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Nonlinear Integral Equations via Ω-Distance Fixed Points in G-b-Metric Spaces

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Abstract. In this paper, we investigate fixed point results for nonlinear contraction mappings defined via an Ω-distance in complete G-b-metric spaces. This approach unifies and extends several classical contraction principles formulated in metric, b-metric, and G-metric settings. By employing suitable contractive inequalities involving Ω-distance, we establish existence and uniqueness results for fixed points under mild assumptions. The obtained theorems generalize and improve a number of recent results in the literature. As an application, we illustrate the usefulness of the developed theory by studying the solvability of a class of nonlinear integral equations.

1. Introduction

Fixed point theory constitutes a fundamental tool in nonlinear analysis and has wide-ranging applications in differential equations, integral equations, optimization, and applied sciences. The celebrated Banach contraction principle guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. Motivated by its importance, several generalizations of metric spaces have been proposed in order to extend the applicability of fixed point results.

Bakhtin [1] introduced contraction principles in quasimetric spaces, while Czerwik [2, 3] developed the theory of b-metric spaces and established fixed point results for both single-valued and multivalued mappings. Later, Mustafa and Sims [13] proposed the notion of G-metric spaces, which has been further extended to G-b-metric spaces. These generalized frameworks have proven to be effective in unifying and extending many classical fixed point theorems.

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In order to formulate more flexible contractive conditions, Shatanawi and Pitea introduced the concept of Ω-distance and obtained several fixed and coupled fixed point results in different generalized metric settings [4, 6]. Subsequent works investigated various generalized contractions using simulation functions, F-contractions, and rational-type conditions in b-metric-like and related spaces [10–15].

Inspired by these developments, the purpose of this paper is to establish new fixed point theorems for self-mappings in complete G-b-metric spaces via Ω-distance. The obtained results extend several known theorems and provide a unified framework covering many existing contraction conditions as special cases. An application to nonlinear integral equations is also presented to demonstrate the effectiveness of the proposed approach.

2. Preliminaries

In this section, we recall some basic definitions and notions that will be used throughout the paper.

Definition 2.1. *Let J be a nonempty set and let $b \geq 1$. A mapping*

$$G : J \times J \times J \to [0, \infty)$$

*is called a G-b-*metric *on J if for all $\mathfrak{z}, \mathfrak{a}, z, u \in J$, the following conditions hold:*

(1) $G(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}) = 0$ *and* $G(\mathfrak{z}, \mathfrak{z}, \mathfrak{a}) > 0$ *for* $\mathfrak{z} \neq \mathfrak{a}$;
(2) $G(\mathfrak{z}, \mathfrak{z}, \mathfrak{a}) \leq G(\mathfrak{z}, \mathfrak{a}, z)$ *whenever* $\mathfrak{a} \neq \mathfrak{k}$;
(3) *G is symmetric in all three variables;*
(4) $G(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) \leq b\Big(G(\mathfrak{z}, u, u) + G(u, \mathfrak{a}, \mathfrak{k})\Big)$.

*The pair (J, G) is called a G-b-*metric space*.*

Definition 2.2. *A sequence $\{\mathfrak{z}_n\}$ in a G-b-metric space (J, G) is said to be G-*Cauchy *if*

$$\lim_{m,n \to \infty} G(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) = 0.$$

The space (J, G) is called complete *if every G-Cauchy sequence converges to a point in J.*

Definition 2.3. *Let (J, G) be a G-b-metric space. A mapping*

$$\Omega : J \times J \times J \to [0, \infty)$$

*is called a G-Ω-*distance *if it satisfies properties analogous to those of a G-metric and is compatible with the convergence structure induced by G.*

3. Main Result

Theorem 3.1. *Let (J, G) be a complete G-b-metric space with coefficient $b \geq 1$ and let $\Omega : J \times J \times J \to [0, \infty)$ be a G-Ω-distance on J. Assume that a mapping $\Xi : J \to J$ satisfies*

$$\begin{aligned} \Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{k}) \leq\ & \alpha_1\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) + \alpha_2\Omega(\mathfrak{z}, \Xi\mathfrak{z}, \Xi\mathfrak{z}) + \alpha_3\Omega(\mathfrak{a}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) \\ & + \alpha_4\Omega(\mathfrak{k}, \Xi\mathfrak{k}, \Xi\mathfrak{k}) + \alpha_5\Omega(\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) + \alpha_6\Omega(\mathfrak{a}, \Xi\mathfrak{z}, \Xi\mathfrak{z}), \end{aligned} \tag{3.1}$$

for all $\mathfrak{z}, \mathfrak{a}, \mathfrak{k} \in J$, *where* $\alpha_i \geq 0$ $(i = 1, \ldots, 6)$ *and*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + b(\alpha_5 + \alpha_6) < \frac{1}{b}. \tag{3.2}$$

Then Ξ *has a unique fixed point in* J.

Proof. Choose an arbitrary point $\mathfrak{z}_0 \in J$ and define a sequence $\{\mathfrak{z}_n\}$ in J recursively by

$$\mathfrak{z}_{n+1} = \Xi\mathfrak{z}_n, \quad n \geq 0.$$

Using condition (3.1) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = \mathfrak{z}_{n-1}$ and $\mathfrak{k} = \mathfrak{z}_{n-1}$, we obtain

$$\begin{aligned}\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq\ & \alpha_1\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) + \alpha_2\Omega(\mathfrak{z}_n, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}) \\ & + \alpha_3\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n) + \alpha_4\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n) \\ & + \alpha_5\Omega(\mathfrak{z}_n, \mathfrak{z}_n, \mathfrak{z}_n) + \alpha_6\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}).\end{aligned}$$

Since $\Omega(\mathfrak{z}_n, \mathfrak{z}_n, \mathfrak{z}_n) = 0$ and by the properties of a G-Ω-distance, we have

$$\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}) \leq \Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n) + \Omega(\mathfrak{z}_n, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}).$$

Hence,

$$\begin{aligned}\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq\ & (\alpha_1 + \alpha_3 + \alpha_4 + b\alpha_6)\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) \\ & + (\alpha_2 + b\alpha_6)\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n).\end{aligned}$$

Rearranging terms, we obtain

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \lambda\, \Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}),$$

where

$$\lambda = \frac{\alpha_1 + \alpha_3 + \alpha_4 + b\alpha_6}{1 - (\alpha_2 + b\alpha_6)}.$$

From condition (3.2), it follows that $0 < \lambda < 1$.

Iterating the above inequality yields

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \lambda^n\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0), \quad n \geq 1.$$

Let $m > n$. By repeated use of the G-Ω-distance inequality, we obtain

$$\Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) \leq \sum_{k=n}^{m-1} b^{k-n}\Omega(\mathfrak{z}_{k+1}, \mathfrak{z}_k, \mathfrak{z}_k) \leq \frac{b\,\lambda^n}{1 - b\lambda}\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0).$$

Since $b\lambda < 1$, we conclude that

$$\lim_{m,n\to\infty} \Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) = 0.$$

Thus, $\{\mathfrak{z}_n\}$ is a G-Cauchy sequence. By completeness of (J, G), there exists $u \in J$ such that

$$\lim_{n\to\infty} \Omega(\mathfrak{z}_n, u, u) = 0.$$

Next, applying (3.1) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = u$ and $\mathfrak{k} = u$, and letting $n \to \infty$, we obtain

$$\Omega(u, \Xi u, \Xi u) = 0,$$

which implies $\Xi u = u$. Hence, u is a fixed point of Ξ.

Finally, suppose that $v \in J$ is another fixed point of Ξ. Then, using (3.1) with $\mathfrak{z} = u$, $\mathfrak{a} = v$ and $\mathfrak{t} = v$, we get

$$\Omega(u, v, v) \leq \alpha_1 \Omega(u, v, v).$$

Since $\alpha_1 < 1$, it follows that $\Omega(u, v, v) = 0$, and hence $u = v$. Therefore, the fixed point of Ξ is unique. □

Example 3.1. *Let $J = [0, \infty)$ and define $G : J \times J \times J \to [0, \infty)$ by*

$$G(\mathfrak{z}, \mathfrak{a}, \mathfrak{t}) = |\mathfrak{z} - \mathfrak{a}| + |\mathfrak{a} - \mathfrak{t}| + |\mathfrak{t} - \mathfrak{z}|, \quad \mathfrak{z}, \mathfrak{a}, \mathfrak{t} \in J.$$

Then (J, G) is a complete G-b-metric space with $b = 1$. Let $\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{t}) = G(\mathfrak{z}, \mathfrak{a}, \mathfrak{t})$.

Define $\Xi : J \to J$ by

$$\Xi\mathfrak{z} = \frac{\mathfrak{z}}{4}, \quad \mathfrak{z} \in J.$$

For all $\mathfrak{z}, \mathfrak{a}, \mathfrak{t} \in J$, we have

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{t}) = \frac{1}{4}\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{t}).$$

Thus, inequality (3.1) holds with

$$\alpha_1 = \frac{1}{4}, \quad \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0,$$

and

$$\alpha_1 < \frac{1}{b} = 1.$$

Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, Ξ has a unique fixed point in J, which is $u = 0$.

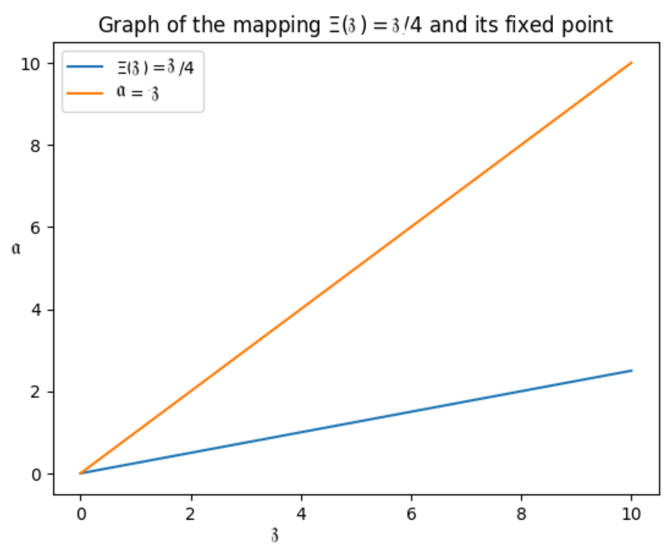

Figure 1. Graph of the self-mapping $\Xi(\mathfrak{z}) = \frac{\mathfrak{z}}{4}$ and the identity map $\mathfrak{a} = \mathfrak{z}$ on $J = [0, \infty)$.The unique intersection point at $\mathfrak{z} = 0$ represents the unique fixed point of Ξ. The slope $\frac{1}{4} < 1$ illustrates the contractive nature required by Theorem 3.1.

Theorem 3.2. *Let (J, G) be a complete G-b-metric space with coefficient $b \geq 1$ and let $\Omega : J \times J \times J \to [0, \infty)$ be a G-Ω-distance on J. Assume that a mapping $\Xi : J \to J$ satisfies, for all $\mathfrak{z}, \mathfrak{a}, \mathfrak{k} \in J$,*

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{k}) \leq \alpha_1\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) + \alpha_2\Omega(\mathfrak{z}, \Xi\mathfrak{z}, \Xi\mathfrak{z}) + \alpha_3\Omega(\mathfrak{a}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) + \alpha_4\Omega(\mathfrak{k}, \Xi\mathfrak{k}, \Xi\mathfrak{k}), \tag{3.3}$$

where $\alpha_i \geq 0$ $(i = 1, 2, 3, 4)$ and

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \frac{1}{b}. \tag{3.4}$$

Then Ξ has a unique fixed point in J.

Proof. Let $\mathfrak{z}_0 \in J$ be arbitrary and define a sequence $\{\mathfrak{z}_n\}$ in J by

$$\mathfrak{z}_{n+1} = \Xi\mathfrak{z}_n, \quad n \geq 0.$$

Applying (3.3) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = \mathfrak{z}_{n-1}$ and $\mathfrak{k} = \mathfrak{z}_{n-1}$, we obtain

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \alpha_1\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) + \alpha_2\Omega(\mathfrak{z}_n, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}) + (\alpha_3 + \alpha_4)\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n).$$

Rearranging terms gives

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \lambda\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}),$$

where

$$\lambda = \frac{\alpha_1 + \alpha_3 + \alpha_4}{1 - \alpha_2}.$$

From condition (3.4), it follows that $0 < \lambda < 1$.

Proceeding as in the proof of Theorem 3.1, for $m > n$ we have

$$\Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) \leq \sum_{k=n}^{m-1} b^{k-n}\Omega(\mathfrak{z}_{k+1}, \mathfrak{z}_k, \mathfrak{z}_k) \leq \frac{b\,\lambda^n}{1 - b\lambda}\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0).$$

Hence,

$$\lim_{m,n\to\infty} \Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) = 0,$$

and thus $\{\mathfrak{z}_n\}$ is a G-Cauchy sequence. Since (J, G) is complete, there exists $u \in J$ such that

$$\lim_{n\to\infty} \Omega(\mathfrak{z}_n, u, u) = 0.$$

Next, applying (3.3) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = u$ and $\mathfrak{k} = u$, and letting $n \to \infty$, we obtain

$$\Omega(u, \Xi u, \Xi u) = 0,$$

which implies $\Xi u = u$. Hence, u is a fixed point of Ξ.

Finally, assume that $v \in J$ is another fixed point of Ξ. Using (3.3) with $\mathfrak{z} = u$, $\mathfrak{a} = v$ and $\mathfrak{k} = v$, we get

$$\Omega(u, v, v) \leq \alpha_1\Omega(u, v, v).$$

Since $\alpha_1 < 1$, it follows that $\Omega(u, v, v) = 0$, and hence $u = v$. Therefore, the fixed point of Ξ is unique. □

Corollary 3.1. *Let (J, G) be a complete G-b-metric space with coefficient $b \geq 1$ and let $\Omega : J \times J \times J \to [0, \infty)$ be a G-Ω-distance on J. Assume that a mapping $\Xi : J \to J$ satisfies*

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{k}) \leq \alpha\,\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) + \beta\Big(\Omega(\mathfrak{z}, \Xi\mathfrak{z}, \Xi\mathfrak{z}) + \Omega(\mathfrak{a}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) + \Omega(\mathfrak{k}, \Xi\mathfrak{k}, \Xi\mathfrak{k})\Big), \tag{3.5}$$

for all $\mathfrak{z}, \mathfrak{a}, \mathfrak{k} \in J$, where $\alpha, \beta \geq 0$ and

$$\alpha + 3\beta < \frac{1}{b}. \tag{3.6}$$

Then Ξ has a unique fixed point in J.

Proof. Condition (3.9) is a particular case of (3.3) in Theorem 3.2 by taking

$$\alpha_1 = \alpha, \quad \alpha_2 = \alpha_3 = \alpha_4 = \beta.$$

The inequality (3.10) ensures that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \alpha + 3\beta < \frac{1}{b}.$$

Hence, all the assumptions of Theorem 3.2 are satisfied. Therefore, Ξ has a unique fixed point in J. □

Theorem 3.3. *Let (J, G) be a complete G-b-metric space with coefficient $b \geq 1$ and let $\Omega : J \times J \times J \to [0, \infty)$ be a G-Ω-distance on J. Assume that a mapping $\Xi : J \to J$ satisfies, for all $\mathfrak{z}, \mathfrak{a}, \mathfrak{k} \in J$,*

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{k}) \leq \alpha\,\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) + \beta\,\Omega(\mathfrak{z}, \Xi\mathfrak{z}, \Xi\mathfrak{z}) + \gamma\,\Omega(\mathfrak{a}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) + \delta\,\Omega(\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{a}), \tag{3.7}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and

$$\alpha + \beta + \gamma + b\delta < \frac{1}{b}. \tag{3.8}$$

Then Ξ has a unique fixed point in J.

Proof. Let $\mathfrak{z}_0 \in J$ be arbitrary and define a sequence $\{\mathfrak{z}_n\}$ in J by

$$\mathfrak{z}_{n+1} = \Xi\mathfrak{z}_n, \quad n \geq 0.$$

Applying (3.7) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = \mathfrak{z}_{n-1}$ and $z = \mathfrak{z}_{n-1}$, we obtain

$$\begin{aligned}\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq{}& \alpha\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) + \beta\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1}) \\ &+ \gamma\,\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n) + \delta\,\Omega(\mathfrak{z}_n, \mathfrak{z}_n, \mathfrak{z}_n).\end{aligned}$$

Since $\Omega(\mathfrak{z}_n, \mathfrak{z}_n, \mathfrak{z}_n) = 0$, the above inequality reduces to

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq (\alpha + \gamma)\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) + \beta\,\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n).$$

Rearranging terms yields

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \lambda\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}),$$

where

$$\lambda = \frac{\alpha + \gamma}{1 - \beta}.$$

From condition (3.8), we have $0 < \lambda < 1$.

Proceeding as in the proof of Theorem 3.2, for $m > n$ we obtain

$$\Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) \le \sum_{k=n}^{m-1} b^{k-n}\Omega(\mathfrak{z}_{k+1}, \mathfrak{z}_k, \mathfrak{z}_k) \le \frac{b\,\lambda^n}{1-b\lambda}\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0).$$

Thus,

$$\lim_{m,n\to\infty} \Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) = 0,$$

which implies that $\{\mathfrak{z}_n\}$ is a G-Cauchy sequence. Since (J, G) is complete, there exists $u \in J$ such that

$$\lim_{n\to\infty} \Omega(\mathfrak{z}_n, u, u) = 0.$$

Now, applying (3.7) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = u$ and $\mathfrak{t} = u$, and letting $n \to \infty$, we get

$$\Omega(u, \Xi u, \Xi u) = 0,$$

which implies $\Xi u = u$. Hence, u is a fixed point of Ξ.

Finally, suppose that $v \in J$ is another fixed point of Ξ. Using (3.7) with $\mathfrak{z} = u$, $\mathfrak{a} = v$ and $\mathfrak{t} = v$, we obtain

$$\Omega(u, v, v) \le \alpha\,\Omega(u, v, v).$$

Since $\alpha < 1$, it follows that $\Omega(u, v, v) = 0$, and hence $u = v$. Therefore, the fixed point of Ξ is unique. □

Theorem 3.4. *Let (J, G) be a complete G-b-metric space with coefficient $b \ge 1$ and let $\Omega : J \times J \times J \to [0, \infty)$ be a G-Ω-distance on J. Assume that a mapping $\Xi : J \to J$ satisfies, for all $\mathfrak{z}, \mathfrak{a}, \mathfrak{t} \in J$,*

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{t}) \le \alpha\,\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{t}) + \beta\Big(\Omega(\mathfrak{z}, \Xi\mathfrak{z}, \Xi\mathfrak{z}) + \Omega(\mathfrak{a}, \Xi\mathfrak{a}, \Xi\mathfrak{a}) + \Omega(\mathfrak{t}, \Xi\mathfrak{t}, \Xi\mathfrak{t})\Big), \tag{3.9}$$

where $\alpha, \beta \ge 0$ and

$$\alpha + 3\beta < \frac{1}{b}. \tag{3.10}$$

Then Ξ has a unique fixed point in J.

Proof. Let $\mathfrak{z}_0 \in J$ be arbitrary and define a sequence $\{\mathfrak{z}_n\}$ in J by

$$\mathfrak{z}_{n+1} = \Xi\mathfrak{z}_n, \quad n \ge 0.$$

Applying (3.9) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = \mathfrak{z}_{n-1}$ and $\mathfrak{t} = \mathfrak{z}_{n-1}$, we obtain

$$\begin{aligned}\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) &\le \alpha\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}) + \beta\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n+1}, \mathfrak{z}_{n+1})\\ &\quad + 2\beta\,\Omega(\mathfrak{z}_{n-1}, \mathfrak{z}_n, \mathfrak{z}_n).\end{aligned}$$

Rearranging terms yields

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \le \lambda\,\Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}),$$

where

$$\lambda = \frac{\alpha + 2\beta}{1-\beta}.$$

From condition (3.10), it follows that $0 < \lambda < 1$.

Proceeding as in the proof of Theorem 3.2, for $m > n$ we obtain

$$\Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) \leq \sum_{k=n}^{m-1} b^{k-n}\Omega(\mathfrak{z}_{k+1}, \mathfrak{z}_k, \mathfrak{z}_k) \leq \frac{b\,\lambda^n}{1-b\lambda}\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0).$$

Hence,

$$\lim_{m,n\to\infty} \Omega(\mathfrak{z}_m, \mathfrak{z}_n, \mathfrak{z}_n) = 0,$$

which implies that $\{\mathfrak{z}_n\}$ is a G-Cauchy sequence. Since (J, G) is complete, there exists $u \in J$ such that

$$\lim_{n\to\infty} \Omega(\mathfrak{z}_n, u, u) = 0.$$

Next, applying (3.9) with $\mathfrak{z} = \mathfrak{z}_n$, $\mathfrak{a} = u$ and $\mathfrak{k} = u$, and letting $n \to \infty$, we get

$$\Omega(u, \Xi u, \Xi u) = 0,$$

which implies $\Xi u = u$. Hence, u is a fixed point of Ξ.

Finally, suppose that $v \in J$ is another fixed point of Ξ. Using (3.9) with $\mathfrak{z} = u$, $\mathfrak{a} = v$ and $\mathfrak{k} = v$, we obtain

$$\Omega(u, v, v) \leq \alpha\,\Omega(u, v, v).$$

Since $\alpha < 1$, it follows that $\Omega(u, v, v) = 0$, and hence $u = v$. Therefore, the fixed point of Ξ is unique. □

4. Application to Nonlinear Integral Equations

4.1. **Problem formulation.** Consider the nonlinear integral equation

$$\mathfrak{z}(t) = \int_0^t K(t,s)\,F(s, \mathfrak{z}(s))\,ds + \int_0^1 H(t,s)\,G(s, \mathfrak{z}(s))\,ds + h(t), \quad t \in [0,1], \tag{4.1}$$

where $F, G : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous nonlinear functions, $K, H : [0,1] \times [0,1] \to \mathbb{R}$ are continuous kernel functions, and $h \in C([0,1], \mathbb{R})$.

4.2. **Functional framework.** Let

$$J = C([0,1], \mathbb{R})$$

and define $G : J \times J \times J \to [0, \infty)$ by

$$G(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) = \sup_{t\in[0,1]} \Big(|\mathfrak{z}(t) - \mathfrak{a}(t)| + |\mathfrak{a}(t) - \mathfrak{k}(t)| + |\mathfrak{k}(t) - \mathfrak{z}(t)|\Big).$$

Then (J, G) is a complete G-metric space and hence a complete G-b-metric space with $b = 1$.

Define

$$\Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}) = G(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}).$$

Define the operator $\Xi : J \to J$ by

$$(\Xi\mathfrak{z})(t) = \int_0^t K(t,s)\,F(s, \mathfrak{z}(s))\,ds + \int_0^1 H(t,s)\,G(s, \mathfrak{z}(s))\,ds + h(t). \tag{4.2}$$

A function $\mathfrak{z} \in J$ is a solution of (4.1) if and only if it is a fixed point of Ξ.

4.3. **Assumptions.** Assume the following conditions hold:

(H1) There exist constants $L_1, L_2 > 0$ such that

$$|F(t,u) - F(t,v)| \leq L_1|u - v|, \quad |G(t,u) - G(t,v)| \leq L_2|u - v|,$$

for all $t \in [0,1]$ and $u, v \in \mathbb{R}$.

(H2) The kernels satisfy

$$\sup_{t\in[0,1]} \int_0^t |K(t,s)|\, ds \leq M_1, \quad \sup_{t\in[0,1]} \int_0^1 |H(t,s)|\, ds \leq M_2.$$

(H3) The smallness condition

$$M_1L_1 + M_2L_2 < \frac{1}{3}$$

holds.

4.4. **Verification of Theorem 3.4.** Let $\mathfrak{z}, \mathfrak{a}, \mathfrak{k} \in J$. For each $t \in [0,1]$, we have

$$\begin{aligned}|\Xi\mathfrak{z}(t) - \Xi\mathfrak{a}(t)| &\leq \int_0^t |K(t,s)|\,|F(s,\mathfrak{z}(s)) - F(s,\mathfrak{a}(s))|\, ds \\ &\quad + \int_0^1 |H(t,s)|\,|G(s,\mathfrak{z}(s)) - G(s,\mathfrak{a}(s))|\, ds \\ &\leq (M_1L_1 + M_2L_2)|\mathfrak{z}(t) - \mathfrak{a}(t)|.\end{aligned}$$

Taking supremum over $t \in [0,1]$, it follows that

$$\Omega(\Xi\mathfrak{z}, \Xi\mathfrak{a}, \Xi\mathfrak{k}) \leq \alpha\, \Omega(\mathfrak{z}, \mathfrak{a}, \mathfrak{k}), \quad \alpha = M_1L_1 + M_2L_2.$$

Since $\alpha < 1$, the contractive condition of Theorem 3.4 is satisfied.

Theorem 4.1. *Under assumptions* (H1)–(H3)*, the nonlinear integral equation* (4.1) *admits a unique solution* $\mathfrak{z}^* \in C([0,1], \mathbb{R})$.

4.5. **Computational convergence.** Let $\{\mathfrak{z}_n\} \subset J$ be defined by

$$\mathfrak{z}_{n+1} = \Xi\mathfrak{z}_n, \quad n \geq 0.$$

Then there exists $\lambda \in (0,1)$ such that

$$\Omega(\mathfrak{z}_{n+1}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \lambda\, \Omega(\mathfrak{z}_n, \mathfrak{z}_{n-1}, \mathfrak{z}_{n-1}),$$

which implies

$$\Omega(\mathfrak{z}_{n+k}, \mathfrak{z}_n, \mathfrak{z}_n) \leq \frac{\lambda^n}{1-\lambda}\Omega(\mathfrak{z}_1, \mathfrak{z}_0, \mathfrak{z}_0).$$

Hence, the successive approximation method converges to the unique solution $\mathfrak{z}^*$ of (4.1).

4.6. **Special cases.** If $H \equiv 0$, then (4.1) reduces to a nonlinear Volterra integral equation. If $K \equiv 0$, it reduces to a nonlinear Fredholm integral equation. If asymmetric kernel effects are present, Theorem 3.3 can be applied.

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References

[1] I.A. Bakhtin, The Contraction Principle in Quasimetric Spaces, Funct. Anal. 30 (1989), 26–37.

[2] S. Czerwik, Nonlinear Set-Valued Contraction Mappings in b-Metric Spaces, Atti Sem. Mat. Fis. Univ. Modena 46 (1998), 263–276. https://cir.nii.ac.jp/crid/1571980075066433280.

[3] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[4] W. Shatanawi, A. Pitea, Fixed and Coupled Fixed Point Theorems of Ω-Distance for Nonlinear Contraction, Fixed Point Theory Appl. 2013 (2013), 275. https://doi.org/10.1186/1687-1812-2013-275.

[5] W. Shatanawi, A. Bataihah, A. Pitea, Fixed and Common Fixed Point Results for Cyclic Mappings of Ω-Distance, J. Nonlinear Sci. Appl. 09 (2016), 727–735. https://doi.org/10.22436/jnsa.009.03.02.

[6] W. Shatanawi, A. Pitea, Ω-Distance and Coupled Fixed Point in G-Metric Spaces, Fixed Point Theory Appl. 2013 (2013), 208. https://doi.org/10.1186/1687-1812-2013-208.

[7] D. Kannan, H.I.O. Ibnouf, Parametric Metric Spaces and Integral-Type Contractions: New Fixed Point Theorems with Applications, Gulf J. Math. 21 (2025), 512–521. https://doi.org/10.56947/gjom.v21i1.3500.

[8] K. Dinesh, E. Sila, K. Zoto, B. Shoba, D. Rizk, A Novel Approach of Digital Parametric Metric Space, J. Interdiscip. Math. 28 (2025), 2677–2686. https://doi.org/10.47974/jim-2313.

[9] K. Dinesh, D. Gerbeti, K. Zoto, H. Ibnouf, Fixed Point Theorems in Fuzzy Metric Spaces via Generalized Rational-Type Contractions, Int. J. Math. Comput. Sci. 20 (2025), 1003–1011. https://doi.org/10.69793/ijmcs/04.2025/zoto.

[10] K. Zoto, I. Vardhami, D. Bajović, Z. D. Mitrović, S. Radenović, On Some Novel Fixed Point Results for Generalized F-Contractions in b-Metric-Like Spaces with Application, Comput. Model. Eng. Sci. 135 (2023), 673–686. https://doi.org/10.32604/cmes.2022.022878.

[11] H. Huang, K. Zoto, Z.D. Mitrović, S. Radenović, Fixed Point Results for Generalized F-Contractions in B-Metric-Like Spaces, Fractal Fract. 6 (2022), 272. https://doi.org/10.3390/fractalfract6050272.

[12] K. Zoto, N. Mlaiki, H. Aydi, Related Fixed Point Theorems via General Approach of Simulations Functions, J. Math. 2020 (2020), 4820191. https://doi.org/10.1155/2020/4820191.

[13] Z. Mustafa, B. Sims, a New Approach to Generalized Metric Spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.

[14] S. Sedghi, N. Shobkolaei, J.R. Roshan, W. Shatanawi, Coupled Fixed Point Theorems in G_b-Metric Spaces, Mat. Vesnik 66 (2014), 190–201.

[15] K. Dinesh, E. Abdalrhim, M.S. Jazmati, M.A. Mohamed, D. Rizk, Fixed Point Theorems of Multivalued Mappings of Integral Type Contraction in Cone Metric Space and Its Applications, Results Nonlinear Anal. 8 (2025), 124–130.