\varsigma_n, x, a, \gamma) = 1, \quad \forall \gamma > 0, \forall a \in \mho. \tag{4.8}$$

Now we claim that $x \in F(\mathcal{S})$, i.e., $\mathcal{S}x = x$. If not, then there exists $b \in \mho$ and $\gamma_0 > 0$ such that

$$\mathfrak{M}(\mathcal{S}x, x, b, \gamma_0) < 1. \tag{4.9}$$

Then again applying the inequality ($\mathfrak{M}5$), we reached to

$$\begin{aligned}
&\mathfrak{M}(\mathcal{S}x, x, b, \gamma_0)\\
&\geq \mathfrak{M}\Big(\mathcal{S}x, x, \varsigma_n, \frac{\gamma_0}{3f(\mathcal{S}x, x, \varsigma_n)}\Big) \star \mathfrak{M}\Big(x, b, \varsigma_n, \frac{\gamma_0}{3g(x, b, \varsigma_n)}\Big) \star \mathfrak{M}\Big(b, \mathcal{S}x, \varsigma_n, \frac{\gamma_0}{3h(b, \mathcal{S}x, \varsigma_n)}\Big)\\
&\geq \mathfrak{M}\Big(x, x, \varsigma_{n-1}, \frac{\gamma_0}{3kf(\mathcal{S}x, x, \varsigma_n)}\Big) \star \mathfrak{M}\Big(x, b, \varsigma_n, \frac{\gamma_0}{3g(x, b, \varsigma_n)}\Big) \star \mathfrak{M}\Big(\varsigma_{n-1}, x, b, \frac{\gamma_0}{3kh(b, \mathcal{S}x, \varsigma_n)}\Big) \quad (\text{using (4.1)}).
\end{aligned}$$

Taking limiting approach and on the appliance of (4.2) and (4.8), we obtain

$$\mathfrak{M}(\mathcal{S}x, x, b, \gamma_0) = 1$$

which contradicts the assumption (4.9).

Next, assume that $\kappa \in \mho$ be such that $\kappa \in F(\mathcal{S})$. Applying (4.1) yields

$$\mathfrak{M}(x, \kappa, a, \gamma) = \mathfrak{M}(\mathcal{S}x, \mathcal{S}\kappa, a, \gamma) \geq \mathfrak{M}(x, \kappa, a, \frac{\gamma}{k}).$$

Hence, using Lemma 4.1, we conclude that

$$x = \kappa,$$

showing that the fixed point of $\mathcal{S}$ is unique. □

4.2. **Application to Marketing Equilibrium problem.** In economic theory, dynamic market equilibrium plays a crucial role in understanding how prices and quantities adjust over time under various influencing factors. The interplay between supply, demand, and price evolution is essential for modeling real-world economic systems. Within this framework, integral equations provide a powerful tool to describe these interactions mathematically.

We illustrate how our established fixed-point result in subsection 4.1 can be applied to determine the unique solution of an integral equation in the context of dynamic economic equilibrium. In various markets, the supply function $Q_\beta$ and the demand function $Q_d$, along with present pricing trends-such as price increases, decreases, and rates of change-significantly influence market behavior. The current price is $\Theta(\alpha)$ and $\frac{d\Theta(\alpha)}{d\alpha}$, $\frac{d^2\Theta(\alpha)}{d\alpha^2}$ are stands for particular interest to economists, as they provide insights into market stability and price fluctuations over time.

Assume,

$$\begin{aligned}
Q_\beta &= g_1 + \gamma_1\Theta(\alpha) + e_1\frac{d\Theta(\alpha)}{d\alpha} + \varsigma_1\frac{d^2\Theta(\alpha)}{d\alpha^2}\\
Q_d &= g_2 + \gamma_2\Theta(\alpha) + e_2\frac{d\Theta(\alpha)}{d\alpha} + \varsigma_2\frac{d^2\Theta(\alpha)}{d\alpha}.
\end{aligned}$$

where $g_i, \gamma_i, e_i$, $i = 1, 2$ are constants.

If prices continuously adjust to equate supply and demand at every moment, the market can be regarded as dynamically stable. Therefore, in equilibrium

$$Q_\beta = Q_d.$$

So,

$$g_1 + \gamma_1\Theta(\alpha) + e_1\frac{d\Theta(\alpha)}{d\alpha} + \varsigma_1\frac{d^2\Theta(\alpha)}{d\alpha^2} = g_2 + \gamma_2\Theta(\alpha) + e_2\frac{d\Theta(\alpha)}{d\alpha} + \varsigma_2\frac{d^2\Theta(\alpha)}{d\alpha}$$
$$\Longrightarrow (\varsigma_1 - \varsigma_2)\frac{d^2\Theta(\alpha)}{d\alpha^2} + (e_1 - e_2)\frac{d\Theta(\alpha)}{d\alpha} + (\gamma_1 - \gamma_2)\Theta(\alpha) + (g_1 - g_2) = 0$$

Letting $\varsigma = (\varsigma_1 - \varsigma_2)$, $e = (e_1 - e_2)$, $\gamma = (\gamma_1 - \gamma_2)$, $g = (g_1 - g_2)$, we can write

$$\varsigma\frac{d^2\Theta(\alpha)}{d\alpha} + e\frac{d\Theta(\alpha)}{d\alpha} + \gamma\Theta(\alpha) = -g.$$

Henceforth, $\Theta(\alpha)$ can be determined by the following IVP:

$$\Theta'' + \frac{e}{\varsigma}\Theta' + \frac{\gamma}{\varsigma}\Theta = -\frac{g}{\varsigma}, \quad \Theta(0) = 0, \Theta'(0) = 0. \tag{4.10}$$

Assuming $\frac{e^2(\alpha)}{\varsigma} = \frac{4\gamma(\alpha)}{\varsigma}$ and letting $\frac{\gamma(\alpha)}{e(\alpha)} = \mu(\alpha)$, where $\mu(\alpha)$ is a continuous function of $\alpha$, the IVP (4.10) can be solved accordingly.

It is easy to show that (4.10) is equivalent to the integral equation

$$\Theta(\alpha) = \int_0^T \psi(\alpha, r)F(\alpha, r, \Theta(r))dr$$

where $F(\alpha, r, \Theta(r))$ represents some transformed function of the original differential equation, and $\psi(\alpha, r)$ is introduced to capture the effect of the coefficients in the differential equation to ensure that the solution remains consistent with the original system. The specific form of $\psi(\alpha, l)$, given as:

$$\psi(\alpha, l) = \begin{cases} le^{\frac{\mu(\alpha - l)}{2}}, & 0 \le l \le \alpha \le T \\ \alpha e^{\frac{\mu(l-\alpha)}{2}}, & 0 \le \alpha \le l \le T. \end{cases}$$

If we define a function $G : [0, T]^2 \times \mathbb{R} \to \mathbb{R}^+$ by $G(\alpha, l, \Theta(l)) = \psi(\alpha, l)\, F(\alpha, l, \Theta(l))$, then the problem (4.10) has the integral solution:

$$\Theta(\alpha) = \int_0^T G(\alpha, r, \Theta(r))\, dr. \tag{4.11}$$

The differential equation (4.10) is equivalent to its integral formulation (4.11), which means that any solution to the integral equation also satisfies the original differential equation.

Next, consider $C[0, T]$ and denote it by $\mho$. Then, following the similar steps of Example 3.2, we can conclude that

$$\mathfrak{M}(\varsigma, y, \nu, s) = \exp\left(-\sup_{\iota\in[0,T]}\frac{\sqrt{|(\varsigma(\iota) - y(\iota))\,(y(\iota) - \nu(\iota))\,(\nu(\iota) - \varsigma(\iota))|}}{s}\right), \quad \forall \varsigma, y, \nu \in \mho, \forall s > 0$$

is a controlled fuzzy 2-metric space with respect to the control functions:

$$f(\varsigma, y, \nu) = \sup_{\iota\in[0,T]}\{\sqrt{|(\varsigma(\iota) - y(\iota))\,(y(\iota) - \nu(\iota))\,(\nu(\iota) - \varsigma(\iota))| + 1}\},$$
$$g(\varsigma, y, \nu) = \sup_{\iota\in[0,T]}\{(\varsigma(\iota) - y(\iota))^2\,(y(\iota) - \nu(\iota))^2\,(\nu(\iota) - \varsigma(\iota))^2 + 2\},$$

$$\text{and} \quad h(\varsigma, y, \nu) = \sup_{\iota\in[0,T]} \{ \sqrt{|(\varsigma(\iota) - y(\iota))\,(y(\iota) - \nu(\iota))|}\,(\nu(\iota) - \varsigma(\iota))^2 + 1\}, \quad \forall \varsigma, y, \nu \in \mho,$$

where the considered $t$-norm is '*product*'. It is easy to prove that $\mho$ is complete in the sense of controlled fuzzy 2-metric $\mathfrak{M}$.

Before establishing the existence and uniqueness of a solution to equation (4.11), we impose certain conditions on the function G to ensure the well-posedness of the integral equation. The following theorem provides sufficient criteria under which equation (4.10) admits a unique solution.

**Theorem 4.2.** *Consider the integral equation (4.11) and suppose that*

*(i) the function $G : [0,T]^2 \times \mathbb{R} \to \mathbb{R}^+$ is continuous;*
*(ii) there exists $K \in (0,1)$ such that for any $\Theta, Q \in \mho$,*
  *(a) $|\int_0^T G(\alpha, l, \Theta(l)) - Q(\alpha)| \leq K \sup_{\alpha\in[0,T]} |\Theta(\alpha) - Q(\alpha)|$;*
  *(b) $\left| \int_0^T \Big( G(\alpha, l, \Theta(l)) - G(\alpha, l, Q(l)) \Big) dl \right| \leq K \sup_{\alpha\in[0,T]} \Big| \Theta(\alpha) - Q(\alpha) \Big|$;*
  *holds for all $\alpha, l \in [0,T]$.*

*Then the equation (4.11) has a unique solution in $C[0,T]$.*

*Proof.* Consider the complete controlled fuzzy 2-metric space $(\mho, \mathfrak{M}, \star)$ discussed above and define a self mapping $F : \mho \to \mho$ by

$$(F\Theta)(\alpha) = \int_0^T G(\alpha, r, \Theta(r))dr, \quad \forall \Theta \in \mho, \ \forall \alpha \in [0,T]. \tag{4.12}$$

Then for any $\varsigma, y, \nu \in \mho$ and $s > 0$, we have

$$\mathfrak{M}(F\varsigma, Fy, \nu, Ks) = \exp\left( - \sup_{\iota\in[0,T]} \frac{\sqrt{|(F\varsigma(\iota) - Fy(\iota))\,(Fy(\iota) - \nu(\iota))\,(\nu(\iota) - F\varsigma(\iota))|}}{Ks} \right)$$

$$\textit{and} \qquad \mathfrak{M}(\varsigma, y, \nu, s) = \exp\left( - \sup_{\iota\in[0,T]} \frac{\sqrt{|(\varsigma(\iota) - y(\iota))\,(y(\iota) - \nu(\iota))\,(\nu(\iota) - \varsigma(\iota))|}}{s} \right).$$

Now we have,

$$\begin{aligned}
&|(F\varsigma(\iota) - Fy(\iota))(Fy(\iota) - \nu(\iota))(\nu(\iota) - F\varsigma(\iota))| \\
&\leq \left| \int_0^T \Big( G(\alpha, l, \varsigma(l)) - G(\alpha, l, y(l)) \Big) dl \right| \cdot \left| \int_0^T G(\alpha, l, y(l))dl - \nu(\iota) \right| \cdot \left| \nu(\iota) - \int_0^T G(\alpha, l, \varsigma(l))dl \right| \\
&\leq K \sup_{\iota\in[0,T]} |\varsigma(\iota) - y(\iota)| \cdot K \sup_{\iota\in[0,T]} |y(\iota) - \nu(\iota)| \cdot K \sup_{\iota\in[0,T]} |\varsigma(\iota) - \nu(\iota)| \\
&\leq K^3 \sup_{\iota\in[0,T]} \Big\{ |\varsigma(\iota) - y(\iota)| \cdot |y(\iota) - \nu(\iota)| \cdot |\varsigma(\iota) - \nu(\iota)| \Big\}.
\end{aligned}$$

Therefore, for all $s > 0$, we have

$$\frac{\sqrt{|(F\varsigma(\iota) - Fy(\iota))\,(Fy(\iota) - \nu(\iota))\,(\nu(\iota) - F\varsigma(\iota))|}}{K^{3/2}s} \leq \sup_{\iota\in[0,T]} \frac{\sqrt{|(\varsigma(\iota) - y(\iota))\,(y(\iota) - \nu(\iota))\,(\varsigma(\iota) - \nu(\iota))|}}{s}, \quad \forall \iota \in [0,T]$$

Taking the supremum over $\iota \in [0,T]$ on both sides, we obtain

$$\sup_{\iota\in[0,T]} \frac{\sqrt{|(F\varsigma(\iota) - Fy(\iota))\,(Fy(\iota) - \nu(\iota))\,(\nu(\iota) - F\varsigma(\iota))|}}{K^{3/2}s} \leq \sup_{\iota\in[0,T]} \frac{\sqrt{|(\varsigma(\iota) - y(\iota)|)\,(y(\iota) - \nu(\iota))\,(\varsigma(\iota) - \nu(\iota))|}}{s}.$$

Equivalently, this shows that

$$\mathfrak{M}(F\varsigma, Fy, \nu, Rs) \geq \mathfrak{M}(\varsigma, y, \nu, s),\ \forall \varsigma, y, \nu \in \mho,\ \forall s > 0$$

where $R = K^{3/2} \in (0,1)$.

Therefore the operator $F$ satisfies the contraction condition of Theorem 4.1 and henceforth the integral equation (4.11) has a unique solution in $C[0,T]$. Consequently the differential equation (4.10) admits a unique solution. □

## 5. Simulation and Soft Computing Interpretation

The paradigm of soft computing emphasizes flexible modeling of systems with uncertainty, gradual convergence, and mutual interaction. In this spirit, we now present a dynamic simulation under the framework of controlled fuzzy 2-metric spaces, highlighting how such structures can naturally accommodate soft computing principles such as approximation, adaptability, and decentralized evolution. For given initial estimates $x_0, y_0, z_0 \in \mathbb{R}$, the system evolves iteratively via symmetric pairwise cooperation. Exact convergence in the Euclidean sense is replaced by convergence in a controlled fuzzy 2-metric: $\mathfrak{M}(x, y, z, \rho) = \frac{\rho}{\rho + D(x,y,z)}$, $\forall x, y, z \in \mathbb{R}$, $\rho > 0$ (please see Example 3.1).

We now simulate a system of three agents, whose estimates evolve iteratively under a symmetric, cooperative influence rule. Let the initial values be:

$$x_0 = 5, \quad y_0 = 10, \quad z_0 = 15.$$

At each discrete time step $t \in \mathbb{N}$, the agents update their positions according to:

$$x_{t+1} = \frac{y_t + z_t}{2}, \quad y_{t+1} = \frac{x_t + z_t}{2}, \quad z_{t+1} = \frac{x_t + y_t}{2}.$$

This iterative process simulates a cooperative decision-making system, where each agent updates its estimate based on the current state of others-capturing the decentralized adjustment mechanisms central to soft computing. The evolving sequence $x_t, y_t, z_t$ models the gradual refinement of decisions under mutual influence, a characteristic feature of distributed systems.

At each time step, the controlled fuzzy 2-metric $\mathfrak{M}(x_t, y_t, z_t, t)$ is evaluated, serving as a quantitative indicator of information alignment among agents, adjusted for imprecision. An increasing trend in $\mathfrak{M}$-values reflects growing synchronization and reduced uncertainty.

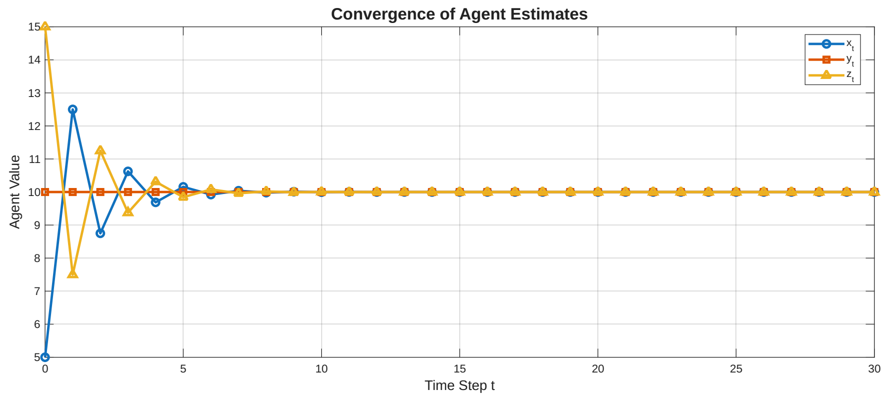


Figure 1. Convergence of agent estimates $x_t, y_t, z_t$ over time $t$.

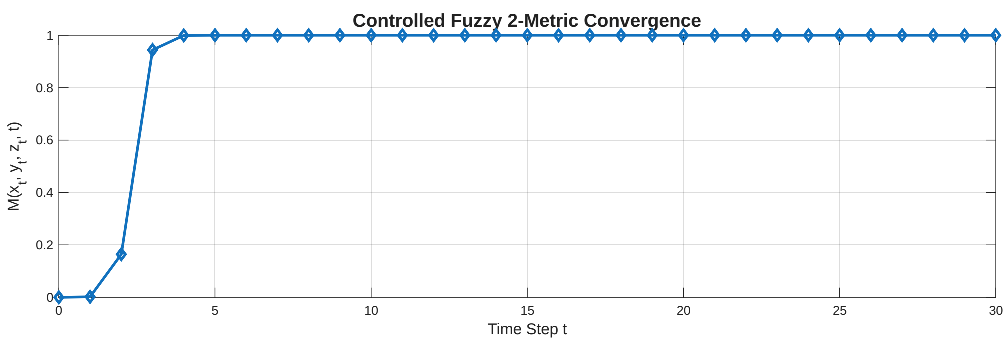


Figure 2. Evolution of the controlled fuzzy 2-metric $\mathfrak{M}(x_t, y_t, z_t, t)$ over time. Increasing values signify reduced imprecision and convergence.

As shown in Figures 1 and 2, the agent estimates converge toward a common value, representing a stable equilibrium. Concurrently, the controlled fuzzy 2-metric value approaches 1, indicating a reduction in the system's uncertainty and a reinforcement of proximity in the fuzzy sense.

These observations confirm that the proposed framework provides an effective and intuitive tool for modeling adaptive systems with uncertainty, reinforcing its relevance to soft computing and decision theory, especially in contexts like dynamic markets and interactive systems governed by approximate reasoning.

## Conclusion

In this article, we introduce the concept of controlled fuzzy 2-metric spaces, a new generalization of fuzzy 2-metric structures achieved by incorporating three distinct control functions that dynamically regulate the proximity among triplets of points. Through illustrative examples, we demonstrate how these control functions refine the structure of fuzzy 2-metrics, extending classical axioms. Furthermore, we apply this framework to the analysis of dynamic market equilibrium, showing that controlled fuzzy distances can effectively represent agents' adaptive behaviors and uncertainty in decision-making. This establishes a clear link between abstract fuzzy analysis and practical modeling in economics and soft computing.

The scientific value of this work lies in providing a unified and adaptive metric model capable of simulating systems where relationships evolve under information uncertainty-an area where traditional metrics fall short. Nevertheless, the current study is limited to theoretical formulation and basic examples; further research may focus on developing computational algorithms, numerical simulations, and applications to learning-based or stochastic decision systems within this framework. Overall, the proposed approach effectively merges fuzzy logic, adaptive control, and metric-based modeling to simulate information-sensitive, consensus-driven dynamics. It thus provides a promising mathematical foundation for analyzing decision-making systems through the lens of soft computing, information theory, and dynamic equilibrium analysis.

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