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SMOOTHNESS TO THE BOUNDARY OF BIHOLOMORPHIC MAPPINGS

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ABSTRACT. Under a plausible geometric hypothesis, we show that a biholomorphic mapping of smoothly bounded, pseudoconvex domains extends to a diffeomorphism of the closures.

1. INTRODUCTION

The Riemann mapping theorem (see [11]) tells us that the function theory of a simply connected, planar domain Ω, other than than the entire plane, can be transferred from Ω to the unit disc D. But, for many questions, one needs to know the behavior of the Riemann mapping at the boundary.

The first person to take up this issue was P. Painlevé. He proved in his thesis that, if the domain Ω has C^∞ boundary, then the Riemann mapping (and its inverse) extends smoothly to the boundary (see [5] for details of this history). Later O. Kellogg gave a proof of this result that connected the Riemann mapping with potential theory. Further on, Stefan Warschawski refined Kellogg's results and gave substantive local boundary analyses of the Riemann mapping.

It was quite some time before any progress was made on this question in the context of several complex variables. The first real theorem of a general nature was proved by C. Fefferman [7]. He showed that a biholomorphic mapping of smoothly bounded, strongly pseudoconvex domains in $\mathbb{C}^n$ extends to a diffeomorphism of the closures. Fefferman's work opened up a flood of developments in this subject. We only mention here that Bell [2] and Bell/Ligocka [4] were able to greatly simplify Fefferman's proof by connecting the problem in a rather direct fashion with the Bergman projection. The work of Bell and Bell/Ligocka led to a number of simplifications, generalizations, and extensions of Fefferman's result. Many different mathematicians have contributed to the development of this work.

The big remaining open problem is this:

> **Problem:** Let Ω_1 and Ω_2 in $\mathbb{C}^n$ be smoothly bounded, (weakly Levi) pseudoconvex domains. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Does Φ extend to a diffeomorphism of the closures?

There are some counterexamples to this question—see for instance [9]—but these definitely do not have smooth boundary. In fact they do not even have C^2 boundary.

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In the present paper we are unable to give a full answer to this main problem. But we present the following somewhat encouraging partial result.

Theorem 1.1. *Let Ω_1, Ω_2 be smoothly bounded, Levi pseudoconvex domains in $\mathbb{C}^n$. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Assume that Φ and Φ^{-1} each satisfy a Lipschitz condition of order exceeding $(n-1)/n$. Then Φ continues to a diffeomorphism of the closures of the domains.*

Corollary 1.2. *Let Ω_1, Ω_2 be smoothly bounded, pseudoconvex domains in $\mathbb{C}^n$. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Assume that Φ and Φ^{-1} each satisfy a Lipschitz condition of order 1. Then Φ continues to a diffeomorphism of the closures of the domains.*

This result is in the nature of a bootstrapping result from partial differential equations. It seems to be the first general result—for *all* pseudoconvex domains—of its kind. And it has some basis in the history of the subject. For Painlevé proved his theorem in dimension one by first establishing a result for C^1 boundary smoothness of the mapping, and then bootstrapping. No less an *eminence gris* than Jacques Hadamard cast public doubt on Painlevé's bootstrapping argument, and Painlevé had to work strenuously to defend his theorem. See [5] for the full history.

It may be noted that the hypothesis of Lipschitz continuity in the theorem is a nontrivial one. Henkin [12] was able to show, prior to Fefferman's celebrated result, that a biholomorphic mapping of smoothly bounded, strongly pseudoconvex domains will extend to be Lipschitz 1/2 to the boundary. He did so by analyzing and estimating the Carathéodory metric. But there are not many results of this type.

We see that the Lipschitz condition in Theorem 1.1 in case $n = 2$ meshes nicely with Henkin's result described in the last paragraph.

A final, rather significant, comment is this. Our arguments here are inspired by those in [2]. Bell uses global regularity ideas of Kohn which exploit weighed L^2 spaces. In the paper [2], a good deal of the work is expended in proving that the complex Jacobian determinant u of the mapping Φ is bounded. This fact is used in turn to prove that the complex Jacobian determinant U of the inverse mapping Φ^{-1} is nonvanishing. As we shall see below, our hypothesis of Lipschitz continuity of order exceeding $(n-1)/n$ obviates these arguments and gets to the necessary result rather quickly. The remaining steps comprise just one paragraph on page 108 of [2]. We have to work a bit harder because we need to set things up in the context of Kohn's weighted L^2 spaces. But the spirit of our arguments follows Bell.

We also warn the reader of the following point. The main thrust of this paper is to prove estimates on the derivatives of the mappings Φ and Φ^{-1}. However our crucial Lemma 4.2, based on an old idea of S. R. Bell, entails taking a good many derivatives of Φ. So it appears as though there are a number of extraneous terms in our calculations that involve derivatives of Φ. But we will go to quite a lot of extra trouble to find a means of absorbing those extra derivatives. In the end they will all be accounted for, and we will obtain valid estimates for the derivatives of Φ.

2. Condition $\boldsymbol{R}$ and Related Ideas

Throughout this paper we shall use the language of Sobolev spaces (see [1], [19]). For s a nonnegative integer and $1 \le p \le \infty$, we let $W_{s,p}$ denote the usual Sobolev space of functions with s weak derivatives in L^p. The norm that we use on the Sobolev space is standard, and we refer to [1] for details.

One of the important innovations that S. R. Bell introduced into this subject is his *Condition R*. It says this:

> **Condition *R*:** Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded domain. We say that Ω satisfies *Condition R* if the Bergman projection P maps $C^\infty(\overline{\Omega})$ to $C^\infty(\overline{\Omega})$. Equivalently, for each $s > 0$, there is an $m(s) > 0$ so that the Bergman projection P maps the Sobolev space $W_{m(s),2}(\Omega)$ to the Sobolev space $W_{s,2}(\Omega)$.

In what follows we shall suppose that $1 < m(1) < m(2) < \cdots \to \infty$ and that each $m(j)$ is an integer.

In the paper [2], Bell proves the following elegant result:

Theorem 2.1. *Let Ω_1, Ω_2 be smoothly bounded, pseudoconvex domains in $\mathbb{C}^n$. Assume that one (but not necessarily both) of these domains satisfies Condition R. Then any biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$ extends to a diffeomorphism of the closures.*

3. Ideas of Kohn

The classical treatment of the $\overline{\partial}$-Neumann problem is based on the traditional Euclidean L^2 inner product—see [8]. Kohn's idea in [14] is to use an inner product with a weight. This is inspired by work of Hörmander [13], and that in turn comes from old ideas of Carleman.

Kohn's setup is this (see [14]). Fix a smoothly bounded domain Ω in $\mathbb{C}^n$. Let λ be a C^∞, nonnegative function on a neighborhood of $\overline{\Omega}$. Usually λ will be strictly plurisubharmonic. With λ fixed and $t \geq 0$, we shall define the $\overline{\partial}$-Neumann problem of weight t, with real $t > 0$. We let $\mathcal{A}$ be the space of all forms on $\overline{\Omega}$ which have C^∞ coefficients up to the boundary. For $\phi, \psi \in \mathcal{A}$, we define

$$\langle \phi, \psi \rangle_{(t)} \equiv \langle \phi, e^{-t\lambda}\psi \rangle \qquad \text{and} \qquad \|\phi\|^2_{(t)} = \langle \phi, \phi \rangle_{(t)} .$$

Here $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{(0)}$ is the usual L^2 inner product.

It is an easily verified fact that the norms $\|\ \ \|_{(t)}$ are equivalent to the norm $\|\ \ \|_0 = \|\ \ \|$. Hence a function is in the completion under any of these norms if and only if it is square integrable. We let $\widetilde{\mathcal{A}}_t$ be the Hilbert space obtained by completing $\mathcal{A}$ under the norm $\|\ \ \|_{(t)}$.

The $\overline{\partial}$-Neumann problem may be set up in the $\langle\ ,\ \rangle_{(t)}$ inner product rather than the usual L^2 inner product $\langle\ ,\ \rangle$. These are familiar ideas, and the details are provided in [14]]. One of the main points that must be noted is that the *formal adjoint* of the operator $\overline{\partial}$, when calculated in the $\langle\ ,\ \rangle_{(t)}$ inner product, is

$$\mathcal{I}_t = \mathcal{I} - t\sigma(\mathcal{I}, d\lambda) .$$

Here σ is the "symbol" in the usual sense of pseudodifferential operators. Also $\mathcal{I}$ is the standard formal adjoint of $\overline{\partial}$ with respect to the standard Euclidean Hermitian inner product and $\mathcal{I}_t$ is the formal adjoint of $\overline{\partial}$ with respect to the inner product $\langle\ ,\ \rangle_{(t)}$. We thus see how the parameter t comes into play. If t is chosen positive and large enough, then certain terms in the usual $\overline{\partial}$-Neumann estimates can be forced to dominate certain others. Again see [14] for the details.

Let Ω_1, Ω_2 be smoothly bounded, pseudoconvex domains and $\Phi : \Omega_1 \to \Omega_2$ a biholomorphic mapping which is bi-Lipschitz of order exceeding $(n-1)/n$. We shall apply the preceding ideas on Ω_2 with $\lambda(z) = \lambda_2(z) = |z|^2$ and on Ω_1 with $\lambda(z) = \lambda_1(z) = |\Phi(z)|^2$. Note that we are applying Kohn's construction twice.[1]

In this context we shall refer to the t-weighted Bergman projection as $P_{t,1}$ (for Ω_1) and $P_{t,2}$ (for Ω_2). We shall also call Bell's regularity condition in the context of Kohn's weighted inner product by the name "Condition R_t." We shall denote the Bergman kernels by $K_{t,1}$ and $K_{t,2}$. As a result of these ideas, the $\overline{\partial}$-Neumann problem on Ω_2, formulated with the indicated weight λ, satisfies favorable estimates (this follows from [14]) as long as t is large enough. Hence Ω_2 satisfies Condition R_t. Bell also makes use of these facts.

These are the tools that we shall need in the next section to get to our result.

In what follows we shall take it that we are working with the Bergman theory for the inner product $\langle\ ,\ \rangle_{(t)}$ for t sufficiently large, and that Ω_2 satisfies Condition R_t. We formulate this last by saying that $P_{t,2} : W_{m(s),2}(\Omega) \to W_{s,2}(\Omega)$ for any $s \geq 0$ and suitable $m(s) \geq s$.

Sometimes, in what follows, we will talk about

(i) a domain Ω with weight λ

but make no reference to

(ii) Ω_1, Ω_2, or the mapping Φ.

We will later apply **(i)** to **(ii)**.

4. The Guts of the Proof

In this section, Ω is a smoothly bounded, pseudoconvex domain equipped with the weight $e^{-t\lambda}$.

Lemma 4.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be smoothly bounded and pseudoconvex. Suppose that the λ from the weight on Ω is smooth on $\overline{\Omega}$. Assume that Ω satisfies Condition R_t with respect to the projection P_t. Let $w \in \Omega$ be fixed. Let K_t denote the Bergman kernel. Then there is a constant $C_w > 0$ so that*

$$\|K_t(w,\,\cdot\,)\|_{\sup} \leq C_w\,.$$

Proof: The function $K(z,\,\cdot\,)$ is harmonic. Let $\varphi : \Omega \to \mathbb{R}$ be a radial, C^∞ function centered at w and supported in Ω so that the radius of the support is comparable to half the distance of w to the boundary. Assume that $\varphi \geq 0$ and $\int \varphi(\zeta)\,dV(\zeta) = 1$. Then the mean value property implies that

$$K_t(z,w) = \int_\Omega K_t(z,\zeta)\varphi(\zeta)\,dV(\zeta) = \int_\Omega K_t(z,\zeta)\big[\varphi(\zeta)e^{t\lambda(\zeta)}\big]e^{-t\lambda(\zeta)}\,dV(\zeta)\,.$$

[1]It is because the weight $e^{-t|\Phi(z)|^2}$ gets differentiated in the proofs below that we must be careful to absorb these error terms.

Of course this last displayed expression equals $P_t\biggl(\varphi(\,\cdot\,)e^{t\lambda(\,\cdot\,)}\biggr)$. Thus

$$\begin{aligned}\|K_t(w,\,\cdot\,)\|_{\sup} &= \sup_{z\in\Omega}|K_t(w,z)| \\ &= \sup_{z\in\Omega}|K_t(z,w)| \\ &= \sup_{z\in\Omega}\left|P_t\bigl[\varphi(\,\cdot\,)e^{t\lambda(\,\cdot\,)}\bigr]\right|.\end{aligned}$$

By Sobolev's theorem, this last is

$$\leq C(\Omega,w)\|P_t\bigl[\varphi(\,\cdot\,)e^{t\lambda(\,\cdot\,)}\bigr]\|_{W_{2n+1,2}}\,.$$

By Condition R_t, this is

$$\leq C(\Omega,w)\cdot\|\varphi(\,\cdot\,)e^{t\lambda(\,\cdot\,)}\|_{W_{m(2n+1),2}}\equiv C_w\,. \qquad \square$$

Remark: It is worth noting that the estimate obtained in this last proof depends on some derivatives of λ on a compact set. In practice this causes no harm. We only need to know that $\|K_t(w,\,\cdot\,)\|_{\sup}$ is bounded so that we can perform an integration by parts in the proof of Lemma 4.2 below.

Lemma 4.2. *Let $u \in C^\infty(\overline{\Omega})$ be arbitrary. Let $s \in \{0,1,2,\dots\}$. Then there is a $v \in C^\infty(\overline{\Omega})$ such that $P_t v = 0$ and the functions u and v agree to order s on $\partial\Omega$.*

Remark: This lemma in the present formulation is not entirely satisfactory. For, in its statement here, we suppose that the weight λ is smooth across the boundary. Yet, in the applications below, the weight is taken to be $|\Phi(z)|^2$, and that is *not* known a priori to be smooth up to the boundary (in fact our goal is to prove that it is smooth up to the boundary).

The way to address this problem is to use an approximation argument. In the case that the domain Ω_1 is convex, the approximation is very simple. We simply replace Ω_1 by $(1-\epsilon)\Omega_1$, $\epsilon > 0$, so that the mapping is smooth across the boundary. The relevant estimates are uniform in ϵ, and the result is correct in the limit.

For non-convex Ω_1, we must take advangage of ideas in [3]. For Bell explains there how to localize the smoothness-to-the-boundary arguments that we present here. As a result, if $p \in \partial\Omega_1$, ν_p is the Euclidean unit outward normal vector at p, and U is a small neighborhood of p, then we may apply our arguments on $\Omega_1^\epsilon \equiv (U\cap\Omega_1)-\epsilon\nu_p$. Thus the mapping will be smooth across the boundary and (a localized version of) Lemma 4.2 will apply without any problem. All the relevant estimates are uniform in ϵ, and our desired result holds in the limit.

Proof: This lemma is the key to Bell's approach to these matters. We will need to expend some effort to adapt Bell's ideas to the new weighted context.

We of course assume that our domain Ω is equipped with an inner product $\langle\ ,\ \rangle_{(t)}$ based on a weight $e^{-t\lambda}$.

After a partition of unity, it suffices to prove the assertion in a small neighborhood W of $z_0 \in \partial\Omega$. After a rotation, we may assume that $\partial\rho/\partial z_1 \neq 0$ on $W\cap\overline{\Omega}$. [Here ρ is a defining function for the domain Ω—see [15].]

Define the differential operator

$$\nu = \frac{\text{Re}\left\{\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial z_j}\right\}}{\sum_{j=1}^n \left|\frac{\partial\rho}{\partial z_j}\right|^2}\,.$$

Observe that $\nu\rho \equiv 1$. Now we shall define v by induction on s. In what follows, we shall make use of the differential operator

$$T = \frac{\partial}{\partial\zeta_1} + t\frac{\partial\lambda}{\partial\zeta_1}\,.$$

For the case $s = 0$, we set

$$w_1 = \frac{\rho u}{T\rho}\,.$$

If W is small then of course $T\rho$ will not vanish. Also define

$$v_1 = Tw_1 = u + \mathcal{O}(\rho)\,.$$

Then we see immediately that u and v_1 agree to order 0 on $\partial\Omega$. Furthermore,

$$\begin{aligned} P_t v_1 &= \int K_t(z,\zeta)Tw_1 e^{-t\lambda}\,dV \\ &= -\int T_\zeta\left[K_t(z,\zeta)e^{-t\lambda}\right]w_1\,dV \\ &= 0\,. \end{aligned}$$

The penultimate equality comes from integration by parts. This operation is justified by Lemma 4.1. Note that T_ζ annihilates $K_t(z,\zeta)e^{-t\lambda(\zeta)}$ by a simple calculation (using the fact that $K_t(z,\zeta)$ is conjugate holomorphic in the ζ variable).

Now suppose inductively that

$$\begin{aligned} w_{s-1} &= w_{s-2} + \theta_{s-1}\cdot\rho^{s-1}\,, \\ v_{s-1} &= Tw_{s-1} \end{aligned}$$

(for some smooth function θ_{s-1}). We construct

$$w_s = w_{s-1} + \theta_s\cdot\rho^s$$

so that

$$v_s \equiv Tw_s$$

agrees to order $s-1$ with u on $\partial\Omega$.

By the inductive hypothesis,

$$\begin{aligned} v_s &= Tw_s \\ &= Tw_{s-1} + T(\theta_s\rho^s) \\ &= v_{s-1} + \rho^{s-1}\left[s\theta_s T\rho + \rho T\theta_s\right]\,. \end{aligned}$$

This expression agrees, by the inductive hypothesis, with u to order $s-1$ on $\partial\Omega$. We now must examine $D(u - v_s)$, where D is any s-order differential operator. There are two cases:

Case 1:: Assume that D involves a tangential derivative D_0. Then we may write $D = D_0 D_1$. Then

$$D(u - v_s) = D_0\alpha\,,$$

where α vanishes on $\partial\Omega$. But then it follows that $D_0\alpha = 0$ because D_0 is tangential.

Case 2:: Now assume that D has no tangential derivative in it. So we take $D = \nu^s$, where ν was defined at the beginning of this discussion. Our job is to choose θ_s so that

$$\nu^s(u - v_s) = 0 \quad \text{on } \partial\Omega\,.$$

So we require that

$$\nu^s(u - v_{s-1}) - \nu^s\left(T(\theta_s \rho^s)\right) = 0 \quad \text{on } \partial\Omega\,.$$

This is the same as

$$\nu^s(u - v_{s-1}) - \theta_s\left(\nu^s T\rho^s\right) = 0 \quad \text{on } \partial\Omega$$

(because terms that contain ρ must vanish on $\partial\Omega$) or

$$\nu^s(u - v_{s-1}) - \theta_s\left(\nu^s \frac{\partial}{\partial\zeta_1}\rho^s\right) - \theta_s\left(\nu^s t\frac{\partial\lambda}{\partial\zeta_1}\rho^s\right) = 0 \quad \text{on } \partial\Omega\,.$$

This may be rewritten as

$$\nu^s(u - v_{s-1}) - \theta_s \cdot s!\frac{\partial\rho}{\partial\zeta_1} - t\cdot\theta_s\cdot\tau\cdot\rho\,,$$

where τ stands for terms that come from the differentiations. The last line may be rewritten as

$$\theta_s = \frac{\nu^s(u - v_{s-1})}{s!\frac{\partial\rho}{\partial\zeta_1} + t\cdot\tau\cdot\rho}\,.$$

If W is small enough then the denominator cannot vanish and we see that we have a well-defined choice for θ_s as desired. □

We note that, in [2], Bell has a particularly elegant way of expressing the content of this last lemma. His formulation will be useful for us later, so we formulate it now. First some notation.

If $\Omega \subseteq \mathbb{C}^n$ is a domain (a connected, open set), then let $W^0_{s,p}(\Omega)$ be the closure of $C^\infty_c(\Omega)$ in $W_{s,p}(\Omega)$. Now we have Bell's formulation of our Lemma 4.2:

Corollary 4.3. *Let Ω be a smoothly bounded, pseudoconvex domain. Let $s, m \in \{0, 1, 2, \ldots\}$. Then there is a linear operator $\Psi^{s,m} : W_{s+m,2}(\Omega) \to W^0_{s,2}(\Omega)$ such that $P_t\Psi^{s,m} = id$. The norm of this operator depends polynomially on t and on derivatives of λ.*

5. A Deeper Analysis

A troublesome feature of Lemma 4.2 and Corollary 4.3 is that the weight λ gets differentiated s times, and λ (in practice) is defined in terms of the mapping Φ. Since our job in the end is to estimate the derivatives of Φ, this looks problematic. We need to develop a way to absorb these extraneous derivatives of Φ.

With that thought in mind, we recall the standard Sobolev embedding theorem for a smooth domain in $\mathbb{R}^N$ (see [1], [19] for details).

Proposition 5.1. [?] *Let $\Omega \subseteq \mathbb{R}^N$ be a smoothly bounded domain. Let $W_{m,p}$ be the standard Sobolev space of functions on Ω having m weak derivatives in the space L^p. Equip $W_{m,p}$ with the usual norm. Then we have the embedding*

$$W_{k,p} \subseteq W_{\ell,q}$$

whenever $k > \ell$, $1 \le p < q \le \infty$, *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{k-\ell}{N} \,.$$

We are particularly interested in domains in $\mathbb{C}^n$. Hence, for us, $N = 2n$. Also we will apply the result in case $k = 4$ and $\ell = 2$. Thus the important inclusion is

$$W_{k+n/2,2} \subseteq W_{k,4} \,.$$

We will generally exploit this embedding in the form of the inequality

$$\|f\|_{k,4} \le C\|f\|_{k+n/2,2} \,.$$

We will also make good use of the following refinement of the Sobolev theorem that is due to Ehrling, Gagliardo, and Nirenberg (see [1] for the details):

Theorem 5.2. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a smoothly bounded domain. Let* $W_{m,p}$ *be the standard Sobolev space of functions on* Ω *having* m *weak derivatives in the space* L^p. *Equip* $W_{m,p}$ *with the usual norm. Let* $\epsilon_0 > 0$. *Let* m *be a positive integer. Then there is a constant* K, *depending on* ϵ_0, m, p, *and* Ω *so that, for any integer* j *with* $0 \le j \le m-1$, *any* $0 < \epsilon < \epsilon_0$, *and any* $u \in W_{m,p}$,

$$\|u\|_{j,p} \le K\epsilon\|u\|_{m,p} + K\epsilon^{-j/(m-j)}\|u\|_{0,p} \,.$$

It is worth taking some time now to do a little analysis. Examine the proof of Lemma 4.2. [Note that, when we apply Lemma 4.2 and Corollary 4.3, we do so on Ω_1 with $\lambda(z) = \lambda_1(z) = |\Phi(z)|^2$.] At each stage we integrate by parts, and therefore a derivative falls on λ (and hence on Φ). Thus we see that the function v that we construct has the form

$$v = q_0 + q_1 t\nabla\Phi + q_2 t^2\nabla^2\Phi + \cdots + q_s t^s\nabla^s\Phi + q_1' t\nabla\overline{\Phi} + q_2' t^2\nabla^2\overline{\Phi} + \cdots + q_s' t^s\nabla^s\overline{\Phi} \,.$$

Here we use the notation $\nabla\Phi$ or $\nabla\overline{\Phi}$ to denote some derivative of some component of Φ or $\overline{\Phi}$ and $\nabla^j\Phi$ or $\nabla^j\overline{\Phi}$ to denote some j^{th} derivative of some component of Φ or $\overline{\Phi}$. Also q_j, q_j' denotes an expression that involves components of Φ, derivatives of ρ, and derivatives of θ_j. Thus we see that

$$\begin{aligned}
\|v\|_{r,2} \quad \le \quad & C \cdot \Big[\int \big|q_0\big|^2 \, dV^{1/2} + \int \big|\nabla^r q_0\big|^2 \, dV^{1/2} + \int |q_1 t\nabla\Phi|^2 \, dV^{1/2} \\
& + \int \big|(t\nabla^r q_1)\Phi\big|^2 + \big|tq_1\nabla^{r+1}\Phi\big|^2 \, dV^{1/2} \\
& + \int |q_2 t\nabla^2\Phi|^2 \, dV^{1/2} + \int \big|(t^2\nabla^r q_2)\Phi\big|^2 + \big|t^2 q_2\nabla^{r+2}\Phi\big|^2 \, dV^{1/2} \\
& + \cdots + \int |q_s t\nabla^s\Phi|^2 \, dV^{1/2} \\
& + \int \big|(t^s\nabla^r q_s)\Phi\big|^2 + \big|t^s q_s\nabla^{r+s}\Phi\big|^2 \, dV^{1/2}\Big]
\end{aligned}$$

plus similar terms involves q_j' and $\nabla^j\overline{\Phi}$.

Using Hölder's inequality, we see that this last is majorized by

$$
\begin{aligned}
C \cdot \Big[& \int |q_0|^2 \, dV^{1/2} + \int \left|\nabla^r q_0\right|^2 dV^{1/2} \\
& + \quad t \int |q_1|^4 \, dV^{1/4} \cdot \int |\nabla \Phi|^4 \, dV^{1/4} + t \int \left|\nabla^r q_1\right|^4 dV^{1/4} \cdot \int \left|\Phi\right|^4 dV^{1/4} \\
& + t \int \left|q_1\right|^4 dV^{1/4} \cdot \int \left|\nabla^{r+1}\Phi\right|^4 dV^{1/4} \\
& \quad + \quad t^2 \int |q_2|^4 \, dV^{1/4} \cdot \int |\nabla^2 \Phi|^4 \, dV^{1/4} + t^2 \int \left|\nabla^r q_2\right|^4 dV^{1/4} \cdot \int \left|\Phi\right|^4 dV^{1/4} \\
& \quad + t^2 \int \left|q_2\right|^4 dV^{1/4} \cdot \int \left|\nabla^{r+2}\Phi\right|^4 dV^{1/4} + \cdots \\
& \quad + \quad t \int |q_s|^4 \, dV^{1/4} \cdot \int |\nabla^s \Phi|^4 \, dV^{1/4} + t^s \int \left|\nabla^r q_s\right|^4 dV^{1/4} \cdot \int \left|\Phi\right|^4 dV^{1/4} \\
& \quad + t^s \int \left|q_s\right|^4 dV^{1/4} \cdot \int \left|\nabla^{r+s}\Phi\right|^4 dV^{1/4} \Big]
\end{aligned}
$$

plus similar terms involves q_j' and $\nabla^j \overline{\Phi}$.

Now we may estimate this more elegantly as

$$C' \cdot (1+t)^s \left[1 + \|\Phi\|_{r+s,4}\right].$$

And then we apply the Sobolev inequality noted above to estimate this at last by

$$C' \cdot (1+t)^s \left[1 + \|\Phi\|_{r+s+n/2,2}\right].$$

At last we apply the Ehrling-Gagliardo-Nirenberg inequality (with $m = r + s + n$, $j = r + s + n/2$) to obtain

$$\leq C''(1+t)^s \left(1 + \epsilon \cdot \|\Phi\|_{r+s+n,2} + \epsilon^{-(r+s+n/2)/(n/2)} \cdot \|\Phi\|_{0,2}\right). \tag{5.2}$$

This inequality is valid for any $0 < \epsilon < 1$ provided that C'' is large enough. Furthermore, C'' depends on s.

We finally note that, in the proof of Lemma 4.2, the definition of w_1 involves a division by $T\rho$ (and the definition of T entails a derivative of Φ). But this can be treated with a Neumann series, or just using the quotient rule.

Now we can reformulate Corollary 4.3 as follows:

Corollary 5.3. Let Ω be a smoothly bounded, pseudoconvex domain. Let $s \in \{0, 1, 2, \dots\}$. Fix any $\epsilon > 0$. Then there is a linear operator $\Psi^s : W_{r+s+n,2}(\Omega) \to W^0_{r,2}(\Omega$ such that $q_t \Psi^s = \mathrm{id}$. Moreover, the norm of this operator does not exceed $C''(1+t)^s \left(1 + \epsilon \cdot \|\Phi\|_{r+s+n,2} + \epsilon^{-(r+s+n/2)/(n/2)} \cdot \|\Phi\|_{0,2}\right)$.

6. The Final Argument

For the rest of this discussion, we let Ω_1 and Ω_2 be fixed, smoothly bounded, pseudoconvex domains in $\mathbb{C}^n$. We fix a strictly plurisubharmonic function $\lambda(z) = \lambda_2(z) = |z|^2$, and we equip Ω_2 with the inner product with weight $e^{-t\lambda(z)}$. We also, as above, equip Ω_1 with the inner product having weight $\lambda(z) = \lambda_1(z) = |\Phi(z)|^2$. We assume that there is a biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$ that extends in a bi-Lipschitzian fashion, of order greater than $(n-1)/n$, to the boundary, and we equip Ω_1 with the inner product with weight $e^{-t\lambda|\Phi(z)|^2}$. We let $P_{t,1}$ and $P_{t,2}$

be the resulting Bergman projections for Ω_1, Ω_2 respectively. We let u denote the complex Jacobian determinant of Φ. And we let U denote the complex Jacobian determinant of Φ^{-1}. So u is a complex-valued holomorphic function on Ω_1 and U is a complex-valued holomorphic function on Ω_2. Finally, for $j = 1, 2$, we let $\delta_j(z) = \delta_{\Omega_j}(z) = \mathrm{dist}_{\mathrm{Euclid}}(z, {}^c\Omega_j)$ for $z \in \Omega_j$.

Lemma 6.1. *The Bergman kernels for the two domains are related by*

$$K_{t,1}(z,\zeta) = u(z) \cdot K_{t,2}(\Phi(z), \Phi(\zeta)) \cdot \overline{u(\zeta)} \,. \tag{6.1.1}$$

Proof: This is a standard change-of-variables argument, using the canonical relationship between the real Jacobian determinant of a biholomorphic mapping and the complex Jacobian determinant of that mapping (see Lemma 1.4.10 of [15]).

Now, if f is a Bergman space function on Ω_1, then we have

$$\begin{aligned}
&\int_{\Omega_1} \left[u(z)K_{t,2}(\Phi(z),\Phi(\zeta))\overline{u(\zeta)}\right] f(\zeta)e^{-t\lambda(\Phi(\zeta))}\, dV(\zeta) \\
&\quad = \int_{\Omega_2} u(z)K_{t,2}(\Phi(z),\xi)\overline{u(\Phi^{-1}(\xi))} \\
&\qquad\qquad \times f(\Phi^{-1}(\xi))u^{-1}(\xi)\overline{u^{-1}(\xi)}e^{-t\lambda(\xi)}\, dV(\xi) \\
&\quad = f(z)u(z)u^{-1}(\Phi(z)) \\
&\quad = f(z)\,.
\end{aligned}$$

Thus we see that the righthand side of (6.1.1) has the reproducing property on Ω_1. It is also conjugate symmetric and is an element of the Bergman space in the first variable. Therefore it must equal $K_{t,1}(z,\zeta)$. □

Lemma 6.2. *For any function $g \in L^2(\Omega_2)$, we have*

$$P_{t,1}\left(u \cdot (g \circ \Phi)\right) = u \cdot \left((P_{t,2}(g) \circ \Phi\right) \,.$$

Proof: We use the preceding lemma to calculate that

$$\begin{aligned}
u(z) \cdot \left((P_{t,2}(g) \circ \Phi\right)(z) &= u(z)\int_{\Omega_2} K_{t,2}(\Phi(z),\zeta)g(\zeta)e^{-t\lambda(\zeta)}\, dV(\zeta) \\
&= u(z)\int_{\Omega_2} u(z)^{-1}K_{t,1}(z,\Phi^{-1}(\xi))\overline{u(\Phi(\xi))^{-1}} \\
&\qquad \times g(\zeta)e^{-t\lambda(\zeta)}\, dV(\zeta) \\
&= u(z)\int_{\Omega_1} u(z)^{-1}K_{t,1}(z,\xi)\overline{u(\xi)^{-1}} \\
&\qquad \times g(\Phi(\xi))e^{-t\lambda(\Phi(\xi))}u(\xi)\overline{u(\xi)}\, dV(\xi) \\
&= \int_{\Omega_1} K_{t,1}(z,\xi)g(\Phi(\xi))u(\xi)e^{-t\lambda(\Phi(\xi))}\, dV(\xi) \\
&= P_{t,1}\left(u \cdot (g \circ \Phi)\right)(z)\,.
\end{aligned}$$

That establishes the result. □

It will be useful to have the following corollary, in which Φ is replaced by Φ^{-1} (and of course the corresponding Bergman kernels switch roles):

Corollary 6.3. *Let U denote the complex Jacobian determinant of Φ^{-1}. Then, for any function $g \in L^2(\Omega_1)$, we have*

$$P_{t,2}\left(U \cdot (g \circ \Phi^{-1})\right) = U \cdot \left((P_{t,1}(g) \circ \Phi^{-1}\right) .$$

Lemma 6.4. *Let $H^\infty(\overline{\Omega}_1)$ denote the space of holomorphic functions on Ω_1 which extend smoothly to $\overline{\Omega}_1$. Let $s \in \{0, 1, 2, \dots\}$. If $h \in H^\infty(\overline{\Omega}_1)$, then let $\phi_s = \Psi^s h$, where Ψ^s is introduced in Corollary 4.3. Then*

$$U \cdot (h \circ \Phi^{-1}) = P_{t,2}(U \cdot (\phi_s \circ \Phi^{-1}))\,.$$

Proof: We calculate, using Corollary 6.3, that

$$P_{t,2}(U \cdot (\phi_s \circ \Phi^{-1})) = U \cdot (P_{t,1}(\phi_s) \circ \Phi^{-1}) = U \cdot (P_{t,1}(\Psi^s h) \circ \Phi^{-1}) = U \cdot (h \circ \Phi^{-1})\,. \ \square$$

The next lemma has nothing to do with Condition R. It is really only calculus.

Lemma 6.5. *Suppose that $\Phi^{-1} : \Omega_2 \to \Omega_1$ is a biholomorphic mapping between smoothly bounded, pseudoconvex domains in $\mathbb{C}^n$. Assume that Φ is bi-Lipschitz of order exceeding $(n-1)/n$. Let U denote the complex Jacobian determinant of Φ^{-1}. For each nonnegative integer s, there is an integer $N = N(s)$ such that the operator*

$$g \longmapsto U \cdot (g \circ \Phi^{-1})$$

is bounded from $W^0_{s+N,2}(\Omega_1)$ to $W^0_{s,2}(\Omega_2)$.

Proof: In what follows we let $d_j(z)$ denote the Euclidean distance of z from the boundary of Ω_j.

Since the components of Φ^{-1} are holomorphic and Lipschitz of order exceeding $(n-1)/n$, the derivatives of Φ^{-1} satisfy finite growth conditions at the boundary (see [10]). That is to say

$$\left|\frac{\partial^\alpha \Phi^{-1}}{\partial w^\alpha}(w)\right| \le C \cdot d_2(w)^{-k+(n-1)/n}\,. \tag{6.5.1}$$

Here α is a multi-index, $k = |\alpha|$, and d_j is the distance of the argument to the boundary of Ω_j, $j = 1, 2$. Estimates like this one go back to Hardy and Littlewood (see [10]).

Now Sobolev's lemma and Taylor's formula tell us that, for $g \in W^0_{s+|\alpha|+n,2}(\Omega_1)$,

$$|D^\alpha g(z)| \le C \cdot \|g\|_{s+|\alpha|+n} d_1(z)^s\,.$$

For a given s, in order to show that an N exists so that $g \mapsto U \cdot (g \circ \Phi^{-1})$ is bounded from $W^0_{s+N,2}(\Omega_1)$ to $W^0_{s,2}(\Omega_2)$, it will suffice to show that there is an integer $m > 0$ such that $d_1(\Phi^{-1}(w))^m \le C \cdot d_2(w)$. That such an m exists is proved by Range [16]. The proof, naturally, consists of applying Hopf's lemma to $\rho \circ \Phi^{-1}$, where ρ is a bounded, plurisubharmonic exhaustion function for Ω_2 of the form $v d_2^{1/m}$ with $v \in C^\infty(\overline{\Omega}_2)$ and $v < 0$ on $\overline{\Omega}_2$. Of course Diederich and Fornæss [6] have proved the existence of such an exhaustion function. Range [17] has given a simpler approach to the matter, with the penalty of assuming greater boundary smoothness.

That completes the proof of the lemma. $\square$

Lemma 6.6. *Let $s \in \{0, 1, 2, \dots\}$. With notation as above,*

$$\|U \cdot (h \circ \Phi^{-1})\|_s \le \|h\|_{s+N} \, .$$

Proof: We note that Kohn's theory (see [2] for the details) entails that $P_{t,2}$ maps $W_{s,2}(\Omega_2)$ to $W_{s,2}(\Omega_2)$ for t sufficiently large and any s.

Now we apply Corollary 6.3 and then Lemma 6.4 to see that

$$\begin{array}{rcl} \|U \cdot (h \circ \Phi^{-1})\|_s & \le & \|P_{t,2}(U \cdot (\phi_s \circ \Phi^{-1}))\|_s \\ & \le & \|U \cdot (\phi_s \circ \Phi^{-1})\|_{m(s)} \\ & \le & \|\phi_s\|_{m(s)+N} \\ & = & \|\Psi^{N,s+N} h\|_{m(s)+N} \\ & \le & (1+t)^{2s+N} C''\big(\epsilon \cdot \|\Phi\|_{m(s)+2N+n,2} \\ & + & \epsilon^{-(r+s+n/2)/(n/2)} \cdot \|\Phi\|_{0,2}\big) \|h\|_{s+2N} \, . \end{array}$$

In the second inequality we use Condition R_t. In the third inequality here we use Lemma 6.4. That completes the argument. $\square$

Proof of Theorem 1.1: The last lemma tells us that $U \cdot (h \circ \Phi^{-1}) \in H^\infty(\overline{\Omega}_2)$ if $h \in H^\infty(\overline{\Omega}_1)$. Taking $h \equiv 1$, we conclude immediately that

$$\|U\|_{s,2} \le C''(1+t)^s \big(1 + \epsilon_s \cdot \|\Phi\|_{m(s)+2N+n,2} + \epsilon_s^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi\|_{0,2}\big) \, .$$

We will later specify ϵ_s to mesh nicely with our other estimates.

Next take $h = w_j$, where w_j is the j^{th} coordinate function on Ω_2. We conclude now that

$$\|U \cdot \big(\Phi^{-1}\big)_j\|_{s,2} \le C''(1+t)^s \big(1 + \epsilon_s \cdot \|\Phi\|_{m(s)+2N+n,2} + \epsilon_s^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi\|_{0,2}\big) \, . \tag{6.7}$$

Fix a point $z \in \Omega_1$. The fact that $\Phi : \Omega_1 \to \Omega_2$ is Lipschitz of order greater than $(n-1)/n$ tells us that

$$|\nabla \Phi(z)| \le C \cdot \delta_1(z)^{-1/n+\epsilon}$$

for some $\epsilon > 0$. Hence the complex Jacobian determinant u of Φ at z is bounded by $d_1^{-1+\epsilon'}(z)$ for some $\epsilon' > 0$. We know from results of Range [16], proved with a direct application of Hopf's lemma, that $d_1(\Phi^{-1}(w))^m \le C \cdot d_2(w)$ for some positive integer m. But in fact the bi-Lipschitz condition of order exceeding $(n-1)/n$ guarantees that m must be 1.

It follows then that U must be of size at least $d_2^{1-\epsilon'}$ if it vanishes at some point of $\partial\Omega_2$ (it cannot vanish in the interior). That contradicts the smoothness of U to the boundary. We conclude then that U cannot vanish. Hence it is bounded from 0 in modulus. So we may see from (6.7) that

$$\begin{aligned} \|\big(\Phi^{-1}\big)_j\|_{s,2} \le C''(1+t)^s \Bigg(& \left[\frac{\epsilon_s}{(1+2t)^{N+n/2}} \right]^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi\|_{0,2} \\ & + \frac{\epsilon_s}{(1+2t)^{N+n/2}} \cdot \|\Phi\|_{2s} \Bigg) \, . \end{aligned} \tag{6.8}$$

Of course a similar argument may be applied with Φ^{-1} replace by Φ and the roles of Ω_1 and Ω_2 reversed to see that

$$\|\big(\Phi\big)_j\|_{s,2} \le C''(1+t)^s \Bigg(\bigg[\frac{\epsilon_s}{(1+2t)^{N+n/2}} \bigg]^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi^{-1}\|_{0,2} + \frac{\epsilon_s}{(1+2t)^{N+n/2}} \cdot \|\Phi^{-1}\|_{2s} \Bigg) . \tag{6.9}$$

Now let $\lambda_\ell = 10^{-\ell}$.

In inequality (6.8), replace s by ℓ and multiply through by λ_ℓ. Likewise, in inequality (6.9), replace s by ℓ and multiply through by λ_ℓ. Now sum over ℓ. The result is

$$\begin{aligned} &\sum_\ell \Big[\lambda_\ell \|\big(\Phi^{-1}\big)_j\|_{\ell,2} + \lambda_\ell \|\big(\Phi\big)_j\|_{\ell,2} \Big] \\ &\quad \le \quad \sum_\ell \Big[C'' \Big(1 + \epsilon_\ell^{-0} (\cdot \|\Phi\|_{0,2} + \epsilon_\ell^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi^{-1}\|_{0,2} \\ &\qquad\qquad + \epsilon_\ell \|\Phi^{-1}\|_{2\ell,2} + \epsilon_\ell \|\Phi\|_{2\ell,2} \Big) \lambda_\ell \Big] . \end{aligned}$$

What is nice about this inequality is that we can now absorb the two ϵ_ℓ terms on the righthand side into the lefthand side. In order to do this, we must note that the term $\| \quad \|_{2\ell,2}$ on the righthand side has coefficient $\epsilon_\ell \lambda_\ell$ while the same term on the lefthand side has coefficient $\lambda_{2\ell}$. So we must choose ϵ_ℓ so that $\epsilon_\ell \lambda_\ell \le (1/2)\lambda_{2\ell}$. Clearly $\epsilon_\ell = (1/2)10^{-\ell}$ will do the job.

The result is that

$$\begin{aligned} &\sum_\ell \Big[\lambda_\ell \|\big(\Phi^{-1}\big)_j\|_{\ell,2} + \lambda_\ell \|\big(\Phi\big)_j\|_{\ell,2} \Big] \\ &\quad \le \quad \sum_\ell C''' \lambda_\ell \cdot \big(\epsilon_\ell^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi\|_{0,2} + \epsilon_\ell^{-(m(s)+2N+n/2)/(n/2)} \cdot \|\Phi^{-1}\|_{0,2} \big) . \end{aligned}$$

We may conclude from this last inequality that $\|\big(\Phi^{-1}\big))j\|_{j,2}$ and $\|\big(\Phi\big))j\|_{j,2}$ are bounded. Thus the bihlomorphic mapping extends to a diffeomorphism of the closures. That is our theorem. $\square$

We remark that, if we strengthen the hypotheses of our theorem to Φ and Φ^{-1} both being Lipschitz 1, then it is immediate that u and U are bounded and the proof simplifies notably. Having a Lipschitz condition of order less than 1 makes things a bit trickier.

7. Concluding Remarks

It would be natural to suppose that a theorem like the one that we prove here is actually valid with only the assumption that Φ and Φ^{-1} are Lipschitz of order ϵ for some $\epsilon > 0$. The methods that we have do not suffice to establish such a result.

We repeat that, of course, the hope is that no Lipschitz hypothesis should be needed. The conclusion should be true all the time. That question will be a topic for future research.

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