Int. J. Anal. Appl. (2026), 24:90

International Journal of Analysis and Applications

Weakly-Heyting Almost Distributive Lattices

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Abstract. The primary goal of this article is to define the Weakly-Heyting Almost Distributive Lattice (WHADL), a novel algebraic structure that extends weakly-Heyting algebras within the broader class of almost distributive lattices. Through illustrative examples, we demonstrate that Heyting Almost Distributive Lattices (HADLs) and WHADLs are distinct structures. Furthermore, we identify and examine several sets of properties that characterize WHADLs, investigate their internal structure via principal ideals, and establish a necessary condition under which an WHADL becomes a HADL.

1. Introduction

Numerous features and correlations that are important in many theoretical and applied disciplines have been revealed through the study of lattices, which are fundamental structures in mathematics. A set of elements can be understood in terms of order, structure, and logical reasoning using lattices as a basic framework. Algebraic structures like Heyting algebras [3], semi-Heyting algebras [10], weakly-Heyting algebra [4], play pivotal roles in lattice theory and logic, offering frameworks to generalize distributive lattices and model intuitionistic reasoning, respectively. The class of almost distributive lattices [9] stands out among these lattice structures as a promising field of study that captures the fine balance between distributivity and deviation.

Received: Jan. 12, 2026.

2020 *Mathematics Subject Classification.* 06D20, 06D99.

Key words and phrases. almost distributive lattice; ideal; Heyting almost distributive lattice; weakly-Heyting almost distributive lattice.

https://doi.org/10.28924/2291-8639-24-2026-90
ISSN: 2291-8639

An almost distributive lattice relaxes the strict distributive laws of traditional lattices, providing flexibility in their algebraic behavior. On the other hand, Heyting algebras enrich Boolean-like structures with an implication operation, making them foundational in intuitionistic logic. A Heyting almost distributive lattice (HADL) [6] arises as a synthesis of these concepts, combining the relaxed distributive properties of ADLs with the logical richness of Heyting algebras. This hybrid structure enables new avenues for exploring non-classical logics and generalized lattice theories.
This paper introduces and investigates the concept of weakly-Heyting almost distributive lattices (WHADLs), a novel algebraic structure that extends weakly-Heyting algebras into the broader framework of almost distributive lattices. WHADLs merge the logical properties of weak Heyting algebras with the structural flexibility of ADLs, making them an interesting subject for both theoretical exploration and practical applications. We begin by defining the axioms and operations that characterize an WHADL. Also, we show that the algebras Heyting almost distributive lattice and weakly-Heyting almost bdistributive lattice are different from each other by means of examples. Following this, we establish several fundamental properties that distinguish WHADLs within the hierarchy of lattice-based structures. Moreover, we analyze WHADLs through their principal ideals, providing deeper insight into their internal organization and relationships and prove that it is a HADL by imposing a condition on an WHADL. By introducing WHADLs, this work aims to contribute to the ongoing development of lattice theory and its intersections with algebraic logic. The results presented here in lay the ground work for further exploration of these lattices in both abstract settings and applied domains. In this section, we recall some essential and frequently used results concerning almost distributive lattices, weakly-Heyting algebras and Heyting almost distributive lattices. These foundational concepts and properties will be referenced throughout the paper.

Definition 1.1. *[9] An algebra $(\mathcal{W}, \vee_*, \wedge_*, 0)$ of type $(2, 2, 0)$ is said to be an Almost Distributive Lattice (ADL) if it satisfies the following identities;*

(1) $\mathsf{w}_1 \vee_* 0 = \mathsf{w}_1$
(2) $0 \wedge_* \mathsf{w}_1 = 0$
(3) $(\mathsf{w}_1 \vee_* \mathsf{w}_2) \wedge_* \mathsf{w}_3 = (\mathsf{w}_1 \wedge_* \mathsf{w}_3) \vee_* (\mathsf{w}_2 \wedge_* \mathsf{w}_3)$
(4) $\mathsf{w}_1 \wedge_* (\mathsf{w}_2 \vee_* \mathsf{w}_3) = (\mathsf{w}_1 \wedge_* \mathsf{w}_2) \vee_* (\mathsf{w}_1 \wedge_* \mathsf{w}_3)$
(5) $\mathsf{w}_1 \vee_* (\mathsf{w}_2 \wedge_* \mathsf{w}_3) = (\mathsf{w}_1 \vee_* \mathsf{w}_2) \wedge_* (\mathsf{w}_1 \vee_* \mathsf{w}_3)$
(6) $(\mathsf{w}_1 \vee_* \mathsf{w}_2) \wedge_* \mathsf{w}_2 = \mathsf{w}_2$

for all $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3 \in \mathcal{W}$.

Example 1.1. *[9] Let $\mathcal{W}$ be a non-empty set and $\mathsf{w}_1, \mathsf{w}_2 \in \mathcal{W}$. Define the binary operation $\mathsf{w}_1 \wedge_* \mathsf{w}_2 = \mathsf{w}_2, \mathsf{w}_1 \vee_* \mathsf{w}_2 = \mathsf{w}_1$. Then the algebra $(\mathcal{W}, \vee_*, \wedge_*)$ forms an Almost Distributive Lattice (ADL), which is referred to as a discrete ADL.*

Unless otherwise noted, an almost distributive lattice $(\mathcal{W}, \vee_*, \wedge_*)$ is referred to as $\mathcal{W}$ throughout this section. The notation $w_1 \leq_* w_2$ indicates that, given $w_1, w_2 \in \mathcal{W}$, w_1 is less than or equal to w_2 if and only if $w_1 = w_1 \wedge_* w_2$; or, alternatively, $w_1 \vee_* w_2 = w_2$,. $\leq_*$ is a partial ordering on $\mathcal{W}$ as a result. If there is no element w_1 such that $m_w < w_1$, then an element m_w is considered maximum.

Theorem 1.1. *[9] For any $m_w \in \mathcal{W}$, the following are equivalent;*

(1) m_w *is maximal*
(2) $m_w \vee_* w_1 = m_w$, *for all* $w_1 \in \mathcal{W}$
(3) $m_w \wedge_* w_1 = w_1$, *for all* $w_1 \in \mathcal{W}$

Theorem 1.2. *[9] For any $w_1, w_2, w_3 \in \mathcal{W}$,*

(1) $w_1 \vee_* w_2 = w_1 \iff w_1 \wedge_* w_2 = w_1$.
(2) $w_1 \vee_* w_2 = w_2 \iff w_1 \wedge_* w_2 = w_1$.
(3) $w_1 \wedge_* w_2 = w_2 \wedge_* w_1 = w_1$ *whenever* $w_1 \leq w_2$.
(4) $\wedge$ *is associative.*
(5) $w_1 \wedge_* w_2 \wedge_* w_3 = w_2 \wedge_* w_1 \wedge_* w_3$.
(6) $(w_1 \vee_* w_2) \wedge_* w_3 = (w_2 \vee_* w_1) \wedge w_3$.
(7) $w_1 \wedge_* w_2 \leq w_2$ *and* $w_1 \leq w_1 \vee w_2$.
(8) $w_1 \wedge_* w_1 = w_1$ *and* $w_1 \vee_* w_1 = w_1$.
(9) If $w_1 \leq w_3$ *and* $w_2 \leq w_3$, *then* $w_1 \wedge w_2 = w_2 \wedge_* w_1$ *and* $w_1 \vee_* w_2 = w_2 \vee_* w_1$.

Definition 1.2. *[9] A non-empty subset I of $\mathcal{W}$ is said to be an ideal, if it satisfies the following:*

(1) For any $w_1, w_2 \in I, w_1 \vee_* w_2 \in I$
(2) For any $w_1 \in I, w_2 \in \mathcal{W}, w_1 \wedge_* w_2 \in I$

Theorem 1.3. *[5] The set $I(\mathcal{W})$ of ideals of $\mathcal{W}$ is a distributive lattice in which the g.l.b and the l.u.b of any two ideals $\mathcal{I}$,$\mathcal{J}$ are $\mathcal{I} \wedge_* \mathcal{J} = \mathcal{I} \cap \mathcal{J}$ and $\mathcal{I} \vee_* \mathcal{J} = \{w_1 \vee_* w_2 \mid w_1 \in \mathcal{I}$ and $w_2 \in \mathcal{J}\}$ respectively.*

Theorem 1.4. *[5] The class $PI(\mathcal{W})$ of all principal ideals of an ADL $\mathcal{W}$ is a sublattice of the distributive lattice $I(\mathcal{W})$ of ideals of $\mathcal{W}$.*

Theorem 1.5. *[5] The following statements are equivalent to each other in $\mathcal{W}$:*

(1). $\mathcal{W}$ is a distributive lattice.
(2). $(\mathcal{W}, \leq_*)$ is directed above poset.
(3). $\vee_*$ is commutative.
(4). $\wedge_*$ is commutative.
(5). $\vee_*$ is right distributive over $\wedge_*$.
(6). The relation $\theta = \{(w_1, w_2) \in \mathcal{W} \times \mathcal{W} \mid w_2 \wedge_* w_1 = w_1\}$ is antisymmetric.

Definition 1.3. *[4] An algebra $(\mathcal{W}, \vee_*, \wedge_*, \rightarrow_*, 0, 1)$ of type (2, 2, 2, 0, 0) is a weakly Heyting algebra (WHA) if $(\mathcal{W}, \vee_*, \wedge_*, 0, 1)$ is a bounded distributive lattice with the following axioms;*

(W_1) $w_1 \rightarrow_* w_1 = 1$

(W_2) $w_1 \rightarrow_* (w_2 \wedge_* w_3) = [(w_1 \rightarrow_* w_2) \wedge_* (w_1 \rightarrow_* w_3)]$

(W_3) $(w_1 \vee_* w_2) \rightarrow_* w_3 = [(w_1 \rightarrow_* w_3) \wedge_* (w_2 \rightarrow_* w_3)]$

(W_4) $[(w_1 \rightarrow_* w_2) \wedge_* (w_2 \rightarrow_* w_3) \leq (w_1 \rightarrow_* w_3)]$

for all $w_1, w_2, w_3 \in \mathcal{W}$.

Definition 1.4. *[8]* $\mathcal{W}$ *with a maximal element* m_w *is said to be a Heyting almost distributive lattice (HADL), if there is a binary operation* $\rightarrow_*$ *on* $\mathcal{W}$ *with the following axioms:*

(H_1) $w_1 \rightarrow_* w_1 = m_w$

(H_2) $(w_1 \rightarrow_* w_2) \wedge_* w_2 = w_2$

(H_3) $w_1 \wedge_* (w_1 \rightarrow_* w_2) = w_1 \wedge_* w_2 \wedge_* m_w$

(H_4) $w_1 \rightarrow_* (w_2 \wedge_* w_3) = [(w_1 \rightarrow_* w_2) \wedge_* (w_1 \rightarrow_* w_3)]$

(H_5) $(w_1 \vee_* w_2) \rightarrow_* w_3 = [(w_1 \rightarrow_* w_3) \wedge_* (w_2 \rightarrow_* w_3)]$

for all $w_1, w_2, w_3 \in \mathcal{W}$.

2. Weakly-Heyting Almost Distributive Lattices

In this section, we introduce weakly-Heyting almost distributive lattices and present several illustrative examples. We then compare these structures with Heyting almost distributive lattices and investigate their algebraic properties. Furthermore, we establish necessary and sufficient conditions under which a weakly-Heyting almost distributive lattice becomes a weakly-Heyting algebra and Heyting almost distributive lattice.

Definition 2.1. *An almost distributive lattice* $(\mathcal{W}, \vee_*, \wedge_*, 0, m_w)$ *is said to be a weakly-Heyting almost distributive lattice (WHADL) if there is a binary operation* $\rightarrow_*$ *on* $\mathcal{W}$ *with the following identities;*

(WH_1): $(w_1 \rightarrow_* w_1) \wedge_* m_w = m_w$

(WH_2): $[w_1 \rightarrow_* (w_2 \wedge_* w_3)] \wedge_* m_w = [(w_1 \rightarrow_* w_2) \wedge_* (w_1 \rightarrow_* w_3)] \wedge_* m_w$

(WH_3): $[(w_1 \vee_* w_2) \rightarrow_* w_3] \wedge_* m_w = [(w_1 \rightarrow_* w_3) \wedge_* (w_2 \rightarrow_* w_3)] \wedge_* m_w$

(WH_4): $[(w_1 \rightarrow_* w_2) \wedge_* (w_2 \rightarrow_* w_3) \wedge_* (w_1 \rightarrow_* w_3)] \wedge_* m_w = [(w_1 \rightarrow_* w_2) \wedge_* (w_2 \rightarrow_* w_3)] \wedge_* m_w$

(WH_5): $(w_1 \rightarrow_* w_2) \wedge_* m_w = [(w_1 \wedge_* m_w) \rightarrow_* (w_2 \wedge_* m_w)]$

for all $w_1, w_2, w_3 \in \mathcal{W}$.

Trough out this section $\mathcal{W}$ stands for a WHADL $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_*, 0, m_w \rangle$ unless otherwise specified.

Remark 2.1. *Every weakly-Heyting algebra is a weakly-Heyting almost distributive lattice. The converse need not be true. For, see the following example.*

Example 2.1. *Let* $\mathcal{W} = \{0, 1, 2\}$ *such that* $0 < 1 < 2$. *Consider the following binary operation* $\rightarrow_*$;

$\rightarrow_*$	0	1	2
0	2	2	2
1	1	2	2
2	0	0	2

Then $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_*, 0, 2\rangle$ *is a weakly-Heyting almost distributive lattice with 2 as its maximal element.*

Example 2.2. *Let* $\mathcal{W} = \{0, 1, 2, 3, 4\}$ *such that* $0 < 1, 2 < 3 < 4$ *and* $1, 2$ *are incomparable. Consider the following binary operation* $\rightarrow_*$;

$\vee_*$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

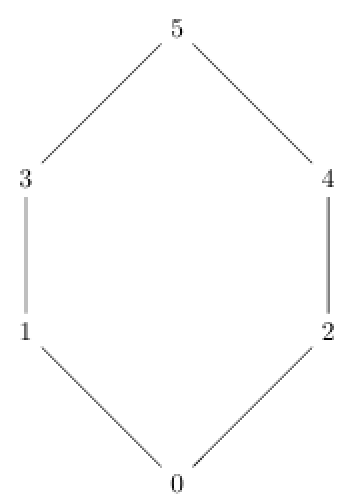
$\wedge_*$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

$\rightarrow_*$	0	1	2	3	4
0	4	4	4	4	4
1	0	4	0	4	4
2	4	4	4	4	4
3	0	4	0	4	4
4	0	3	0	3	4

Then $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_*, 0, 4\rangle$ *is a weakly-Heyting almost distributive lattice with 4 as its maxima element.*

In Example 2.3, it is shown that an implication operation defined on a lattice can satisfy the conditions of both a Heyting ADL and a weakly-Heyting ADL. Furthermore, an implication operation may define a Heyting ADL without necessarily making it a weakly-Heyting ADL. Conversely, another choice of implication on the same lattice may fail to form a Heyting ADL while still satisfying the conditions of a weakly-Heyting ADL. This demonstrates that the classification of a lattice as a Heyting ADL or a weakly Heyting ADL depends on the specific implication chosen, highlighting the refined relationship between these algebraic structures.

Example 2.3. *Let* $\mathcal{W} = \{0, 1, 2, 3, 4, 5\}$ *whose Hasse-diagram is*



Let us consider the following three implication operators on $\mathcal{W}$:

$\rightarrow_1$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	4	5	4	5
2	3	3	5	3	5	5
3	4	1	4	5	4	5
4	3	3	2	3	5	5
5	0	1	2	3	4	5

$\rightarrow_2$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	2	5	5	5
2	0	1	5	3	5	5
3	0	4	2	5	4	5
4	0	3	2	3	5	5
5	0	1	2	3	4	5

$\rightarrow_3$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	4	5	4	5
2	0	3	5	4	5	5
3	0	1	4	5	2	5
4	0	2	3	4	5	5
5	0	1	2	3	4	5

1. Trivially, $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_1, 0, 5\rangle$ is a Heyting almost distributive lattice and weakly-Heyting almost distributive lattice with 5 as its maximal element.
2. Notice that $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_2, 0, 5\rangle$ with 5 as its maximal element, is not a Heyting almost distributive lattice because H_3 fails for the pairs $(1,4)$ and $(3,1)$. However,it qualifies as a weakly-Heyting almost distributive lattice and the axioms WH_1 to WH_5 from Definition 3.1, can be easily verified.
3. Notice that $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_3, 0, 5\rangle$ with 5 as its maximal element, is not a weakly-Heyting almost distributive lattice because WH_4 fails for the pairs $(2,1,3)$,$(3,4,1)$, $(4,5,1)$ and $(5,2,3)$. However, it qualifies as a Heyting almost distributive lattice and the axioms H_1 to H_5 from Definition 1.4, can be easily verified.

In Lemma 2.1 and 2.2, we derive some properties of WH-ADL, which are frequently used to further develop the theory.

Lemma 2.1. *Let* $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3 \in \mathcal{W}$ *such that* $\mathsf{w}_1 \wedge_* \mathsf{m_w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m_w}$. *Then*

(1) $(\mathsf{w}_3 \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m_w} \leq_* (\mathsf{w}_3 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m_w}$

(2) $(\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m_w} \leq_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m_w}$

Proof. Let $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3 \in \mathcal{W}$. Then

$$
\begin{aligned}
(1): (\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} &= [\mathsf{w}_3 \to_* (\mathsf{w}_1 \wedge_* \mathsf{w}_2)] \wedge_* \mathsf{m}_\mathsf{w} \\
&\quad (\text{by Def. } 2.1(WH_2)) \\
&= (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \\
&\quad (\text{since } \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w})
\end{aligned}
$$

Therefore $(\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$. Similarly,
(2): $(\mathsf{w}_2 \to_* \mathsf{w}_3) \wedge_* (\mathsf{w}_1 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w}$

$$
\begin{aligned}
&= [(\mathsf{w}_2 \vee_* \mathsf{w}_1) \to_* \mathsf{w}_3] \wedge_* \mathsf{m}_\mathsf{w} && (\text{by Def. } 2.1 WH_3) \\
&= [(\mathsf{w}_2 \vee_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}] \to_* [\mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}] && (\text{by Def. } 2.1 WH_5) \\
&= [(\mathsf{w}_1 \vee_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}] \to_* [\mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}] && (\text{by Theorem } 1.2(6)) \\
&= (\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}) \to_* (\mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}) && (\text{since } \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}) \\
&= (\mathsf{w}_1 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} && (\text{by Def. } 2.1 WH_5)
\end{aligned}
$$

Therefore $(\mathsf{w}_2 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_1 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w}$.

□

Lemma 2.2. *For any* $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3 \in \mathcal{W}$, *the following holds;*

(1) *If* $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}$, *then* $(\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{m}_\mathsf{w}$.
(2) *If* $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}$, *then* $(\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} = (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$.
(3) $(\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_1 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \leq_* [\mathsf{w}_1 \to_* (\mathsf{w}_3 \vee_* \mathsf{w}_2)] \wedge_* \mathsf{m}_\mathsf{w}$.

Proof. Let $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3 \in \mathcal{W}$. Then
(1): $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}$

$$
\begin{aligned}
&\Longrightarrow (\mathsf{w}_1 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} && (\text{by Lemma } 2.1(1)) \\
&\Longrightarrow \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{m}_\mathsf{w} && (\text{by Def. } 2.1(WH_1))
\end{aligned}
$$

Therefore $(\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{m}_\mathsf{w}$.
(2): $(\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$

$$
\begin{aligned}
&= [\mathsf{w}_3 \to_* (\mathsf{w}_1 \wedge_* \mathsf{w}_2)] \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} && (\text{by Def. } 2.1 WH_2) \\
&= (\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} && (\text{since } \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}) \\
&= [(\mathsf{w}_3 \vee_* \mathsf{w}_2) \to_* \mathsf{w}_1] \wedge_* \mathsf{m}_\mathsf{w} && (\text{by Def. } 2.1 WH_3) \\
&= [(\mathsf{w}_3 \vee_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}] \to_* [\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}] && (\text{by Def. } 2.1 WH_5) \\
&= [(\mathsf{w}_2 \vee_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w}] \to_* [\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}] && (\text{by Theorem } 1.2(6)) \\
&= (\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} && (\text{since } \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w} \leq_* \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w})
\end{aligned}
$$

Therefore $(\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$. From Definition 2.1 (WH_4), it follows that $(\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$. Hence $(\mathsf{w}_3 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} = (\mathsf{w}_3 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_2 \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$.

(3): It holds that $(\mathsf{w}_2 \wedge_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \leq_* (\mathsf{w}_3 \vee_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$. From Lemma 2.1 (1) and Definition 2.1(WH_4), it follows that $(\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* (\mathsf{w}_1 \to_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \leq_* [\mathsf{w}_1 \to_* (\mathsf{w}_3 \vee_* \mathsf{w}_2)] \wedge_* \mathsf{m}_\mathsf{w}$. □

Theorem 2.1. *Let $\langle \mathcal{W}, \vee_*, \wedge_*, \to_*, 0, \mathsf{m}_\mathsf{w} \rangle$ be a WHADL and define $\mathsf{w}_1 \to_n \mathsf{w}_2 = (\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathfrak{n}$. Then, given a maximal element $\mathfrak{n} \in \mathcal{W}$, $\langle \mathcal{W}, \vee_*, \wedge_*, \to_n, 0, \mathfrak{n} \rangle$ is a WHADL.*

Proof. Let $\mathsf{w}_1, \mathsf{w}_2 \in \mathcal{W}$. Define $\mathsf{w}_1 \to_n \mathsf{w}_2 = (\mathsf{w}_1 \to_* \mathsf{w}_2) \wedge_* \mathfrak{n}$. Then one can routinely prove that $\langle \mathcal{W}, \vee_*, \wedge_*, \to_n, 0, \mathfrak{n} \rangle$ is a WHADL. □

Theorem 2.2. *An ADL $\mathcal{W}$ is a WHADL if and only if $PI(\mathcal{W})$ is a weakly-Heyting algebra.*

Proof. Suppose $\mathcal{W}$ is a WHADL, clearly $PI(\mathcal{W})$ is a bounded distributive lattice and $PI(\mathcal{W})$ is a weakly-Heyting algera is obtained by defining the binary operation $\to_*$ on $\mathcal{W}$ as $(\mathsf{w}_1] \to_* (\mathsf{w}_2] = (\mathsf{w}_1 \to_* \mathsf{w}_2]$, for any $\mathsf{w}_1, \mathsf{w}_2 \in \mathcal{W}$. On the other hand if $PI(\mathcal{W})$ is weakly-Heyting algebra, then $\mathcal{W}$ is a WHADL is obtained by defining the binary operation $\to_*$ on $\mathcal{W}$ as $\mathsf{w}_1 \to_* \mathsf{w}_2 = \mathfrak{e} \wedge_* \mathsf{m}_\mathsf{w}$and $(\mathsf{w}_1] \to_* (\mathsf{w}_2] = (\mathfrak{e}]$, for $\mathsf{w}_1, \mathsf{w}_2, \mathfrak{e} \in \mathcal{W}$. □

An WHADL becomes a weakly-Heyting algebra once it satisfies the lattice structure. Hence, by Theorem 2.3, we obtain several equivalent conditions under which an WHADL becomes a weakly-Heyting algebra.

Theorem 2.3. *Let $\mathcal{W}$ be an WH-ADL then the following are equivalent:*

(1). $\mathcal{W}$ is a weakly-Heyting algebra.
(2). $\mathcal{W}$ is a distributive lattice.
(3). $(\mathcal{W}, \leq_*)$ is directed above poset.
(4). $\vee_*$ is commutative.
(5). $\wedge_*$ is commutative.
(6). $\vee_*$ is right distributive over $\wedge_*$.
(7). The relation $\theta = \{(\mathsf{w}_1, \mathsf{w}_2) \in \mathcal{W} \times \mathcal{W} \mid \mathsf{w}_2 \wedge_* \mathsf{w}_1 = \mathsf{w}_1\}$ is antisymmetric.

Remark 2.2. *In an WHADL we can observe that $(\mathsf{m}_\mathsf{w} \to_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \neq \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}$.*

Example 2.4. *Let $\mathcal{W} = \{0, 1, 2, 3\}$ be a four element chain such that $0 < 1 < 2 < 3$, illustration of $\to_*$ described below;*

$\to_*$	*0*	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	1	3	3
3	1	1	3	3

Then $\langle \mathcal{W}, \vee_*, \wedge_*, \rightarrow_*, 0, 3\rangle$ is an WHADL with 3 as its maximal element and clearly we can observe that $(3 \rightarrow_* 2) \wedge_* 3 \neq 2 \wedge_* 3$.

Lemma 2.3. *Let $\mathcal{W}$ be an WHADL with $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$, then the following holds;*

(1). If $\mathsf{w}_1 \wedge_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}$, then $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w}$

(2). $\mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$.

Proof. Consider $\mathcal{W}$ with $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$.

(1). Let $\mathsf{w}_1 \wedge_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w}$.

$$\Rightarrow [\mathsf{w}_2 \rightarrow_* (\mathsf{w}_1 \wedge_* \mathsf{w}_2)] \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \qquad \text{(by Lemma 2.1(1))}$$

$$\Rightarrow (\mathsf{w}_2 \rightarrow_* \mathsf{w}_1) \wedge_* (\mathsf{w}_2 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \qquad \text{(by Def. 2.1}WH_2\text{)}$$

$$\Rightarrow (\mathsf{w}_2 \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w} \qquad \text{(by Def. 2.1}WH_1\text{)}$$

Since $\mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{m}_\mathsf{w}$. Then by Lemma 2.1(2), $(\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$ So $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_2 \rightarrow_* \mathsf{w}_3) \wedge_* \mathsf{m}_\mathsf{w}$.

(2). $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w}$

$$\Rightarrow \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$$

$$\Rightarrow \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \text{ (by Def. 2.1}WH_4\text{)}$$

$$\Rightarrow \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$$

On the other hand , $\mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{m}_\mathsf{w}$.

$$\Rightarrow (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \text{ (by Lemma 2.1(2))}$$

$$\Rightarrow \mathsf{w}_1 \wedge_* (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$$

Hence $\mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$.

□

The relationship between WHADLs and HADLs is particularly interesting because it is possible to recover the structure of a HADL by imposing additional conditions on an WHADL. Specifically, if an WHADL satisfies a certain set of stronger axioms typically related to the behavior of implication it becomes a HADL. This transformation illustrates how the logical strength of a system can be algebraically characterized by adjusting structural properties. The following Theorem shows a tight relation between WHADL's and HADL's:

Theorem 2.4. *Let $\mathcal{W}$ be an WHADL. Then $\mathcal{W}$ is a HADL iff for every $\mathsf{w}_1 \in \mathcal{W}$, $(\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}$.*

Proof. If $\mathcal{W}$ is a HADL, clearly $(\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_1) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w}$. On the other hand, it suffices to show that $\mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$, then
$\mathsf{w}_3 \wedge_* \mathsf{w}_1 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}$. Suppose $\mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w} \leq (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w}$. Then $\mathsf{w}_1 \wedge_* \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_1 \wedge_* (\mathsf{w}_1 \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* (\mathsf{m}_\mathsf{w} \rightarrow_* \mathsf{w}_2) \wedge_* \mathsf{m}_\mathsf{w} = \mathsf{w}_1 \wedge_* \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}$. So $\mathsf{w}_1 \wedge_* \mathsf{w}_3 \wedge_* \mathsf{m}_\mathsf{w} \leq \mathsf{w}_2 \wedge_* \mathsf{m}_\mathsf{w}$. □

3. Conclusions

In conclusion, we have introduced the concept of a Weakly-Heyting Almost Distributive Lattice (WHADL) as a novel algebraic structure that extends weakly-Heyting algebras within the framework of almost distributive lattices. Through illustrative examples, we demonstrated that WHADLs and Heyting Almost Distributive Lattices (HADLs) are distinct structures, highlighting key differences in their algebraic behavior. Furthermore, we characterized WHADLs by identifying sets of defining properties and examined their internal structure via principal ideals. Finally, we showed that by imposing a specific condition on a WHADL, it can be strengthened into a HADL.

Acknowledgments: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2287/2568).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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