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SECOND HANKEL DETERMINANT FOR BI-UNIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH HOHLOV OPERATOR

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Abstract. In the present paper, we consider a subclass of the function class Σ of bi-univalent analytic functions in the open unit disk Δ associated with Hohlov operator and we obtain the functional $|a_2a_4 - a_3^2|$ for the function class Σ . Our result gives corresponding $|a_2a_4 - a_3^2|$ for the subclasses of Σ defined in the literature.

1. Introduction

Let $\mathcal{A}$ be the class of functions given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1.1}$$

and analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let Ω be the family of functions $f \in \mathcal{A}$ which are univalent in Δ and satisfying the normalization conditions (see[4]):

$$f(0) = f'(0) - 1 = 0.$$

The well-known Koebe one-quarter theorem (see[4]) asserts that the image of Δ under every univalent function $f \in \Omega$ contains a disk of radius $\frac{1}{4}$. Thus, the inverse of $f \in \Omega$ is a univalent analytic function on the disk $\Delta_\rho := \{z : z \in \mathbb{C} \text{ and } |z| < \rho;\ \rho \geq \frac{1}{4}\}$. Therefore, for each function $f(z) = w \in \Omega$, there is an inverse function $f^{-1}(w)$ of $f(z)$ defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (w \in \Delta_\rho)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in \Omega$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent function in Δ given by (1.1). The concept of bi-univalent analytic functions was introduced by Lewin [14] in 1967 and he showed

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that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [18], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\})$is presumably still an open problem. In [3](see also [2, 7, 20, 22, 23]), certain subclasses of the bi-univalent analytic functions class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

In 1976, Noonan and Thomas [19] defined the qth Hankel determinant of f for $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Further, Fekete and Szegö [6] considered the Hankel determinant of $f \in A$ for $q = 2$ and $n = 1$, $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$. They made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ with μ real. The well known result due to them states that if $f \in \mathcal{A}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if} \quad \mu \geq 1, \\ 1 + 2\ exp(\frac{-2\mu}{1-\mu}) & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if} \quad \mu \leq 0. \end{cases}$$

Furthermore, Hummel [9, 10] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is convex functions and also Keogh and Merkes [13] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike and convex in Δ. Here we consider the Hankel determinant of $f \in \mathcal{A}$ for $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

For the functions $f,\ g \in \mathcal{A}$ and given by the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \Delta),$$

the Hadamard product (or convolution) of f and g denoted by $f * g$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \Delta).$$

By using the Hadamard product (or convolution), Hohlov (cf.[11]) introduced and studied the linear operator $\mathcal{I}_c^{a,b} : \Omega \to \Omega$ defined by

$$\mathcal{I}_c^{a,b} f(z) = z\, {}_2F_1(a, b; c; z) * f(z) \quad (f \in \Omega, z \in \Delta),$$

where ${}_2F_1(z)$ known as *Gaussian hypergeometric function* is defined by
(1.3)

$${}_2F_1(z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (a, b \in \mathbb{C}, c \in \mathbb{C}\backslash\mathbb{Z}_0^- =: \{0, -1, -2, \dots\})$$

and $(\lambda)_n$ is the *Pochhamer symbol* or *shifted factorial*, written in terms of the *gamma function* Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0 \\ \lambda(\lambda+1)....(\lambda+n-1), & n \in \mathbb{N} := \{1,2,3,.....\}. \end{cases}$$

Note that ${}_2F_1(z)$ is symmetric in a and b and that the series (1.3) terminates if at least one of the numerator parameter a and b is zero or a negative integer.Observe that for the function f of the form (1.1), we have

$$\begin{aligned} \mathcal{I}_c^{a,b} f(z) &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Phi_n a_n z^n \qquad (z \in \Delta), \end{aligned} \tag{1.4}$$

where

$$\Phi_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

Making use of Hohlov operator we consider a new subclass of Σ due to Panigarhi and Murugusundaramoorthy[20] as given below

Definition 1.1. [20] A function $f \in \Sigma$ and of the form (1.1)is said to be in the class $\mathcal{M}_{\Sigma}^{a,b;c}(\beta,\lambda)$ if the following conditions are satisfied:

$$\Re\left[(1-\lambda)\frac{\mathcal{I}_c^{a,b} f(z)}{z} + \lambda\left(\mathcal{I}_c^{a,b} f(z)\right)'\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 1, z \in \Delta) \tag{1.5}$$

and

$$\Re\left[(1-\lambda)\frac{\mathcal{I}_c^{a,b} g(w)}{w} + \lambda\left(\mathcal{I}_c^{a,b} g(w)\right)'\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 1, w \in \Delta) \tag{1.6}$$

where the function g is the inverse of f given by (1.2).

It is of interest to note that by taking $a = b$ and $c = 1$ we state the following subclass $\mathcal{F}_{\Sigma}(\beta,\lambda)$ due to Frasin et al.[7].

Example 1.2. [7] *A function $f \in \Sigma$ and of the form (1.1) is said to be in the class $\mathcal{F}_{\Sigma}(\beta,\lambda)$ if the following conditions are satisfied:*

$$\Re\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 1, z \in \Delta) \tag{1.7}$$

and

$$\Re\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 1, w \in \Delta) \tag{1.8}$$

where the function g is the inverse of f given by (1.2).

It is of interest to note that by taking $a = b; c = 1$ and $\lambda = 1$ we state the following subclass $\mathcal{H}_{\Sigma}(\beta)$ due to Srivastava et al.[22]. By taking $a = b; c = 1$ and we state the following :

Example 1.3. [22] *A function $f \in \Sigma$ and of the form (1.1) is said to be in the class $\mathcal{H}_\Sigma(\beta)$ if the following conditions are satisfied:*

$$\Re\left[f'(z)\right] > \beta \quad (0 \le \beta < 1, z \in \Delta)$$

and

$$\Re\left[g'(w)\right] > \beta \quad (0 \le \beta < 1, w \in \Delta)$$

where the function g is the inverse of f given by (1.2).

The object of the present paper is to determine the functional $|a_2a_4 - a_3^2|$ for the function $f \in \mathcal{M}_\Sigma^{a,b;c}(\beta, \lambda)$. Our result gives corresponding $|a_2a_4 - a_3^2|$ for the subclasses of Σ defined in the Examples 1.2 and 1.3.

2. Coefficient bounds for the function class $\mathcal{M}_\Sigma^{a,b;c}(\beta, \lambda)$

We need the following lemma for our investigation.

Lemma 2.1. *(see [4], p. 41) Let $\mathcal{P}$ be the class of all analytic functions $p(z)$ of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{2.1}$$

satisfying $\Re(p(z)) > 0$ $(z \in \Delta)$ and $p(0) = 1$. Then

$$|p_n| \le 2 \ (n = 1, 2, 3, ...).$$

This inequality is sharp for each n. In particular, equality holds for all n for the function

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

Lemma 2.2. *If the function $p \in \mathcal{P}$ is given by the series*

$$2p_2 = p_1^2 + x(4 - p_1^2), \tag{2.2}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2z), \tag{2.3}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

Lemma 2.3. [8] *The power series for p given in (2.1) converges in Δ to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{2.4}$$

and $c_{-k} = \overline{c_k}$, are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k z}), \ \rho_k > 0, \ t_k \text{ real}$$

and $t_k \ne t_j$ for $k \ne j$ in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \ge m$.

In the following theorem we determine the second hankel coefficient results for

Theorem 2.4. *Let $f \in \mathcal{M}_{\Sigma}^{a,b;c}(\beta,\lambda)$ be given by (1.1). Then*

(2.5)
$$|a_2a_4-a_3^2| \leq \begin{cases} 4(1-\beta^2)\left[\frac{(1+\lambda)^3\Phi_2^3+4(1-\beta)^2(1+3\lambda)\Phi_4}{(1+\lambda)^4(1+3\lambda)\Phi_2^4\Phi_4}\right], & \beta \in \left[0, 1-\sqrt{\frac{(1+\lambda)^3\ \Phi_2^3}{8(1+3\lambda)\Phi_4}}\right] \\ \\ \frac{9(1+\lambda)^2(1-\beta)^2\Phi_2^2}{2(1+3\alpha)\Phi_4[(1+\lambda)^3\Phi_2^3-2(1-\beta)^2(1+3\lambda)\Phi_4]}, & \beta \in \left(1-\sqrt{\frac{(1+\lambda)^3\ \Phi_2^3}{8(1+3\lambda)\Phi_4}}, 1\right). \end{cases}$$

Proof. Since $f \in \mathcal{M}_{\Sigma}^{a,b;c}(\beta,\lambda)$, there exists two functions $\phi(z)$ and $\psi(z) \in \mathcal{P}$ satisfying the conditions of Lemma 2.1 such that

$$(1-\lambda)\frac{\mathcal{I}_c^{a,b}f(z)}{z} + \lambda\left(\mathcal{I}_c^{a,b}f(z)\right)' = \beta + (1-\beta)\phi(z) \tag{2.6}$$

and

$$(1-\lambda)\frac{\mathcal{I}_c^{a,b}g(w)}{w} + \lambda\left(\mathcal{I}_c^{a,b}g(w)\right)' = \beta + (1-\beta)\psi(z) \tag{2.7}$$

where

$$\phi(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ... \tag{2.8}$$

and

$$\psi(w) = 1 + d_1w + d_2w^2 + d_3w^3 + ... \tag{2.9}$$

. Equating the coefficients in (2.6) and (2.7)gives

$$(1+\lambda)\Phi_2a_2 = (1-\beta)c_1 \tag{2.10}$$

$$(1+2\lambda)\Phi_3a_3 = (1-\beta)c_2 \tag{2.11}$$

$$(1+3\lambda)\Phi_4a_4 = (1-\beta)c_3 \tag{2.12}$$

and

$$-(1+\lambda)\Phi_2a_2 = (1-\beta)d_1 \tag{2.13}$$

$$(1+2\lambda)\Phi_3(2a_2^2-a_3) = (1-\beta)d_2 \tag{2.14}$$

$$-(1+3\lambda)\Phi_4(5a_2^3-5a_2a_3+a_4) = (1-\beta)d_3 \tag{2.15}$$

From (2.10) and (2.13) gives

$$a_2 = \frac{1-\beta}{(1+\lambda)\Phi_2}c_1 = -\frac{1-\beta}{(1+\lambda)\Phi_2}d_1 \tag{2.16}$$

which implies

$$c_1 = -d_1$$

Now from(2.11) and (2.14), we obtain

$$a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2\ \Phi_2^2}\ c_1^2 + \frac{(1-\beta)}{4(1+2\lambda)\Phi_3}(c_1-c_2). \tag{2.17}$$

On the other hand, subtracting (2.15) from (2.12) and using (2.16), we get

(2.18)
$$a_4 = \frac{1}{2(1+3\lambda)\Phi_4}\left[\frac{-5(1+3\lambda)(1-\beta)^3\Phi_4}{(1+\lambda)^3\Phi_2^3}c_1^3 + \frac{5(1+3\lambda)(1-\beta)\Phi_4}{(1+\lambda)\Phi_2}a_3c_1 + (1-\beta)(c_3-d_3)\right].$$

Thus we establish that

$$|a_2a_4 - a_3^2| = \left| -\frac{(1-\beta)^4}{(1+\lambda)^4\Phi_2^4}c_1^4 + \frac{(1-\beta)^3c_1^2(c_2-d_2)}{8(1+\lambda)^2(1+2\lambda)\Phi_2^2\Phi_3} + \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}c_1(c_3-d_3) - (1-\beta)^2(c_2-d_2)^2 \right|. \tag{2.19}$$

According to Lemma2.2 we have

$$2c_2 = c_1^2 + x(4-c_1^2), \quad \text{and} \quad 2d_2 = d_1^2 + x(4-d_1^2),$$

hence we have

$$c_2 = d_2 \tag{2.20}$$

and further

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2z),$$
$$4d_3 = d_1^3 + 2(4-d_1^2)d_1x - d_1(4-d_1^2)x^2 + 2(4-d_1^2)(1-|x|^2z)$$

$$c_3 - d_3 = \frac{1}{2}c_1^3 + c_1(4-c_1^2)x - \frac{1}{2}c_1(4-c_1^2)x^2 \tag{2.21}$$

$$|a_2a_4 - a_3^2| = \left| \frac{-(1-\beta)^4}{(1+\lambda)^4\Phi_2^4}c_1^4 + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}c_1^4 + \frac{(1-\beta)^2c_1^2(4-c_1^2)x}{2(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} - \frac{(1-\beta)^2c_1^2(4-c_1^2)x^2}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} \right| \tag{2.22}$$

Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$ since $\phi \in \mathcal{P}$ so $|c_1| \leq 2$.Thus,applying triangle inequality on (2.19),with $\mu = |x| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^4}{(1+\lambda)^4\Phi_2^4}c^4 + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}c^4 + \frac{(1-\beta)^2c^2(4-c^2)\mu}{2(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} + \frac{(1-\beta)^2c^2(4-c^2)\mu^2}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} = F(\mu) \tag{2.23}$$

Differentiating $F(\mu)$, we get

$$F'(\mu) = \frac{(1-\beta)^2c_1^2(4-c_1^2)}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} + \frac{(1-\beta)^2c^2(4-c^2)\mu}{2(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}$$

By using elementary calculus, one can show that $F'(\mu) > 0$ for $\mu > 0$ hence F is an increasing function and thus ,the upper bound for $F(\mu)$ corresponds to $\mu = 1$,in which case

$$F(\mu) = F(1) = \left[\frac{(1-\beta)^4}{(1+\lambda)^4\Phi_2^4} + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}\right]c^4 + \frac{3(1-\beta)^2c^2(4-c^2)}{4(1+\lambda)(1+3\lambda)\Phi_4\Phi_2} = G(c) \tag{2.24}$$

Assume that $G(c)$ has a maximum value in an interior of $c \in [0,2]$, by elementary calculations we find

$$G'(c) = \left[\frac{4(1-\beta)^4}{(1+\lambda)^4\Phi_2^4} - \frac{2(1-\beta)^2}{(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}\right]c^3 + \frac{6(1-\beta)^2c}{(1+\lambda)(1+3\lambda)\Phi_4\Phi_2}. \tag{2.25}$$

Then $G'(c) = 0$ implies the real critical point $c_{01} = 0$ or $c_{02} = \sqrt{\frac{3(1+\lambda)^3\ \Phi_2^3}{(1+\lambda)^3\Phi_2^3 - 2(1-\beta)^2(1+3\lambda)\Phi_4}}$.

After some calculations we concluded following cases:

Case 1: When $\beta \in \left[0, 1 - \sqrt{\frac{(1+\lambda)^3\ \Phi_2^3}{8(1+3\lambda)\Phi_4}}\right]$, we observe that $c_{02} \geq 2$, that is, c_{02} is out of the interval (0, 2). Therefore the maximum value of $G(c)$ occurs at $c_{01} = 0$ or $c = c_{02}$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0, 2]$. Since G is an increasing function in the interval [0, 2], maximum point of G must be on the boundary of $c \in [0, 2]$, that is, $c = 2$. Thus, we have

$$\max_{0 \leq c \leq 2} G_1(p) = G(2) = 4(1-\beta^2)\left[\frac{(1+\lambda)^3\Phi_2^3 + 4(1-\beta)^2(1+3\lambda)\Phi_4}{(1+\lambda)^4(1+3\lambda)\Phi_2^4\Phi_4}\right]$$

Case 2: When $\beta \in \left(1 - \sqrt{\frac{(1+\lambda)^3\ \Phi_2^3}{8(1+3\lambda)\Phi_4}}, 1\right)$, we observe that $c_{02} \leq 2$, that is, c_{02} is interior of the interval [0, 2]. Since $G''(c_{02}) < 0$, the maximum value of $G(c)$ occurs at $c = c_{02}$. Thus, we have

$$\begin{aligned}\max_{0 \leq c \leq 2} G(c) = G(c_{02}) &= G\left(\sqrt{\frac{3(1+\lambda)^3\ \Phi_2^3}{(1+\lambda)^3\Phi_2^3 - 2(1-\beta)^2(1+3\lambda)\Phi_4}}\right) \\ &= \frac{9(1+\lambda)^2(1-\beta)^2\Phi_2^2}{2(1+3\alpha)\Phi_4[(1+\lambda)^3\Phi_2^3 - 2(1-\beta)^2(1+3\lambda)\Phi_4]}.\end{aligned}$$

□

Concluding Remarks: Suitably specializing the parameter λ one can state the Hankel coefficients for various subclasses of$\mathcal{M}_\Sigma^{a,b;c}(\beta, \lambda)$. In fact, by choosing $a = b$ and $c = 1$ we have $\Phi_2 = 1; \Phi_3 = 1; \Phi_4 = 1$ hence we state the Hankel determinant coefficients for the function $f \in \mathcal{F}_\Sigma(\beta, \lambda)$ studied in[7] as given below:

(2.26)
$$|a_2a_4 - a_3^2| \leq \begin{cases} 4(1-\beta^2)\left[\frac{(1+\lambda)^3 + 4(1-\beta)^2(1+3\lambda)}{(1+\lambda)^4(1+3\lambda)}\right], & \beta \in \left[0, 1 - \sqrt{\frac{(1+\lambda)^3}{8(1+3\lambda)}}\right] \\ \frac{9(1+\lambda)^2(1-\beta)^2}{2(1+3\alpha)[(1+\lambda)^3 - 2(1-\beta)^2(1+3\lambda)]}, & \beta \in \left(1 - \sqrt{\frac{(1+\lambda)^3}{8(1+3\lambda)}}, 1\right). \end{cases}$$

Also by choosing $\lambda = 1$ one can easily derive Hankel determinant $|a_2a_4 - a_3^2|$ for the functions $f \in \mathcal{H}_\Sigma$ studied by Srivastava et al.[22].

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