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The Interconnection Between Stability, D-Stability, μ-Values With Applications to Linear Systems

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Abstract. This study looks at how D-stability relates to the structured singular value for certain types of structured matrices, which helps us to understand dynamic systems influenced by structured and unstructured uncertainties. We share new results on D-stability, ensuring that eigenvalues remain in a specific area of the complex plane $\mathbb{C}$ even with permitted changes. The calculations of a singular value and a structured singular value are important measures to evaluate how well a system can handle changes, perform, and remain stable when faced with certain types of uncertainty. We establish a theoretical connection between these concepts by characterizing D stable regions within the μ-analysis framework. Numerical tests support our findings, showing how singular values and structured singular values behave with our method compared to traditional techniques.

1. Introduction

Matrix proximity challenges involve developing computational approaches and numerical algorithms to find a matrix not only **near** to a given matrix, but must possess certain properties as per the requirement of the problem. In matrix proximity challenges, the aim is to reduce the gap using appropriate matrix norms. The common choice is of $\|\cdot\|_2$ and $\|\cdot\|_F$, where $\|\cdot\|$ denotes the matrix-norm, subscripts 2 represent the matrix 2-norm, and F represent the matrix Frobenius norm.

When considering a specific characteristic P for matrix M that lacks this property, the matrix proximity challenge is to find another suitable matrix $\hat{M}$ nearest to M. The matrix B holds the

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property P, and also the approximated distance between matrices M and $\hat{M}$. In such a case, the matrix proximity challenge mathematically may be expressed as

$$min||M - X|| \; s.t \; X \in \hat{P},$$

with $\hat{P}$ represents a collection of matrices that satisfy the property P.

matrix proximity challenges can be extended to matrix polynomials [1]. In these cases, the goal is to consider $M(\alpha)$, expressed as $M(\alpha) = \sum_{i=0}^{d} M_i\alpha^i$, which is a polynomial with matrix coefficients. The aim is then to identify another matrix polynomial $\hat{M}(\alpha) = \sum_{i=0}^{d} \hat{M}_i\alpha^i$, and subsequently investigate and evaluate the associated minimization problem

$$min \sum_{i=0}^{d} ||M_i - \hat{M}_i||^2.$$

In [2], the authors proposed a solution to the low-rank matrix completion problem, where the objective is to reconstruct a low-rank matrix based on partially observed entries. A low-rank completion problem with the help of Riemannian optimization and using the Grassmann manifold was addressed in [3]. An augmented Lagrangian algorithm was developed by Borsdrof [4] to study and analyze matrix proximity challenges. The structured total least norm algorithms [5] were developed to deal with a specific class of matrix proximity challenges.

In his work on calculating the nearest correlation matrix, Dykstra [6] introduced an alternating projection approach, which was then proposed in [7]. A dual approach to study and analyze matrix proximity challenges was developed in [8,9]. We further refer to see [10–15] for a more and comprehensive details and efficient techniques to study and solve the matrix proximity challenges.

An analytical method to solve and analyze matrix proximity challenges was presented in [16]. The more generic analytical method was developed [17] to solve matrix proximity challenges with the characterization in term of best approximation of matrix elements. For the computation of a nearest distance matrix for a given arbitrary distance matrix an analytical method was presented in [18]. A bisection method in [19] was developed to study and analyze matrix proximity challenges.

In our current study, we analyze the matrix proximity challenge. for a given n-dimensional real-valued matrix, we mainly aim to evaluate the nearest correlation matrix such that having a unit diagonal, and non-negative eigenvalues. The suggested approach relies on a collection of methods from mathematical optimization and linear algebra to determine the necessary nearest correlation matrix. We further to the analysis on the spectral properties of dynamical systems under consideration. The idea presented in this paper is to compute nearest correlation matrix while maximizing the negative eigenvalues in such a way that all eigenvalues become non-negative. This is achieved by solving and analyzing the gradient system of ordinary differential equations obtained from mathematical optimization.

In section titled with New results on stability and D-stability, we consider that the dynamical system under consideration possesses a coefficient matrix which is a nearest correlation matrix. This work introduces novel theoretical and computational insights into the relationships among stability, structured singular values, and D-stabilityin dynamical systems. Leveraging tools from matrix analysis and system theory, we derive conditions for stability by optimizing the eigenvalues of perturbed structured matrices. Additionally, we integrate D-stability analysis with structured singular value computations, ensuring that all singular values remain within the interval $(\infty, 1]$ for robust performance.

According to the literature, a n-dimensional matrix $M \in \mathbb{R}^{n,n}$ is stable or positive stable if $Re(\lambda_i(M)) > 0$, $\forall i = 1 : n$. An n-dimensional real-valued matrix M is regarded as a D-stable matrix for a positive diagonal matrix $D = \text{diag}(p_i) > 0$, for all i, if $\text{Re}(\lambda_i(DM)) > 0$ for all $i = 1 : n$. In the classical works, [20, 21] proposed the idea of D-stability. The notion of D-stability has numerous applications across various fields, including economics, engineering, and mathematical modeling, see [22–28, 33].

The verification of D-stability is a hard problem. For $n = 3$, a three dimensional matrix, a complete description to the verification of D-stability was presented in [30]. For $n = 4$, the theoretical results D-stability were proved and presented in [31]. In [32], several necessary conditions for D-stability and a specific class of structured matrices that are D-stable were provided, applicable to matrices with dimension $n > 4$. But, unfortunately the characterization of D-stability for structured matrices in higher dimensions is an open problem. The classical and theoretical results on the verification of D-stable matrices are provided in [22–29].

In control system theory, a mathematical tool that was initially presented in [34], the idea of approximating the single structured value and its upper and lower bounds is well-established. The primary benefit of computing structured singular values lies in its ability to tackle fundamental problems related to the analysis of linear time-invariant systems. From an application perspective, it facilitates the discussion of robustness, performance, stability, and instability in dynamical systems. Structured singular values are calculated, considering a set of block-diagonal uncertainty. The set of block-diagonal uncertainties covers a number of repeated real, repeated complex, number of full block, and mixture of both real and complex uncertainties. For more details on examples and applications of structured singular values and its lower and upper bounds, we refer [35–42].

The analytical approach to computing the structured singular value is highly complex, and determining its exact value is known to be an NP-hard problem, as discussed in [39]. Due to this computational challenge, significant efforts have been directed toward developing numerical methods that provide upper and lower approximations of the structured singular value. For instance, MATLAB's **mussv** function estimates upper bounds using techniques such as diagonal balancing and Linear Matrix Inequalities (LMIs), as detailed in [43, 44].

A notable study in [45] establishes sufficient conditions for a real-valued matrix to be classified as D-stable. Further research in [45,46] demonstrates that verifying D-stability for higher-dimensional matrices poses significant mathematical difficulties. However, theoretical analysis confirms that such results are more tractable in lower dimensions, particularly in two or three-dimensional spaces.

The interplay between D-stable matrices and structured singular values has been extensively explored in [47,48]. Additionally, [49] presents further insights into the connections among D-stable matrices, strongly D-stable matrices, and structured singular values, specifically for complex-valued matrices in n-dimensional spaces.

The robust stability of Boolean networks (BNs) for two kinds of EPs, that is, edge removal and edge sign switch is presented in [59] while first converting them to a multibit function perturbations (FPs). Then a parameterized reachability matrix is formulated for the verification of the robust stability. A comprehensive study on the compositional method to alleviate the high computational cost of large-scale Boolean networks was presented and developed in [60].

In [57], a systems of linear equations with fuzzy coefficients and variables based approach is developed to address the modeling of multiproduct supply and demand equilibrium under uncertainty. The proposed methodology consists upon the reformulation of an equilibrium conditions with fuzzy arithmetic and examine the existence and nature of fuzzy solutions. In [58], linear algebra for economic geography models was proposed, and mainly a sufficient statistics was developed to nominal as well as the real wage exposure to productivity shocks in a constant elasticity economic geography model. The exposure measures summarize the first order general equilibrium elasticity in each location with respect to the productivity shocks for all possible locations.

Article Overview. Section 2 of this paper presents a comprehensive review of matrix proximity challenges, where the goal is to minimize the distance between given matrices under specific norms, such as the spectral norm (2-norm) and the Frobenius norm.

In Section 3, we introduce a novel approach for computing the closest correlation matrix using eigenvalue optimization techniques. The solution to this optimization problem is formulated as a system of ordinary differential equations (ODEs), which are then solved numerically via Euler's method to analyze spectral behavior.

Section 4 develops new theoretical results on stability analysis and D-stability in dynamical systems. Our methodology integrates advanced tools from linear algebra, mathematical optimization, and system theory to establish rigorous analytical guarantees.

Finally, in Section 5 numerical experiments conducted to examine the properties of singular values and structured singular values for structured matrices, along with their practical implications.

2. Matrix Proximity Challenges

In matrix proximity challenges one aims to find an arbitrary matrix (say) $M \in \mathbb{R}^{m,n}$ which is nearest candidate to class of structured matrices, the distance between matrices is measured by a matrix norm. Consider $\hat{S}$ denotes $\mathbb{R}^{m,n}$, a matrix norm on $\hat{S}$ is represented by $\|\cdot\|$. Assume that P denotes a matrix property, then a distance function d for matrix M is given by

$$d(M) = min\{\|E\| : M + E \in \hat{S} \ has \ property \ P\}.$$

For $d(M)$, in the matrix proximity challenge we aim to:
(1) Find an explicit formula for $d(M)$.
(2) Find $X = M + M_{min}$, with E_{min} is such that minimum in $d(M)$ is attained. Then, the important question to answer is either X is a unique matrix.
(3) Write algorithms to estimate $d(M)$ and X.

Remark 2.1. *Matrix norm selection in $d(m)$ ensures computational feasibility in proximity analysis.*

In matrix proximity challenges, usually matrix 2-norm and matrix Frobenius norm are used to study and analyze the problem. The matrix 2-norm for $M \in \mathbb{R}^{m,n}$ is

$$\|M\|_2 = \rho(M^T M)^{\frac{1}{2}},$$

with the spectral radius of a matrix denoted by $\rho(\cdot)$.

The matrix Frobenius norm of $M \in \mathbb{R}^{m,n}$ is defined as

$$\|M\|_F = \left(\sum_{i,j} |m_{ij}|^2\right)^{\frac{1}{2}} = Trace(M^T M)^{\frac{1}{2}}.$$

Remark 2.2. *These matrix norms ($\|\cdot\|_2$, $\|\cdot\|_F$) maintain unitary invariance properties"*

As it well-known that for a given $M \in \mathbb{R}^{m,n}$,

$$\|M\|_2 \leq \|M\|_F,$$

and this further implies that $d_2(M) \leq d_F(M)$, the distance measured in matrix 2-norm, and matrix Frobenius norm. The use of $\|\cdot\|_1$ and $\|\cdot\|_\infty$ in $d(M)$ leads to intractable matrix proximity challenges.

Remark 2.3. *For rank-1 matrices M, the relationship $\|M\|_2 = \|M\|_F$ holds. Further, if E_{min} has a rank-1, then $d_2(M) = d_F(M)$.*

Many of the matrix proximity challenges solutions can be respected in term of decomposition of matrices. For instance:
(1) **Symmetric/Skew-Symmetric Decomposition:** Any matrix $M \in \mathbb{R}^{n,n}$ can be expressed as the sum of a symmetric and a skew-symmetric matrix:

$$M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T).$$

(2) **Polar Decomposition:** For any matrix $M \in \mathbb{C}^{m,n}$ where $m \geq n$, it can be taken into consideration as

$$M = UH,$$

where $U \in \mathbb{C}^{m,n}$ is a matrix with orthonormal columns, and $H \in \mathbb{C}^{n,n}$ is a unique Hermitian positive semidefinite matrix.

(3) **Singular Value Decomposition (SVD):** Any matrix $M \in \mathbb{R}^{m,n}$ with $m \geq n$ can be decomposed as

$$M = U\Sigma V^T,$$

where $U \in \mathbb{C}^{m,m}$ and $V \in \mathbb{C}^{n,n}$ are unitary matrices, and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, with $\sigma_i \geq 0$ for all i. In applications mostly real-valued matrices are studied. In such case $\hat{S}$ contains $\mathbb{R}^{m,n}$, and following problem is the matrix proximity challenge

Given $M \in \mathbb{R}^{n,n}$, the distance function is denoted by $d(M)$ and is defined as

$$d(M) = min\{\|E\| : M + E \in \mathbb{R}^{n,n} \textit{ is symmetric}\}.$$

The above matrix proximity challenge for given $M \in \mathbb{R}^{n,n}$ was solved in [50], and they obtain that

$$d(M) = \|M_k\| = \frac{1}{2}\|M - M^T\|, \;\; X = M_H = \frac{1}{2}\|M + M^T\|.$$

For a given $M \in \mathbb{R}^{n,n}$ let $\delta(M) = \|M - X\|$ with X, a positive semi-definite matrix. The following approximation problem was given by Highmam [51].

Theorem 2.1. *Let $M \in \mathbb{R}^{n,n}$, and consider that $M_H = UH$. Then $X_F = (M_H + H)/2$ is unique approximate to M, and*

$$\delta_F(M)^2 = \sum\nolimits_{\lambda_i < 0} \lambda_i(M_H)^2 + \|M_k\|_F^2.$$

The challenge of locating nearest matrix having orthonormal columns to given $M \in \mathbb{R}^{m,n}$, is an important problem in matrix proximity challenge. In such a problem, the distance function for M is

$$d(M) = min\{\| E \|: E \in \mathbb{R}^{m,n}, \; (M + E)^T(M + E) = I\}.$$

The following minimization problem for given $M, B \in \mathbb{R}^{n,n}$

$$min\{\| M - BQ \|_F: Q \in \mathbb{R}^{n,n}, \; Q^TQ = I\}$$

is an orthogonal procrustes problem.

Remark 2.4. *The solution to above matrix proximity challenge are given by Theorem* 4.1 *[52].*

For a matrix $M \in \mathbb{C}^{n,n}$, the problem of its proximity to normality in relation to the Frobenius norm was addressed in [53]. The matrix M can be expressed as the sum of three components:

$$M = D + H + S,$$

where $D = \text{diag}(m_{kk})$, with $h_{kk} = 0$ and $s_{kk} = 0$ for the off-diagonal entries of H and S, respectively.

$$h_{jk} = \begin{cases} (m_{jk} + exp(2i\theta_{jk})\bar{m}_{kj})/2, & m_{jj} \neq m_{kk}, \; (j \neq k) \\ m_{jk}, & m_{jj} = m_{kk}, \; (j \neq k) \end{cases}$$

$$s_{jk} = \begin{cases} (m_{jk} - exp(2i\theta_{jk})\bar{m}_{kj})/2, & m_{jj} \neq m_{kk}, \; (j \neq k) \\ 0, & m_{jj} = m_{kk}, \; (j \neq k), \end{cases}$$

here $\theta_{jk} = arg(m_{kk} - m_{jj})$.

The following theorem [53, 54] is on nearness normal problem.

Theorem 2.2. *Let $M \in \mathbb{C}^{m,n}$, and let X denote its closest normal matrix in the Frobenius norm. If $X = ZDZ^*$, then the matrix Z^*MZ is a $\triangle H$-matrix, with $D = diag(Z^*MZ)$.*

Remark 2.5. *In matrix decomposition $M = D + H + S$, if $H = 0$, then decomposition of M is called a $\triangle H-$matrix.*

Following theorem by Cauchy 1964, Gabriel 1979, gives further insights to matrix nearness normal problem.

Theorem 2.3. *Let $M \in \mathbb{C}^{n,n}$ and consider a decomposition of X such that $X = ZDZ^*$, where Z is a unitary matrix and D is a diagonal matrix. The matrix X is the nearest normal matrix to M in the Frobenius norm if and only if*

$$\|diag(Z^*MZ)\|_F = \max \|diag(Q^*MQ)\|_F,$$

*where the **maximum** is taken over all unitary matrices Q such that $Q^*Q = I$, and*

$$D = diag(Z^*MZ).$$

For a given matrix $M \in \mathbb{C}^{n,n}$, such that λ be its eigenvalue, then it is not much hard to determine nearest matrix to M. The nearness problem can be transformed to problem

$$\min\left\{\frac{\|E\|}{\|M\|} : det(M + E) = 0\right\}.$$

This is possible because λ is from spectrum of $A + E$ and hence $det(M - \lambda I) + E) = 0$.
The challenge of locating a nearest matrix in case of repeated eigenvalue is much more complicated, see [55, 56].

3. Computing Non-Negative Spectrum

In this section, we aim to determine the behavior of smallest eigenvalue $\lambda_1^{\min}$ for a given matrix $M \in \mathbb{R}^{n\times n}$. We construct a structured matrix E such that $\text{diag}(E) = 0$, and further

$$\|E\|_F = \sqrt{\sum_{i,j} e_{ij}^2} \leq 1.$$

We omit for sake of bravery the dependency of E on t, that is, $E = E(t)$ for all $t \in \mathbb{R}^+$. In order to investigate the behavior of $\lambda_1^{\min}$, it is essential to examine the spectrum of the perturbed matrix $(M + \alpha E(t))$, where $\alpha > 0$ is a positive parameter. By defining $E = E(t)$, with $t \in \mathbb{R}^+$ and ensuring that $\text{diag}(E) = 0$, we are able to study the behavior of the smallest eigenvalue $\lambda_1^{\min}$. Further, it demands to determine a direction $Z = \frac{d}{dt}E(t), t \in \mathbb{R}^+$, so that one can easily analyze how fast or slow $\lambda_1^{\min}$ grows. We study the eigenvalue problem for a given $M \in \mathbb{R}^{n\times n}$:

$$(M + \alpha E(t))v(t) = \lambda^{\min}(t)v(t),$$

with $\lambda^{\min}$, the smallest eigenvalue, and $v(t)$ being the corresponding eigenvector. Here, we assume $v(t)$ such that $\|v(t)\|_2 \leq 1,\ \forall t \in \mathbb{R}^+$.

Assumption 3.1. *For computational tractability and relevance to economic applications, we restrict our analysis to real-valued eigenvectors $v(t) \in \mathbb{R}^n$ for all $t \in \mathbb{R}^+$. This assumption is justified in practical scenarios where:*

(1) *The coefficient matrices arise from real economic data,*
(2) *The dominant eigenvalues determining system behavior are real, and*
(3) *The complex eigenvalues typically appear in conjugate pairs with real parts determining stability.*

This restriction does not affect the generality of our stability conclusions.

The eigenvalue problem for $M \in \mathbb{R}^{n\times n}$ presented above can be stated in the manner described below:

$$v^T(t)\,(M+\alpha E(t)) = \lambda^{\min}(t)v^T(t).$$

The time-derivative yields:

$$(M+\alpha E(t))\,\frac{d}{dt}\Big(v(t)\Big)+\alpha\frac{d}{dt}\Big(E(t)\Big)v(t) = \frac{d}{dt}\Big(\lambda^{\min}(t)\Big)v(t)+\lambda^{\min}(t)\frac{d}{dt}\Big(v(t)\Big).$$

This further takes the form,

$$v^T(t)\,(M+\alpha E(t))\,\frac{d}{dt}\Big(v(t)\Big)+\alpha v^T(t)\frac{d}{dt}\Big(E(t)\Big)v(t)$$
$$=$$
$$\frac{d}{dt}\Big(v^T(t)v(t)\Big)+\lambda^{\min}(t)v^T(t)v(t)+\lambda^{\min}(t)v^T(t)\frac{d}{dt}\Big(v(t)\Big).$$

Since, $v^T(t)v(t) = \langle v(t), v(t)\rangle = \|v(t)\|_2^2 = 1$, thus the above expression reduces to

$$v^T(t)\,(M+\alpha E(t))\,\frac{d}{dt}\Big(v(t)\Big)+\alpha v^T(t)\frac{d}{dt}\Big(E(t)\Big)v(t) = \frac{d}{dt}\Big(\lambda^{\min}(t)\Big)+\lambda^{\min}(t)v^T(t)+\frac{d}{dt}\Big(v(t)\Big).$$

This yields,

$$\lambda^{\min}(t)v^T(t)\frac{d}{dt}\Big(v(t)\Big)+\alpha v^T(t)\frac{d}{dt}\Big(E(t)\Big)v(t) = \frac{d}{dt}\Big(\lambda^{\min}(t)\Big)+\lambda^{\min}(t)v^T(t)\frac{d}{dt}\Big(v(t)\Big). \tag{1}$$

By utilizing the fact that $v^T(t)\,(M+\alpha E(t))\,v(t) = \lambda^{\min}(t)v^T(t)v(t)$, we have:

$$\lambda^{\min}(t)v^T(t)v(t) = v^T(t)\,(M+\alpha E(t))\,.$$

Thus, finally we have that:

$$\frac{d}{dt}\Big(\lambda^{\min}(t)\Big) = \alpha v^T(t)\frac{d}{dt}\Big(E(t)\Big)v(t).$$

By taking $v^T(t)\frac{d}{dt}\Big(v(t)\Big) \to$ into Equation (1), and $z = \frac{d}{dt}\Big(E(t)\Big)$, we obtain the following maximization problem.

3.1. **Maximization Problem.** The resolution of the following maximization problem identifies a direction $Z = \frac{d}{dt}(E(t))$, such that the solution to the system of ordinary differential equations reflects the growth of $\lambda_1^{\min}(t)$. The maximization problem is given by

$$\max\Big(v_1^T Z v_1\Big)$$
$$\text{Subject to}$$
$$\langle Z, E(t)\rangle \geq 0$$
$$\text{diag}(E) = 0,$$

where $v_1 \in \mathbb{R}^{n,1}$ represents an eigenvector related to the smallest eigenvalue $\lambda_1^{\min}(t)$. The resolution to the aforementioned maximization problem is provided by the following Lemma.

Lemma 3.1. *For $t \in \mathbb{R}$, the perturbation matrix $E(t)$ is such that*

$$\|E(t)\|_F \leq \sqrt{\sum_{i,j} e_{ij}^2} \leq 1.$$

Consider that $v_1(t)$ are the non-zero eigenvectors corresponding to $\lambda_1^{\min}(t)$. Therefore, the solution to the optimization problem can be expressed as

$$Z = \frac{d}{dt}\Big(E(t)\Big) = P_r\Big(v_2(t)v_2^T(t)\Big) - \langle P_r\Big(v_2(t)v_2^T(t)\Big), E(t)\rangle E(t),$$

where $P_r(\cdot)$ denotes the projection of Z onto the manifold of matrices $E(t), t \in \mathbb{R}^+$.

Definition 3.1. *The projection operator $P_r(\cdot) : \mathbb{R}^{n\times n} \to \mathcal{M}_0$ maps any matrix onto the manifold $\mathcal{M}_0$ of matrices with zero diagonal elements. Specifically, for any matrix $A \in \mathbb{R}^{n\times n}$,*

$$Pr(A) = A - diag(diag(A))$$

where $diag(A)$ extracts the diagonal elements of A, and $diag(diag(A))$ creates a diagonal matrix from these elements. This operator ensures that the perturbation matrix $E(t)$ maintains the constraint $diag(E(t)) = 0$ throughout the optimization process.

3.2. **The System of Ordinary Differential Equations.** The structured matrix Z with structure:

$$Z = P_r\Big(v_1(t)v_1^T(t)\Big) - \langle P_r\Big(v_1(t)v_1^T(t)\Big), E(t)\rangle E(t).$$

is the solution matrix to the maximization problem. The system of ordinary differential equations is $\frac{d}{dt}(E(t)) = E(t)$ given by

$$\frac{d}{dt}\Big(E(t)\Big) = P_r\left(v_1(t)v_1^T(t)\right) - \left\langle P_r\left(v_1(t)v_1^T(t)\right), E(t)\right\rangle.$$

The mathematical solution of the above system of ordinary differential equations has the following properties, that is,

(P_1) $\frac{d}{dt}(\lambda_1^{\min}(E(t))) > 0$, that is, the smallest eigenvalue $\lambda_1^{\min}(E(t))$ has a monotonically increasing behavior.

(P_2) $\frac{d}{dt}(E(t)) = 0$ iff $\frac{d}{dt}(\lambda_1^{\min}(t)) = 0$.

(P_3) $\frac{d}{dt}(\lambda_1^{\min}(t)) = 0$ iff $E(t) \propto P_r\Big(v_1(t)v_1^T(t)\Big)$.

Next, we analyze the behavior of $\lambda_1^{\min}$, $\lambda_2^{\min}$. Assume $\lambda_2^{\min}$ is a negative eigenvalue, but we aim to have that

$$\lambda_1^{\min} > 0, \ \lambda_2^{\min} > 0.$$

For this purpose, we need to determine a matrix $Z = \frac{d}{dt}(E(t))$, $t \in \mathbb{R}^+$ such that the solution to the system of ordinary differential equations shows a maximum growth of $\lambda_1^{\min}$, $\lambda_2^{\min}$. the optimization problem which gives a maximum increase in $\lambda_1^{\min}$ and $\lambda_2^{\min}$ is given by

$$\begin{aligned}
&\max\left(v_1^T Z v_1\right)\\
&\text{Subject to}\\
&v_2^T Z v_2 = v_1^T Z v_1\\
&\langle Z, E(t)\rangle \geq 0\\
&\operatorname{diag}(E) = 0.
\end{aligned}$$

The solution to the maximization problem is given as

$$Z = \frac{d}{dt}E(t) = (1-\mu)v_1 v_1^T - \mu v_2 v_2^T - \left\langle v_1 v_1^T - v_2 v_2^T, E(t)\right\rangle - \left\langle v_1^T v_1, E(t)\right\rangle.$$

The perturbed matrix $(M + \alpha E(t))$ is an upper triangular matrix. The solution to the maximization problem can also be rewritten as

$$Z = \frac{d}{dt}E(t) = (1-\mu)P_r(v_1 v_1^T) + \mu P_r(v_2 v_2^T) - \beta E(t).$$

Remark 3.1. *The solution $Z = \frac{d}{dt}(E(t))$ can be determined using Euler's method, that is,*

$$E_{n+1} = E_n + h\frac{d}{dt}(E_n).$$

The eigenvalue equation

$$A = (M + \alpha E(t))v(t) = \lambda^{\min}(t)v(t),$$

explains the non-negative eigenvalues associated with the perturbed system. The matrix A is positive semi-definite and is given by

$$A = M + \alpha E(t).$$

4. New Results on Stability and D-Stability

This section presents novel results on stability analysis and D-stability in dynamical systems. Our theoretical investigations demonstrate that a system's stability can be verified by computing eigenvalues satisfying $\operatorname{Re}(\lambda_i(\cdot)) > 0$ for all i.

Building on this foundation, we examine D-stability through the spectral analysis of matrix products. For a given matrix M and an arbitrary positive diagonal matrix D, we establish stability conditions by ensuring $\operatorname{Re}(\lambda_i(DM)) > 0$ for all eigenvalues. This approach provides a complete characterization of D-stability for the dynamical systems under investigation.

The following Theorem 4.1 demonstrates that the dynamical system with the coefficient matrix

$$(M + \alpha E(t))x(t), \quad x(t) \in \mathbb{R}^n$$

is stable, meaning all of its eigenvalues have strictly positive real parts.

Theorem 4.1. *The dynamical system*

$$\frac{dx}{dt} = (M + \alpha E(t))x(t), \quad x(t) \in \mathbb{R}^n$$

is stable if

$$Re\left[\lambda(M + \alpha E(t))\right] > 0, \quad \forall t \in \mathbb{R}^+.$$

Proof. The dynamical system is stable if all eigenvalues of $(M + \alpha E(t))$ have positive real parts. By the construction in Section 3, the perturbation matrix $E(t)$ is designed to maximize the smallest eigenvalue $\lambda_{\min}(t)$, ensuring that $\lambda_{\min}(t) > 0$.

Since $(M + \alpha E(t))$ is Hermitian by construction, all eigenvalues are real, and the condition $\mathrm{Re}(\lambda_i(M + \alpha E(t))) > 0$ is equivalent to $\lambda_i(M + \alpha E(t)) > 0$ for all i.

For any non-zero vector x, the quadratic form

$$x^T(M + \alpha E(t))x = \sum_{i=1}^{n} \lambda_i |\langle x, v_i \rangle|^2 > 0$$

where v_i are the eigenvectors, confirming positive definiteness and hence stability. □ □

Lemma 4.1 show that real part of each eigenvalue of perturbed matrix $(M + \alpha E(t))$ is strictly positive for all $t \in \mathbb{R}^+$.

Lemma 4.1. *The eigenvalue $\lambda \in \sigma(M + \alpha E(t))$, where $\sigma(\cdot)$ denotes the spectrum of $M + \alpha E(t)$. Then*

$$\mathrm{Re}(\lambda(M + \alpha E(t))) > 0, \quad \forall t \in \mathbb{R}^+.$$

Proof. The eigen-pair (λ, x) exists for $M + \alpha E(t)$. Then, we have that

$$x^T(M + \alpha E(t))x = x^T \lambda x. \tag{4.1}$$

This implies that

$$\mathrm{Re}(\lambda) = \frac{x^T(M + \alpha E(t))x}{x^T x} > 0, \tag{4.2}$$

By the optimization procedure described in Section 3, the perturbation $E(t)$ is constructed to ensure that the smallest eigenvalue $\lambda_{\min}(M + \alpha E(t))$ becomes positive. Since the optimization process in Equation (7) maximizes $\lambda_{\min}(t)$, and the algorithm terminates when $\lambda_{\min}(t) > \delta > 0$ for some tolerance δ, we have established that $M + \alpha E(t)$ has all positive eigenvalues.

Therefore, for any eigenvalue $\lambda \in \sigma(M + \alpha E(t))$ and corresponding normalized eigenvector x, we have

$$x^T(M + \alpha E(t))x = \lambda \|x\|^2 = \lambda > 0$$

which implies $\mathrm{Re}(\lambda) > 0$. □ □

Theorem 4.2 demonstrates that a dynamical system is stable in the event that a matrix B exists whose rank equals the dimension of the given matrix M.

Theorem 4.2. *The dynamical system*

$$\frac{dx}{dt} = (M + \alpha E(t))x, \quad x \in \mathbb{R}^{n,1}$$

is stable if $B^T(M + \alpha E(t))B$ *is positive definite with* $\mathrm{rank}(B) = n$*, where* n *is the dimension of* $(M + \alpha E(t))$.

Proof. Assume that $\lambda_i(M + \alpha E(t)) \neq 0, \forall i$. Then

$$\mathrm{rank}(B) = \mathrm{rank}((M + \alpha E(t))B) = \mathrm{rank}(B^T(M + \alpha E(t))B).$$

Since, we know that $(M + \alpha E(t))$ is a non-singular matrix, this suggests that the rank of $B^T(M + \alpha E(t))B$ is equal to n, which is exactly the rank of B. □

The following Theorem 4.3 show that dynamical system is stable if the absolute value of eigenvalue of perturbed matrix is its spectral radius.

Theorem 4.3. *Let* $(M + \alpha E(t))$ *such that* $\mathrm{Re}(\lambda(M + \alpha E(t))) > 0$*. Consider* (λ, x) *to be an eigen-pair of* $(M + \alpha E(t))$ *and* $|\mathrm{Re}(\lambda)| = \rho(M + \alpha E(t))$*. Then,*

$$|x| > 0 \quad \textit{and} \quad (M + \alpha E(t))|x| = \rho(M + \alpha E(t))|x|.$$

Proof. For given $(M + \alpha E(t))$, we have $z = (M + \alpha E(t))|x|$. This implies that

$$z = (M + \alpha E(t))|x| \geq |(M + \alpha E(t))x| = |\mathrm{Re}(\lambda)|||x|.$$

From above, it further follows that

$$z = |\mathrm{Re}(\lambda)|||x| = \rho((M + \alpha E(t)))|x|.$$

Let

$$\hat{z} = z - \rho((M + \alpha E(t)))|x| \geq 0.$$

For $\hat{z} \geq 0$, we have that

$$\rho((M + \alpha E(t)))|x| = (M + \alpha E(t))|x| > 0.$$

This means that

$$\rho((M + \alpha E(t))) > 0, \quad \text{and} \quad |x| > 0.$$

Alternatively, if $\hat{z} \neq 0$, then

$$\begin{aligned} 0 < (M + \alpha E(t))\hat{z} &= (M + \alpha E(t))z - \rho(M + \alpha E(t))(M + \alpha E(t))|x| \\ &= (M + \alpha E(t))z - \rho(M + \alpha E(t))z. \end{aligned}$$

This ensures that

$$(M + \alpha E(t))z > \rho(M + \alpha E(t))z > \rho(M + \alpha E(t))z.$$

This further implies that

$$\rho(M + \alpha E(t)) > \rho(M + \alpha E(t)),$$

which is not possible, and hence $\hat{z} \neq 0$. □

The following theorem 4.4 show that for the dynamical system to be stable the perturbed matrix $(M+\alpha E(t))^m$ is a positive matrix for $m \geq 1$.

Theorem 4.4. *The dynamical system*

$$\frac{dx}{dt} = (M+\alpha E(t))x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+$$

is stable, that is, $\mathrm{Re}(\lambda_i(M+\alpha E(t))) > 0, \forall i$. *Then, for some* $m \geq 1$, $(M+\alpha E(t))^m$ *is positive.*

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the set of eigenvalues of $(M+\alpha E(t))$. Then $\{\lambda_i^m\}_{i=1}^n$ are the eigenvalues of $(M+\alpha E(t))^n$. Furthermore, we have that

$$\rho((M+\alpha E(t))^m) = (\rho(M+\alpha E(t)))^m,$$

and it is strictly positive. The rest of the eigenvalues have modulus which is strictly less than $\rho((M+\alpha E(t))^m)$. Thus $(n-1)$ number of eigenvalues are strictly less than $\rho(M+\alpha E(t))$. This ensures that $\rho((M+\alpha E(t))$ is the the remaining number of eigenvalues. □

The following Theorem 4.5 show that dynamical system under consideration is D-stable if the structured singular value of the perturbation matrix $(M+\alpha E(t))-2$ lie in $[0,1)$.

Theorem 4.5. *The dynamical system*

$$\frac{dx}{dt} = (M+\alpha E(t))x, \quad x \in \mathbb{R}^{n,\,1}, \quad t \in \mathbb{R}^+$$

is D-stable if

$$0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{(M+\alpha E(t))^2}\right) < 1.$$

Proof. The provided matrix $(M+\alpha E(t))$ is D-stable if $\mathrm{Re}(\lambda_i(M+\alpha E(t))) > 0$, $\forall i$ and

$$\prod_{i=1}^{n} \lambda_i((M+\alpha E(t))^2 + D^2) \neq 0, \ \forall i.$$

Here, $D = \mathrm{diag}(d_{11}, d_{22}, \ldots, d_{nn})$ such that $d_{ii} > 0$, $\forall i = 1, \ldots, n$. It is enough to show that

$$\prod_{i=1}^{n} \lambda_i((M+\alpha E(t))^2 + D^2) \neq 0, \quad \forall i.$$

Because in turn this will yield that

$$0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{(M+\alpha E(t))^2}\right) < 1.$$

As,

$$\prod_{i=1}^{n} \lambda_i\left((M+\alpha E(t))^2 + D^2\right) \neq 0 \implies \prod_{i=1}^{n} \lambda_i\left((M+\alpha E(t))^2\right) - D^2\,(M+\alpha E(t))^{-1}\,D\,(M+\alpha E(t)) \neq 0.$$

Furthermore, we have that

$$\prod_{i=1}^{n} \lambda_i((M+\alpha E(t))^{-2} - D^{-2}) \neq 0 \Rightarrow \prod_{i=1}^{n} \lambda_i\left(I_n - (M+\alpha E(t))^{-2}\hat{D}\right) \neq 0,$$

where $\hat{D} = D$, is a matrix of positive diagonal elements. Thus,

$$\prod_{i=1}^{n} \lambda_i \Big(I_n - (M + \alpha E(t))^{-2} \hat{D} \Big) \neq 0 \implies 0 \leq \mu_{\mathbb{B}_1} \left(\frac{1}{(M + \alpha E(t))^2} \right) < 1.$$

□

The following Theorem 4.6 show that the dynamical system under consideration is D-stable if real parts of all the eigenvalues of $(D(M + \alpha E(t)) + (M + \alpha E(t))D)$ are strictly positive and structured singular values of matrix B lie in in $[0, 1)$.

Theorem 4.6. *The dynamical system*

$$\frac{dx}{dt} = (M + \alpha E(t))x, \quad x \in \mathbb{R}^{n,\,1}, \quad t \in \mathbb{R}^+$$

is D-stable if $\mathrm{Re}(\lambda_i(D(M + \alpha E(t)) + (M + \alpha E(t))D)) > 0, \forall i$ *and* $0 \leq \mu_{\mathbb{B}_1}(B) < 1$, *where B is obtained as*

$$B = \Big(\boldsymbol{i} I_n + D(M + \alpha E(t)) + (M + \alpha E(t))^T D \Big)^{-1} \Big(\boldsymbol{i} I_n - D(M + \alpha E(t)) - (M + \alpha E(t))^T D \Big).$$

Proof. Our goal is to demonstrate that $(M + \alpha E(t))$ is a D-stable matrix for $t \in \mathbb{R}^+, \lambda > 0$ iff

$$\mathrm{Re}(\lambda_i(D(M + \alpha E(t)) + (M + \alpha E(t))^T D)) > 0, \ \forall i, \ \forall D,$$

and

$$0 \leq \mu_{\mathbb{B}_1}(B) < 1.$$

Consider that $\Delta \in \mathbb{B}_1$ with a block diagonal structure defined as

$$\Delta = (\mathbf{i} I_n - D)(\mathbf{i} I_n + D)^{-1}.$$

As

$$\mathrm{Re}(\lambda_i(D(M + \alpha E(t)) + (M + \alpha E(t))^T D)) \neq 0, \forall i, \forall D.$$

This further yields that

$$\mathrm{Re}(\lambda_i(D(M + \alpha E(t)) + (M + \alpha E(t))^T D + \mathbf{i} D)) \neq 0, \forall i.$$

$$\Longleftrightarrow$$

$$\mathrm{Re}(\lambda_i(D(M + \alpha E(t)) + (M + \alpha E(t))^T D + (\mathbf{i} I_n + \Delta)^{-1}(\mathbf{i} I_n - \Delta))) \neq 0, \forall i, \forall \Delta \in \mathbb{B}_1.$$

This further reduces to inequality

$$\mathrm{Re}(\lambda_i(\mathbf{i} I_n + D(M + \alpha E(t)) + (M + \alpha E(t))^T D) - (\mathbf{i} I_n - D(M + \alpha E(t)) - (M + \alpha E(t))^T D)) \neq 0, \forall i, \forall \Delta \in \mathbb{B}_1.$$

Thus, finally we have that

$$\mathrm{Re}(\lambda_i(\mathbf{i} I_n + D(M + \alpha E(t)) + (M + \alpha E(t))^T D)(\mathbf{i} I_n - D(M + \alpha E(t)) - (M + \alpha E(t))^T D)) \neq 0, \forall i, \forall D, \forall \Delta \in \mathbb{B}_1.$$

The last condition is a necessary condition that

$$0 \leq \mu_{\mathbb{B}_1}(B) < 1,$$

where

$$B = (\mathbf{i}I_n + D(M+\alpha E(t)) + (M+\alpha E(t))^T D)^{-1}(\mathbf{i}I_n - D(M+\alpha E(t)) - (M+\alpha E(t))^T D).$$

□

The following Theorem 4.7 show the D-stability of dynamical system for a perturbation matrix B obtained from coefficient matrix $(M+\alpha E(t))$ for $t \in \mathbb{R}^+$.

Theorem 4.7. *The dynamical system*

$$\frac{dx}{dt} = (M+\alpha E(t))x, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}^+$$

is D-stable if $(M+\alpha E(t))$ *is a stable matrix, and*

$$0 \le \mu_{\mathbb{B}_1}(B) < 1, \ B = (\boldsymbol{i}I_n + D(M+\alpha E(t))^{-1}(\boldsymbol{i}I_n - (M+\alpha E(t)).$$

Proof. The perturbation matrix $(M+\alpha E(t))$ is D-stable iff

$$\mathrm{Re}(\lambda_i((M+\alpha E(t)) + \mathbf{i}D)) \neq 0, \quad \forall i, \ \forall D.$$

Assume that $(M+\alpha E(t))$ is D-stable, that is,

$$\mathrm{Re}(\lambda_i((M+\alpha E(t)) + \mathbf{i}D)) \neq 0, \quad \forall i.$$

Consider an admissible perturbation matrix with a block diagonal structure Δ defined as

$$\Delta = (\mathbf{i}I_n - D)(\mathbf{i}I_n + D)^{-1}, \text{ then } D = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

This is true for all $\Delta \in \mathbb{B}_1$, the set of block-diagonal matrices. Since,

$$\mathrm{Re}\,(\lambda_i((M+\alpha E(t)) + \mathbf{i}D)) \neq 0, \forall i.$$

$$\Rightarrow \mathrm{Re}\left(\lambda_i((M+\alpha E(t)) + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta))\right) \neq 0, \forall i, \forall \Delta \in \mathbb{B}_1.$$

Furthermore,

$$\sigma_N((M+\alpha E(t)) + (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)) = \sigma_N((\mathbf{i}I_n + (M+\alpha E(t)) - (\mathbf{i}I_n - (M+\alpha E(t)) \Delta), \forall \Delta \in \mathbb{B}_1,$$

where $\sigma_N(\cdot)$ denotes the number of non-zero singular values of a matrix. The above equality further implies that

$$((M+\alpha E(t)) + (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)) \sim ((\mathbf{i}I_n + (M+\alpha E(t)) - (\mathbf{i}I_n - (M+\alpha E(t)))\Delta), \quad \forall \Delta \in \mathbb{B}_1.$$

Also, we have that

$$((\mathbf{i}I_n + (M+\alpha E(t))) - ((\mathbf{i}I_n - (M+\alpha E(t))\Delta) = (I_n - ((\mathbf{i}I_n + (M+\alpha E(t))^{-1})(\mathbf{i}I_n - (M+\alpha E(t))\Delta, \quad \forall \Delta \in \mathbb{B}_1.$$

Thus,

$$\mathrm{Re}\left(\lambda_i\left(I_n - (\mathbf{i}I_n + (M+\alpha E(t)))^{-1}(\mathbf{i}I_n - (M+\alpha E(t))\Delta)\right)\right) \neq 0, \forall \Delta \in \mathbb{B}_1.$$

Which is a necessary condition that

$$0 \leq \mu_{\mathbb{B}_1}(B) < 1.$$

Conversely, we assume that

$$0 \leq \mu_{B_1}(B) < 1,\ B = (\mathbf{i}I_n + (M+\alpha E(t))^{-1}(\mathbf{i}I_n - (M+\alpha E(t)).$$

We need to prove that the matrix $(M+\alpha E(t))$, $t \in \mathbb{R}^+$ is a matrix that is D-stable. For the μ-value, the adequate condition is that if $\mu_{B_1}(B) < 1$, then

$$\mathrm{Re}(\lambda_i(\mathbf{i}I_n - B\Delta)) \neq 0, \forall \Delta \in \mathbb{B}_1.$$

In other words,

$$\mathrm{Re}(\lambda_i(I_n - (\mathbf{i}I_n + (M+\alpha E(t))^{-1}(\mathbf{i}I_n - (M+\alpha E(t))\Delta)) \neq 0, \forall \Delta \in \mathbb{B}_1.$$

The last inequality reduces to

$$\mathrm{Re}(\lambda_i(M+\alpha E(t)) + \mathbf{i}D) \neq 0, \forall i,$$

which proves that $(M+\alpha E(t))$, $t \in \mathbb{R}^+$ is a D-stable matrix. □

Theorem 11, stated below, presents theoretical results on the D-stability of dynamical system such that matrix product $(I_n + M + \alpha E(t))(I_n - M - \alpha E(t))$ has structured singular values bounded by 1.

Theorem 4.8. *The dynamical system*

$$\frac{dx}{dt} = (M+\alpha E(t))x,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}^+$$

is D-stable if $(M+\alpha E(t))$ is stable, and

$$0 \leq \mu_{\mathbb{B}_1}\left[(I_n + M + \alpha E(t))(I_n - M - \alpha E(t))\right] < 1.$$

Proof. Consider that $A = (I_n + M + \alpha E(t))^{-1}(I_n - M - \alpha E(t))$. Then, one can prove that

$$\sigma_{max}[(I_n + M + \alpha E(t))^{-1}(I_n - M - \alpha E(t))D^{-1}] < 1.$$

for some $D \in \hat{D}$.

The above inequality can be re-written as

$$\lambda_{\max}\left[D^{-1}\left(I_n - M - \alpha E(t)\right)^T\left(I_n + M + \alpha E(t)\right)^{-T}D^2(I_n + M + \alpha E(t))^{-1}(I_n - M - \alpha E(t))D^{-1}\right] < 1.$$

This inequality is equivalent to

$$D^{-1}\left(I_n - M - \alpha E(t)\right)^T\left(I_n + M + \alpha E(t)\right)^{-T}D^2\left(I_n + M + \alpha E(t)\right)^{-1}\left(I_n - M - \alpha E(t)\right)D^{-1} - I_n < 0.$$

$$\Longrightarrow\ (M+\alpha E(t))^{\top}D^2 + D^2(M+\alpha E(t)) > 0.$$

Then, by using definitions of structured singular value and using simple matrix algebra, we have that

$$0 \leq \mu_{\mathbb{B}_1}\left[\left(I_n + M + \alpha E(t)\right)^{-1}\left(I_n - M - \alpha E(t)\right)\right] \leq 1.$$

□

Algorithm 1 Compute structured singular values and pseudo spectrum

Initialize matrix M, grid parameters, and generate X, Y using meshgrid
Initialize r_n, s_v, and ss_v as zero matrices
for $i = 1$ **to** grid points **do**
    for $j = 1$ **to** grid points **do**
        $z \leftarrow X(i,j) + \mathrm{i} \cdot Y(i,j)$
        $r_n(i,j) \leftarrow 1/\|(zI - M)^{-1}\|$
        $s_v(i,j) \leftarrow \sigma_{\max}(zI - M)$
        $ss_v(i,j) \leftarrow 1/(\sigma_{\min}(zI - M) + \varepsilon)$
    end for
end for
while counter ≤ 4 **do**
    if counter $= 1$ **then**
        Plot $\log_{10}(r_n)$
    else if counter $= 2$ **then**
        Plot $\log_{10}(s_v)$
    else
        Plot $\log_{10}(ss_v)$
    end if
    counter $\leftarrow$ counter $+1$
end while

Convergence Analysis: Algorithm 1 converges when the change in eigenvalues between successive iterations falls below a specified tolerance:

$$|\lambda_{\min}^{(k+1)} - \lambda_{\min}^{(k)}| < \varepsilon$$

where $\varepsilon = 10^{-6}$ is typically used. The algorithm exhibits linear convergence with rate proportional to the eigenvalue separation. For well-conditioned $n \times n$ matrices, convergence is typically achieved within $O(n^2)$ iterations.

Computational Complexity: Each iteration requires $O(n^3)$ operations for eigenvalue computation and $O(n^2)$ for the matrix updates, resulting in total complexity of $O(n^5)$ for typical convergence.

5. Numerical Experimentation

This section presents numerical experiments conducted to compute eigenvalues, singular values, structured singular values, and pseudo-spectra for structured matrices arising in economic modeling. The pseudo-spectral plots in the complex plane are visualized as level sets corresponding to the resolvent norm $\|(M - zI_n)^{-1}\|$, illustrating variations across different ϵ-values.

Example 1. A Vector Auto-Regression (VAR) model captures dynamic relationships among multiple time series variables. In this framework, each variable is regressed on its own lagged values as well as those of the other included variables. A simple VAR model given as

$$X_t = M_{t-1} X_{t-1},$$

where

$$X_t = \begin{bmatrix} m(t) \\ q(t) \\ i(t) \end{bmatrix}; \; M_{t-1} = \begin{bmatrix} 1.213 & -0.235 & -0.187 \\ -0.202 & 1.142 & -0.094 \\ 0.577 & 0.894 & 1.017 \end{bmatrix}; \; X_{t-1} = \begin{bmatrix} m(t-1) \\ q(t-1) \\ i(t-1) \end{bmatrix}.$$

The matrix M_{t-1} captures the dynamic inter dependencies in a macroeconomic system where $m(t)$ represents money supply growth, $q(t)$ denotes GDP growth rate, and $i(t)$ represents the interest rate level. The coefficient 1.213 indicates that current money supply positively influences future money supply (monetary persistence), while -0.235 and -0.187 represent the contractionary effects of GDP growth and interest rates on future money supply, respectively.

The eigenvalues of M_{t-1} determine the stability and convergence properties of this economic system. Eigenvalues with positive real parts (as ensured by our D-stability analysis) guarantee that the economic variables converge to a stable equilibrium following any shock, indicating a robust macroeconomic framework capable of self-correction. The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of M_{t-1} are presented in Figure 1.

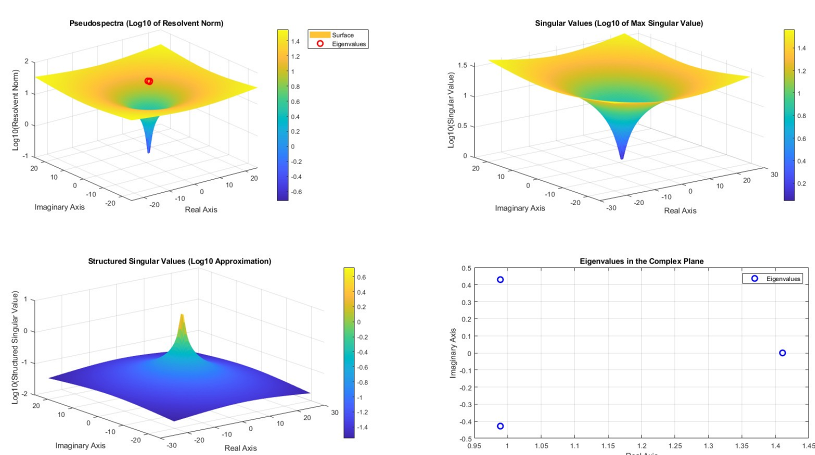

Figure 1. Spectral properties of matrix M_{t-1} in Example-1.

Example 2. Consider an economy with three dimensional and five sectors having real-valued consumption matrices as

$$M_1 = \begin{bmatrix} 0.01 & 0.002 & 0.04 \\ 0.02 & 0.004 & 0 \\ 0 & 0.01 & 0.02 \end{bmatrix}; \; M_2 = \begin{bmatrix} 1.0206 & 0.0020 & 0.1022 & 0.0216 & 0.0019 \\ 0.0634 & 1.0008 & 0.0486 & 0.0025 & 0.0703 \\ 0.0010 & 0.0153 & 1.0008 & 0.0110 & 0.0020 \\ 0.0039 & 0.0221 & 0.0039 & 1.0009 & 0.0816 \\ 0.0307 & 0.0008 & 0.0351 & 0.0110 & 1.0009 \end{bmatrix}.$$

The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of M_1 and M_2 are presented in Figure 2.

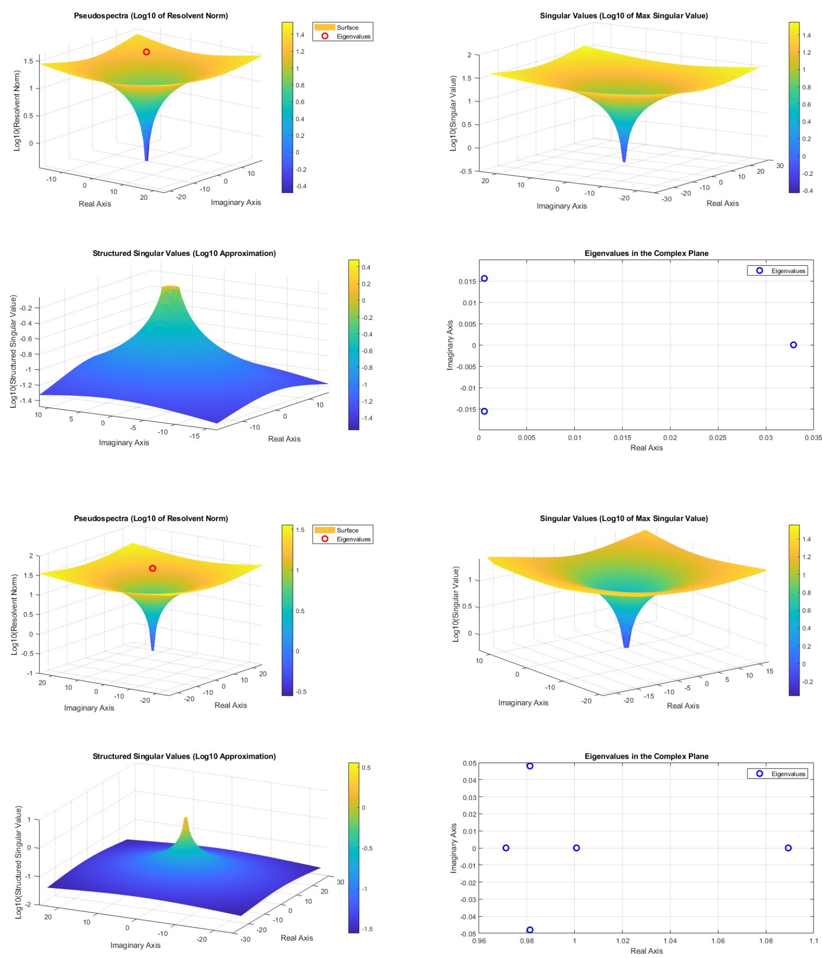

FIGURE 2. Spectral properties of consumption matrices $M_1, \; M_2$ in Example-2.

Example 3. We consider 50, 500 and 1000 dimensional real valued matrices which are randomly generated by MATLAB command **rand**. The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of 50, 500 and 1000 dimensional real valued matrices are presented in Figure 3.

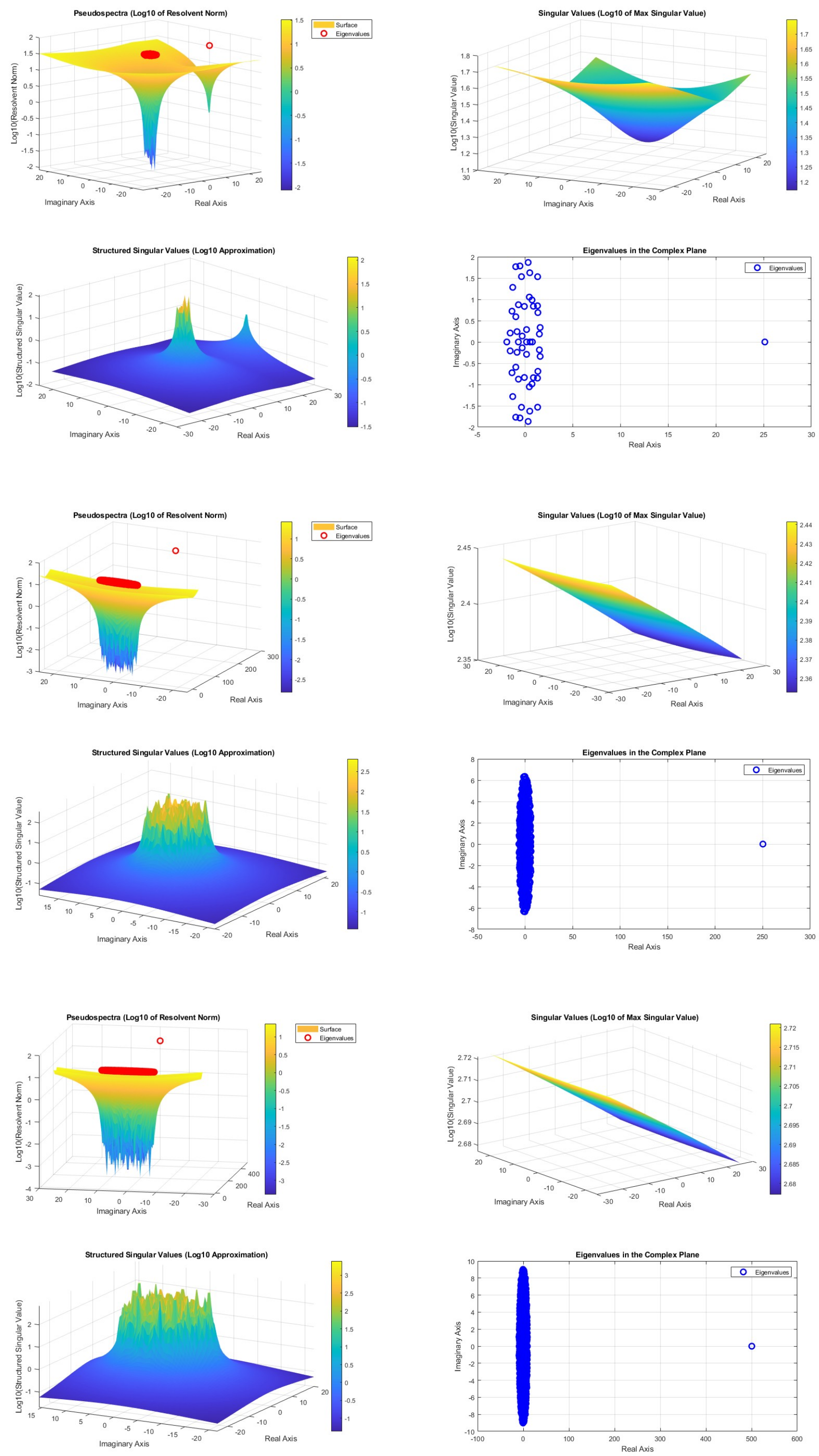

FIGURE 3. Spectral properties of 50, 500 and 1000 real valued matrices in Example-3.

Example 4. A sustainably model in economics and engineering represents the interdependence between production, energy, and transportation to environmental flows like as emission. This is also including the linkages with social impacts, for instance, welfare and employment. We consider Leontief input-output model of the form

$$x = Mx + y \ \ or \ \ x = (I - M)^{-1}y,$$

where x is the output vector, y is the vector of demand, and M is the sustainability matrix (the matrix of coefficients). We consider a M to be of size 3×3 given as follows:

$$M = \begin{bmatrix} 0.40 & 0.20 & 0.10 \\ 0.30 & 0.40 & 0.20 \\ 0.10 & 0.10 & 0.30 \end{bmatrix}.$$

The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of M are presented in Figure 6.

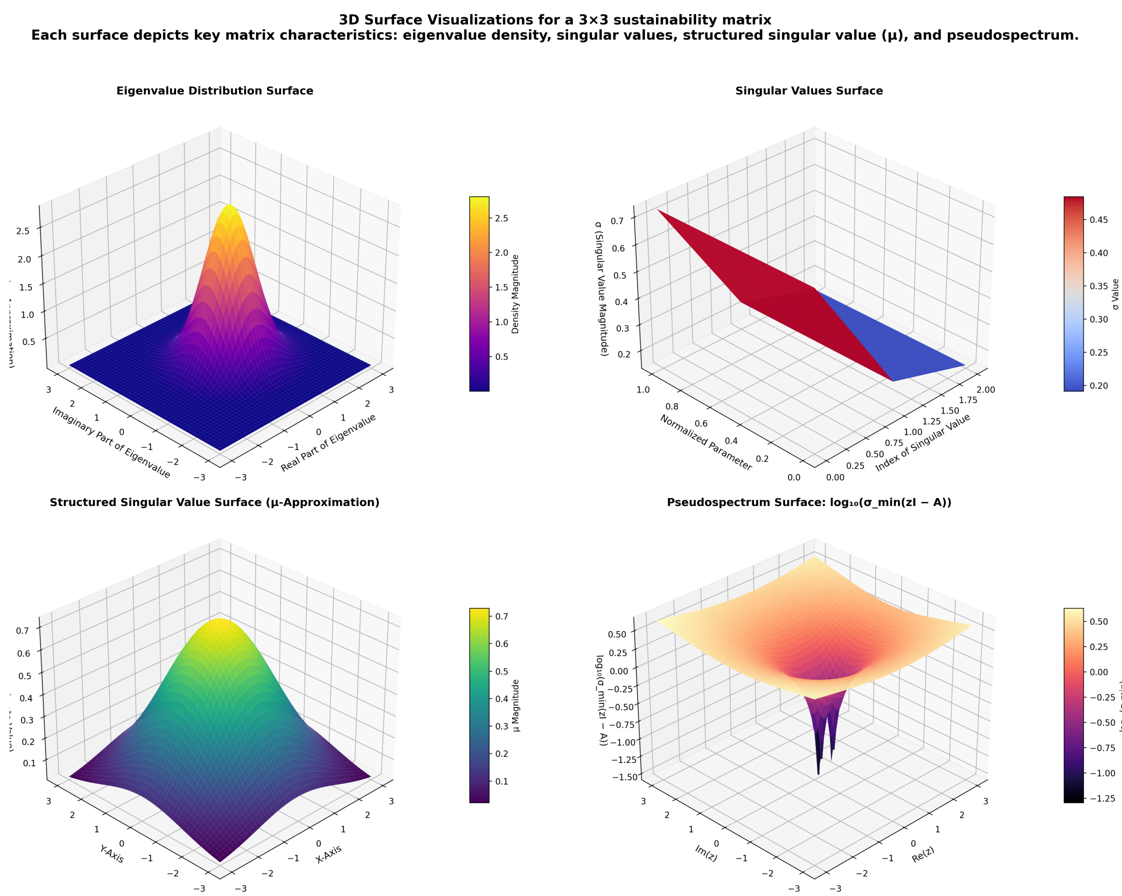

Figure 4. Spectral properties of sustainability matrix M in Example-4.

6. Conclusion

In this work, we present the computation of a closest correlation matrix with unit diagonal and non-negative spectrum for a given matrix M. We present a reformulation to an eigenvalue optimization problem whose solution gives ordinary differential equations in a gradient system. We can create a matrix with a non-negative spectrum by mathematically optimizing eigenvalues and singular values. The novel theoretical results are establish for a deeper understanding of the interconnection between D-stability theory and the structured singular value μ-theory. Our proposed results provide theoretical advancements and practical tools for the analysis of stability, instability, D-stability, and structured singular values. The significant gaps are filled between regional eigenvalue stability and resilient operation of dynamical systems confronted to structured perturbations. The key contributions to this paper are:

Theoretical Insights: Theoretical findings regarding the relationship between stability, D-stability and μ-analysis are derived with use includes numerous concepts from system theory, matrix analysis, and linear algebra. Our findings demonstrate how the structured singleton values may characterize and enforce D-stable regions in the complex plane.

Numerical Validation: The numerical testing were performed to confirm the practical benefits of the suggested methodology. Further investigation examines spectral characteristics and organized singular values in structured matrix systems with EigTool.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Gnazzo, V. Noferini, L. Nyman, F. Poloni, Riemann-Oracle: A General-Purpose Riemannian Optimizer to Solve Nearness Problems in Matrix Theory, Found. Comput. Math. (2025). https://doi.org/10.1007/s10208-025-09731-9.

[2] B. Vandereycken, Low-Rank Matrix Completion by Riemannian Optimization, SIAM J. Optim. 23 (2013), 1214–1236. https://doi.org/10.1137/110845768.

[3] F. Goyens, C. Cartis, A. Eftekhari, Nonlinear Matrix Recovery Using Optimization on the Grassmann Manifold, Appl. Comput. Harmon. Anal. 62 (2023), 498–542. https://doi.org/10.1016/j.acha.2022.11.001.

[4] R. Borsdorf, An Algorithm for Finding an Optimal Projection of a Symmetric Matrix Onto a Diagonal Matrix, SIAM J. Matrix Anal. Appl. 35 (2014), 198–224. https://doi.org/10.1137/12088906x.

[5] J.B. Rosen, H. Park, J. Glick, Total Least Norm Formulation and Solution for Structured Problems, SIAM J. Matrix Anal. Appl. 17 (1996), 110–126. https://doi.org/10.1137/s0895479893258802.

[6] R.L. Dykstra, An Algorithm for Restricted Least Squares Regression, J. Am. Stat. Assoc. 78 (1983), 837–842. https://doi.org/10.1080/01621459.1983.10477029.

[7] N.J. Higham, Computing the Nearest Correlation Matrix–A Problem from Finance, IMA J. Numer. Anal. 22 (2002), 329–343. https://doi.org/10.1093/imanum/22.3.329.

[8] S. Boyd, L. Xiao, Least-Squares Covariance Matrix Adjustment, Technical Report, Department of Electrical Engineering, Stanford University, 2004.

[9] J. Malick, A Dual Approach to Semidefinite Least-Squares Problems, SIAM J. Matrix Anal. Appl. 26 (2004), 272–284. https://doi.org/10.1137/s0895479802413856.

[10] H.H. Bauschke, J.M. Borwein, W. Li, Strong Conical Hull Intersection Property, Bounded Linear Regularity, Jameson's Property (G), and Error Bounds in Convex Optimization, Math. Program. 86 (1999), 135–160. https://doi.org/10.1007/s101070050083.

[11] H.H. Bauschke, J.M. Borwein, P. Tseng, Bounded Linear Regularity, Strong CHIP, and CHIP Are Distinct Properties, J. Convex Anal. 7 (2000), 395–412. https://eudml.org/doc/121081.

[12] J.M. Borwein, A.S. Lewis, Partially Finite Convex Programming, Part I: Quasi Relative Interiors and Duality Theory, Math. Program. 57 (1992), 15–48. https://doi.org/10.1007/bf01581072.

[13] J.M. Borwein, H. Wolkowicz, A Simple Constraint Qualification in Infinite Dimensional Programming, Math. Program. 35 (1986), 83–96. https://doi.org/10.1007/bf01589443.

[14] C.A. Micchelli, F.I. Utreras, Smoothing and Interpolation in a Convex Subset of a Hilbert Space, SIAM J. Sci. Stat. Comput. 9 (1988), 728–746. https://doi.org/10.1137/0909048.

[15] R. Mifflin, Semismooth and Semiconvex Functions in Constrained Optimization, SIAM J. Control. Optim. 15 (1977), 959–972. https://doi.org/10.1137/0315061.

[16] J.B. Keller, Closest Unitary, Orthogonal and Hermitian Operators to a Given Operator, Math. Mag. 48 (1975), 192–197. https://doi.org/10.1080/0025570x.1975.11976482.

[17] K. Lau, W. Riha, Characterization of Best Approximations in Normed Linear Spaces of Matrices by Elements of Finite-Dimensional Linear Subspaces, Linear Algebr. Appl. 35 (1981), 109–120. https://doi.org/10.1016/0024-3795(81)90268-8.

[18] R. Mathar, The Best Euclidian Fit to a Given Distance Matrix in Prescribed Dimensions, Linear Algebr. Appl. 67 (1985), 1–6. https://doi.org/10.1016/0024-3795(85)90181-8.

[19] R. Byers, A Bisection Method for Measuring the Distance of a Stable Matrix to the Unstable Matrices, SIAM J. Sci. Stat. Comput. 9 (1988), 875–881. https://doi.org/10.1137/0909059.

[20] K.J. Arrow, M. McManus, A Note on Dynamic Stability, Econometrica 26 (1958), 448–454. https://doi.org/10.2307/1907624.

[21] J. Quirk, R. Ruppert, Qualitative Economics and the Stability of Equilibrium, Rev. Econ. Stud. 32 (1965), 311–326. https://doi.org/10.2307/2295838.

[22] G. Giorgio, Stable and Related Matrices in Economic Theory, Control Cybern. 32 (2003), 397–410.

[23] F. Hahn, Chapter 16 Stability, in: Handbook of Mathematical Economics, Elsevier, 1982: pp. 745–793. https://doi.org/10.1016/S1573-4382(82)02011-6.

[24] M.C. Kemp, Y. Kimura, Introduction to Mathematical Economics, Springer New York, 1978. https://doi.org/10.1007/978-1-4612-6278-7.

[25] P.K. Newman, Some Notes on Stability Conditions, Rev. Econ. Stud. 27 (1959), 1–9. https://doi.org/10.2307/2296045.

[26] J.P. Quirk, R. Saposnik, Introduction to General Equilibrium Theory and Welfare Economics, McGraw-Hill, 1968.

[27] J.E. Woods, Mathematical Economics, Longman Publishing Group, 1978.

[28] E. Kaszkurewicz, A. Bhaya, Matrix Diagonal Stability in Systems and Computation, Birkhäuser, Boston, 2000. https://doi.org/10.1007/978-1-4612-1346-8.

[29] D.O. Logofet, Stronger-Than-Lyapunov Notions of Matrix Stability, or How "Flowers" Help Solve Problems in Mathematical Ecology, Linear Algebr. Appl. 398 (2005), 75–100. https://doi.org/10.1016/j.laa.2003.04.001.

[30] B. Cain, Real, 3×3, D-Stable Matrices, J. Res. Nationol Bur. Stondords - B. Math. Sci. 80B (1976), 75–77.

[31] G.V. Kanovei, D.O. Logofet, D-Stability of 4-by-4 Matrices, Comput. Math. Math. Phys. 38 (1998), 1369–1374. http://mi.mathnet.ru/zvmmf1812.

[32] C.R. Johnson, Sufficient Conditions for D-Stability, J. Econ. Theory 9 (1974), 53–62. https://doi.org/10.1016/0022-0531(74)90074-X.

[33] J.E. Woods, Mathematical Economics: Topics in Multi-Sectoral Economics, Longman, London, 1978.

[34] A. Packard, J. Doyle, The Complex Structured Singular Value, Automatica 29 (1993), 71–109. https://doi.org/10.1016/0005-1098(93)90175-s.

[35] B. Bernhardsson, Real Perturbation Values and Real Quadratic Forms in a Complex Vector Space, Linear Algebr. Appl. 270 (1998), 131–154. https://doi.org/10.1016/s0024-3795(98)80015-3.

[36] J. Chen, M. Fan, C. Nett, Structured Singular Values with Nondiagonal Structures. I. Characterizations, IEEE Trans. Autom. Control. 41 (1996), 1507–1511. https://doi.org/10.1109/9.539434.

[37] Jie Chen, M. Fan, C. Nett, Structured Singular Values with Nondiagonal Structures. II. Computation, IEEE Trans. Autom. Control. 41 (1996), 1511–1516. https://doi.org/10.1109/9.539435.

[38] D. Hinrichsen, A.J. Pritchard, Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness, Springer, 2010.

[39] M. Karow, μ-Values and Spectral Value Sets for Linear Perturbation Classes Defined by a Scalar Product, SIAM J. Matrix Anal. Appl. 32 (2011), 845–865. https://doi.org/10.1137/090774896.

[40] M. Karow, D. Hinrichsen, A.J. Pritchard, Interconnected Systems with Uncertain Couplings: Explicit Formulae for μ-Values, Spectral Value Sets, and Stability Radii, SIAM J. Control. Optim. 45 (2006), 856–884. https://doi.org/10.1137/050626508.

[41] D. Kressner, B. Vandereycken, Subspace Methods for Computing the Pseudospectral Abscissa and the Stability Radius, Technical Report, ANCHP-MATHICSE, Mathematics Section, Ecole Polytechnique Fédérale de Lausanne, 2012.

[42] K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control, Prentice Hall, Upper Saddle River, 1996.

[43] M. Fan, A. Tits, J. Doyle, Robustness in the Presence of Mixed Parametric Uncertainty and Unmodeled Dynamics, IEEE Trans. Autom. Control. 36 (1991), 25–38. https://doi.org/10.1109/9.62265.

[44] P.M. Young, M.P. Newlin, J.C. Doyle, Practical Computation of the Mixed μ Problem, in: 1992 American Control Conference, IEEE, 1992: pp. 2190–2194. https://doi.org/10.23919/acc.1992.4792521.

[45] C.R. Johnson, Sufficient Conditions for D-Stability, J. Econ. Theory 9 (1974), 53–62. https://doi.org/10.1016/0022-0531(74)90074-x.

[46] C. Yu, M.K. Fan, Decentralized Integral Controllability and D-Stability, Chem. Eng. Sci. 45 (1990), 3299–3309. https://doi.org/10.1016/0009-2509(90)80221-y.

[47] J. Chen, M.K. Fan, C. Yu, On D-Stability and Structured Singular Values, Syst. Control. Lett. 24 (1995), 19–24. https://doi.org/10.1016/0167-6911(94)00036-u.

[48] J. Lee, T.F. Edgar, Real Structured Singular Value Conditions for the Strong D-Stability, Syst. Control. Lett. 44 (2001), 273–277. https://doi.org/10.1016/s0167-6911(01)00147-5.

[49] M. Rehman, T.H. Rasulov, F. Amir, D-Stability, Strong D-Stability and μ-Values, Lobachevskii J. Math. 45 (2024), 1227–1233. https://doi.org/10.1134/s1995080224600754.

[50] K. Fan, A.J. Hoffman, Some Metric Inequalities in the Space of Matrices, Proc. Am. Math. Soc. 6 (1955), 111–116. https://doi.org/10.1090/s0002-9939-1955-0067841-7.

[51] N.J. Higham, Computing a Nearest Symmetric Positive Semidefinite Matrix, Linear Algebr. Appl. 103 (1988), 103–118. https://doi.org/10.1016/0024-3795(88)90223-6.

[52] N.J. Higham, Matrix Nearness Problems and Applications, University of Manchester, 1988.

[53] A. Ruhe, Closest Normal Matrix Finally Found!, BIT 27 (1987), 585–598. https://doi.org/10.1007/bf01937278.

[54] R. Gabriel, Matrizen Mit Maximaler Diagonale Bei Unitärer Similarität, J. Reine Angew. Math. 1979 (1979), 31–52. https://doi.org/10.1515/crll.1979.307-308.31.

[55] J.H. Wilkinson, On Neighbouring Matrices with Quadratic Elementary Divisors, Numer. Math. 44 (1984), 1–21. https://doi.org/10.1007/bf01389751.

[56] J.W. Demmel, On Condition Numbers and the Distance to the Nearest Ill-Posed Problem, Numer. Math. 51 (1987), 251–289. https://doi.org/10.1007/bf01400115.

[57] V. Liern, S.E. Parada-Rico, L.A. Conde-Solano, Linear Equation Systems Under Uncertainty: Applications to Multiproduct Market Equilibrium, Mathematics 13 (2025), 2566. https://doi.org/10.3390/math13162566.

[58] B. Kleinman, E. Liu, S.J. Redding, The Linear Algebra of Economic Geography Models, AEA Pap. Proc. 114 (2024), 328–333. https://doi.org/10.1257/pandp.20241043.

[59] W. Li, H. Li, X. Yang, Robust Stability of Boolean Networks Subject to Edge Perturbations, IEEE Trans. Syst. Man, Cybern.: Syst. 54 (2024), 5617–5626. https://doi.org/10.1109/tsmc.2024.3408155.

[60] H. Li, X. Pang, Stability Analysis of Large-Scale Boolean Networks via Compositional Method, Automatica 159 (2024), 111397. https://doi.org/10.1016/j.automatica.2023.111397.