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Separation Axioms on Upper Sets of BE-Algebras

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Abstract. This paper investigates the topological properties of upper sets in BE-algebras by introducing three novel separation axioms BET_0, BET_1, and BET_2 inspired by the classical T_0, T_1, and T_2 separation axioms in general topology. Using the structure of BE-algebras and their induced upper sets, we define a topology generated by a subbasis of sets of the form $A(x,y) = \{z \in X \mid x * (y * z) = 0\}$. We establish the necessary and sufficient conditions for a BE-space to satisfy each of these axioms. Several illustrative examples are provided to demonstrate the distinctions among the BET_i-spaces. Furthermore, we examine interrelations among the axioms and their implications for algebraic structures, such as involutory BE-algebras and fuzzy ideals. Our results contribute to the integration of algebraic and topological concepts, offering new insights into the study of BE-structured spaces.

1. Introduction

The study of algebraic structures such as BCK-algebras, BCI-algebras, and BE-algebras has received considerable attention in recent decades due to their theoretical depth and diverse applications in logic, computer science, and information systems. Among these, BE-algebras, introduced by Kim and Kim in 2007 [7] as a generalization of BCK-algebras, have provided a fertile ground for investigating logical connectives and various ideal-related structures.

In algebraic logic, BE-algebras are defined by a binary operation together with a distinguished constant, thereby offering a framework for modeling logical implication and deduction. They are particularly significant due to their role in describing logical systems and characterizing ordered algebraic structures. One of the essential directions in the study of BE-algebras is the analysis of their ideal theory, filters, congruences, and the associated order-theoretic properties.

An important concept arising from the order-theoretic perspective of BE-algebras is the notion of upper sets. In this context, an upper set is a subset closed under upward closure, often described

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by algebraic conditions such as

$$A(u,v) = \{z \in X \mid u * (v * z) = 1\}.$$

These sets naturally align with ideals and filters and play a central role in exploring the structure and behavior of *BE*-algebras from both algebraic and topological viewpoints. Foundational studies by Kim and Lee [8], Ahn and So [4,5], and others have laid the groundwork for this line of investigation. In parallel, topological concepts—particularly the separation axioms—provide powerful tools for distinguishing elements or subsets within a space. Recall that a set X together with a family $\tau = \{U\}$ of subsets is called a *topological space*, denoted by (X,τ), if the following conditions hold: (1) $X, \varnothing \in \tau$, (2) the intersection of any finite number of members of τ belongs to τ, and (3) the union of any arbitrary subfamily of τ belongs to τ. The members of τ are called the *open sets* of X.

In general topology, separation axioms T_0 (Kolmogorov), T_1 (Fréchet), and T_2 (Hausdorff) characterize the extent to which points or sets can be separated by neighborhoods or open sets. Translating these axioms into the framework of BE-algebras requires the formulation of analogous notions that accommodate the algebraic operations and partial order structures inherent in these algebras. In 2017, Mehrshad and Golzarpoor [9] examined some fundamental properties of T_0, T_1, and T_2-type topologies on *BE*-algebras.

The present research investigates the class of upper sets derived from BE-algebras and explores their topological properties through the lens of separation axioms. In particular, we define and analyze BET_0, BET_1, and BET_2 spaces—generalizations of the classical separation axioms adapted to upper set topologies in *BE*-algebras. These topologies are constructed using sets of the form $A(x,y)$ as a basis, and our study centers on the algebraic conditions under which the corresponding separation properties hold.

To this end, we establish necessary and sufficient conditions for upper set topologies in BE-algebras to satisfy each type of separation axiom. Special emphasis is placed on subclasses of BE-algebras, such as involutory *BE*-algebras. Furthermore, we investigate how these axioms interact with ideal structures, congruences, and morphisms, thereby providing a deeper understanding of the algebraic–topological interface.

Separation axioms have been studied in other generalized frameworks, such as biminimal structure spaces. In particular, Tunapan [3] investigated CT_0, CT_1, C-Hausdorff, C-regular, and C-normal spaces in the setting of biminimal structures, establishing fundamental properties and relationships among these axioms. This line of research highlights the versatility of separation principles beyond classical topological spaces, thereby motivating further exploration within algebraic systems. Inspired by these developments, our present study extends the investigation of separation axioms to the upper set topologies on *BE*-algebras, providing a novel algebraic–topological perspective that parallels and enriches these earlier works.

In summary, this work focuses on the systematic construction and characterization of upper set topologies on BE-algebras, emphasizing the separation axioms BET_0, BET_1, and BET_2. We establish new characterizations, examine their connections with ideals and congruences, and explore their behavior under algebraic morphisms. These investigations not only broaden the theoretical framework of BE-algebras but also strengthen the interplay between algebraic and topological methods, offering potential applications in logical reasoning, fuzzy systems, and theoretical computer science.

2. Preliminaries

In order to proceed with the study, we first recall some essential concepts and results that will be used throughout this paper. These preliminaries include the basic notions of BE-algebras, upper sets, and classical separation axioms in topology, which provide the theoretical foundation for the subsequent sections.

Definition 2.1. *[7] A BE-algebra $(X, *, 1)$ is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:*

(BE1) $x * x = 1$

(BE2) $x * 1 = 1$

(BE3) $1 * x = x$

(BE4) $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in X$.

Definition 2.2. *[4] Let $(X, *, 1)$ be a BE-algebra and for any $x, y, z \in X$, define the upper set of x and y by*

$$A(x, y) = \{z \in X \mid x * (y * z) = 1\}.$$

*The set $A(x, y)$ is called the **upper set** of x and y.*

Theorem 2.1. *[5] If $(X, *, 1)$ is a BE-algebra, then $1, x, y \in A(x, y)$ for all $x, y \in X$.*

Theorem 2.2. *[6] If $(X, *, 1)$ is a BE-algebra, then $A(x, y) = A(y, x)$ for all $x, y \in X$.*

Example 2.1. *From the example of a BE-algebra $(X, *, 1)$ in [10] as shown in the following Cayley table:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

We can determine the upper sets on X as follows:

$$A(1, 1) = \{1\},$$

$$A(1, a) = A(a, 1) = A(a, a) = \{1, a\},$$

$$A(1,b) = A(b,1) = A(b,b) = \{1,b\},$$
$$A(1,c) = A(c,1) = A(c,c) = \{1,c\},$$
$$A(a,b) = A(b,a) = \{1,a,b\},$$
$$A(a,c) = A(c,a) = \{1,a,c\},$$
$$A(b,c) = A(c,b) = \{1,b,c\}.$$

Hence, the collection of all upper sets of X is

$$\mathcal{A} = \{A(x,y) \mid x,y \in X\} = \{A(1,1),\ A(1,a),\ A(1,b),\ A(1,c),\ A(a,b),\ A(a,c),\ A(b,c)\}.$$

Moreover, we observe that $\mathcal{P}(X) = \{\emptyset, \{1\}, \{a\}, \{b\}, \{c\}, \{1,a\}, \{1,b\}, \{1,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{1,a,b\}, \{1,a,c\}, \{1,b,c\}, \{a,b,c\}, X\}$. *So that* $\mathcal{A} \neq \mathcal{P}(X)$.

Definition 2.3. *[9] Let (X,τ) be a topological space. We recall the following classical separation axioms:*

- T_0: *For each $x,y \in X$ with $x \neq y$, there exists an open neighborhood of one point not containing the other.*
- T_1: *For each $x,y \in X$ with $x \neq y$, there exist open neighborhoods $U,V \in \tau$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.*
- T_2: *For each $x,y \in X$ with $x \neq y$, there exist disjoint open neighborhoods $U,V \in \tau$ such that $x \in U$ and $y \in V$.*

*A topological space satisfying T_i is called a T_i-*space*. In particular, a T_2-space is also known as a* Hausdorff space*.*

Definition 2.4. *[1,2] Let X be a nonempty set and let $P(X)$ denote the power set of X.*

1. *A subfamily $m_X \subseteq P(X)$ is called a* minimal structure *(briefly, m-structure) on X if $\emptyset \in m_X$ and $X \in m_X$. In this case, (X, m_X) is called an m-*space*, where members of m_X are said to be m_X-open and their complements are said to be m_X-closed.*
2. *If m_1^X and m_2^X are two minimal structures on the same set X, then the triple (X, m_1^X, m_2^X) is called a* biminimal structure space *(briefly, bim-space).*

Definition 2.5. *[3] Let (X, m_1^X, m_2^X) be a biminimal structure space. Then:*

1. *(X, m_1^X, m_2^X) is called a CT_0-*space *if for each pair of distinct points $x,y \in X$, there exist open sets $A \in m_1^X$ and $B \in m_2^X$ such that either $x \notin B$ or $y \notin A$.*
2. *(X, m_1^X, m_2^X) is called a CT_1-*space *if for each pair of distinct points $x,y \in X$, there exist open sets $A \in m_1^X$ and $B \in m_2^X$ such that $x \in A$, $y \in B$, $x \notin B$, and $y \notin A$.*
3. *(X, m_1^X, m_2^X) is called a* C-Hausdorff space *if for each pair of distinct points $x,y \in X$, there exist open sets $A \in m_1^X$ and $B \in m_2^X$ such that $x \in A$, $y \in B$, and $A \cap B = \emptyset$.*
4. *(X, m_1^X, m_2^X) is called a* C-regular space *if for any $x \in X$ and any nonempty subset $F \subseteq X$ with $x \notin F$, there exist open sets $A \in m_1^X$ and $B \in m_2^X$ such that $x \in A$, $F \subseteq B$, and $A \cap B = \emptyset$.*

(5) (X, m_1^X, m_2^X) *is called a* C-normal space *if for any pair of nonempty disjoint subsets* $A, B \subseteq X$, *there exist open sets* $U \in m_1^X$ *and* $V \in m_2^X$ *such that* $A \subseteq U$, $B \subseteq V$, *and* $U \cap V = \emptyset$.

3. Separation Axioms in *BE*-Structures

In the study of algebraic structures equipped with topological properties, a fundamental question arises as to how classical topological notions can be translated and adapted to algebraic settings. In particular, the formulation of separation axioms plays a crucial role in distinguishing between elements or subsets in a structured space. Within BE-algebras, the introduction of topologies induced by upper sets offers a fertile ground for exploring such axioms from an order-theoretic and algebraic viewpoint.

Several authors have proposed generalized forms of open sets and neighborhood systems in BE-algebras, leading naturally to the development of new topological constructs [4,5,9]. These constructs serve as a bridge between algebraic ideals and topological separation, providing a framework to investigate when and how points can be separated by algebraically defined open sets. The investigation of separation axioms such as BET_0, BET_1, and BET_2 emerges as a natural extension in this context.

Inspired by classical topology, the notions of T_0, T_1, and T_2 spaces are reformulated using the algebraically constructed upper sets $A(x, y)$ in BE-algebras. The resulting generalized topological spaces are denoted as BET_0, BET_1, and BET_2 spaces, respectively. These serve as algebraic analogues of their classical counterparts, but within the framework induced by BE-structure.

In what follows, we propose three new classes of topological spaces—namely, BET_0, BET_1, and BET_2 spaces—that are adapted from the classical separation axioms but defined with respect to the upper set topologies generated by BE-algebras. These definitions utilize the algebraic basis sets $A(x, y)$ derived from the internal operation of the BE-algebra and are interpreted through the order-theoretic and topological lenses. We then provide conditions under which each axiom is satisfied and examine their interrelationships. Our goal is to contribute to the broader understanding of topological behavior in BE-structured environments and to illustrate how algebraic identities affect topological separability.

Definition 3.1. *Let* $(X, *, 1)$ *be a BE-algebra and let* $P(X)$ *denote the power set of* X. *A subfamily* $T_X \subseteq P(X)$ *is said to be a BE-structure if the following conditions hold:*

(1) $\{1\} \in T_X$ *and* $X \in T_X$,

(2) *If* $A, B \in T_X$, *then* $A \cap B \in T_X$.

In this case, the pair (X, T_X) *is called a* BE-space. *The elements of* T_X *are referred to as* open sets, *while their complements in* X *are called* closed sets.

Example 3.1. *Let* $X = \{1, 2, 3\}$. *Then the power set of* X *is*

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Consider the following collections of subsets of X:

$$\mathcal{T}_X^1 = \Big\{\{1\}, \{1,2\}, \{2,3\}\Big\}, \quad \mathcal{T}_X^2 = \Big\{\{1\}, \{1,2\}, \{1,2,3\}\Big\},$$

$$\mathcal{T}_X^3 = \Big\{\{1\}, \{2,3\}, \{1,2,3\}\Big\}, \quad \mathcal{T}_X^4 = \Big\{\{1\}, \{1,3\}, \{1,2,3\}\Big\}.$$

It follows that $\mathcal{T}_X^2$ *and* $\mathcal{T}_X^4$ *are BE-structures. Hence, the structures* $(X, \mathcal{T}_X^2)$ *and* $(X, \mathcal{T}_X^4)$ *are BE-spaces. However,* $\mathcal{T}_X^1$ *is not a BE-structure since* $X \notin \mathcal{T}_X^1$*, and* $\mathcal{T}_X^3$ *is not a BE-structure because* $\{1\} \in \mathcal{T}_X^3$ *and* $\{2,3\} \in \mathcal{T}_X^3$*, but* $\{1\} \cap \{2,3\} = \emptyset \notin \mathcal{T}_X^3$.

Example 3.2. *By the example in [6] a BE-algebra* $(X, *, 1)$ *define the binary operation* $*$ *on X by the following Cayley table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	b	1	d	1
c	1	1	1	1	d
d	1	1	b	c	1

We obtain the upper sets as follow:

$$\begin{aligned}
&A(1,1) = \{1\}\\
&A(1,a) = A(a,1) = A(a,a) = \{1,a\},\\
&A(1,b) = A(b,1) = A(b,b) = A(a,b) = A(b,a) = \{1,b,d\},\\
&A(1,c) = A(c,1) = A(a,c) = A(c,a) = A(b,c) = A(c,b) = \{1,a,b,c\},\\
&A(1,d) = A(d,1) = A(a,d) = A(d,a) = \{1,a,d\},\\
&A(c,d) = A(d,c) = X.
\end{aligned}$$

hence, the set of all upper sets of X is

$$\mathcal{A} = \Big\{A(1,1), A(1,a), A(1,b), A(1,c), A(1,d), A(c,d)\Big\},$$

and hence $(X, \mathcal{A})$ *is a BE-space.*

Theorem 3.1. *Let* $(X, *, 1)$ *be a BE-algebra, and a distinct elements* $x, y \in X$ *such that* $x * y = y$*. Define*

$$\mathcal{T}_X = \{A(x,y) \mid x, y \in X\} \cup \{X\}.$$

Then $\mathcal{T}_X$ *constitutes a BE-structure on X.*

Proof. (1) Since $1 \in X$, we have $A(1,1) = \{1\} \in \mathcal{T}_X$, and by definition $X \in \mathcal{T}_X$.

(2) Let $A(a,b), A(c,d) \in \mathcal{T}_X$. We now show that $A(a,b) \cap A(c,d) \in \mathcal{T}_X$.

Case 1.: If $a = 1$, then $A(1, b) = \{1, b\}$. Observe that:

$$A(1, b) \cap A(1, d) = \{1, b\} \cap \{1, d\} = \{1\} \in \mathcal{T}_X,$$
$$A(1, b) \cap A(1, c) = \{1, b\} \cap \{1, c\} = \{1\} \in \mathcal{T}_X,$$
$$A(1, b) \cap A(c, d) = \{1, b\} \cap \{1, c, d\} = \{1\} \in \mathcal{T}_X.$$

Case 2.: If $b = 1$, then $A(a, 1) = \{1, a\}$. Similarly, we obtain:

$$A(a, 1) \cap A(1, d) = \{1, a\} \cap \{1, d\} = \{1\} \in \mathcal{T}_X,$$
$$A(a, 1) \cap A(c, 1) = \{1, a\} \cap \{1, c\} = \{1\} \in \mathcal{T}_X,$$
$$A(a, 1) \cap A(c, d) = \{1, a\} \cap \{1, c, d\} = \{1\} \in \mathcal{T}_X.$$

Case 3.: If $a \neq 1$ and $b \neq 1$, then $A(a, b) = \{1, a, b\}$. Thus,

$$A(a, b) \cap A(1, d) = \{1, a, b\} \cap \{1, d\} = \{1\} \in \mathcal{T}_X,$$
$$A(a, b) \cap A(c, 1) = \{1, a, b\} \cap \{1, c\} = \{1\} \in \mathcal{T}_X,$$
$$A(a, b) \cap A(c, d) = \{1, a, b\} \cap \{1, c, d\} = \{1\} \in \mathcal{T}_X.$$

Therefore, by (1) and (2), $\mathcal{T}_X$ is closed under finite intersections and contains X. Hence, $\mathcal{T}_X$ constitutes a BE-structure on X. □ □

Definition 3.2. *Let $(X, \mathcal{T}_X)$ be a BE-space and $\emptyset \neq Y \subseteq X$. We said to be $(Y, \mathcal{T}_Y)$ is a BE-*subspace *of $(X, \mathcal{T}_X)$ if $\mathcal{T}_Y$ is a BE-structure on Y.*

Example 3.3. *By the example 1,*

$$\mathcal{T}_X = \{A(x, y) \mid x, y \in X\} \cup \{X\} = \{\{1\}, \{1, a\}, \{1, b\}, \{1, c\}, \{1, a, b\}, \{1, a, c\}, \{1, b, c\}, X\} \text{ and}$$

$(X, \mathcal{T}_X)$ is a BE-space. Choosing $Y = A(1, a) = \{1, a\} \in \mathcal{T}_X$. Define

$$\mathcal{T}_Y = \{A(1, a) \cap A(u, v) \mid A(u, v) \in \mathcal{T}_X\}$$

Then, we obtain

$\mathcal{T}_Y = \{\{1, a\} \cap \{1\}, \{1, a\} \cap \{1, a\}, \{1, a\} \cap \{1, b\}, \{1, a\} \cap \{1, c\}, \{1, a\} \cap \{1, a, b\},$
$\{1, a\} \cap \{1, a, c\}, \{1, a\} \cap \{1, b, c\}, \{1, a\} \cap X\} = \{\{1\}, \{1, a\}\}$

We observe that $\{1\} \in \mathcal{T}_Y$ and $Y = \{1, a\} \in \mathcal{T}_Y$, with $\{1\} \cap \{1, a\} = \{1\} \in \mathcal{T}_Y$. Therefore, $\mathcal{T}_Y$ is a BE-structure. Thus, we conclude that $(Y, \mathcal{T}_Y)$ is a BE-subspace of $(X, \mathcal{T}_X)$.

Definition 3.3. *A BE-space $(X, \mathcal{T}_X)$ is said to be a BET_i-**space** for $i = 0, 1, 2$ if for any two distinct elements $x, y \in X$, the following conditions hold:*

*(i) BET_0-**space:** There exist open sets A and B in $\mathcal{T}_X$ such that $x \in A$, $y \in B$, and*

$$x \notin B \quad or \quad y \notin A.$$

(ii) BET_1***-space:*** *There exist open sets* $A, B \in \mathcal{T}_X$ *such that*

$$x \in A, \quad y \notin A, \quad y \in B, \quad \text{and} \quad x \notin B.$$

(iii) BET_2***-space:*** *There exist disjoint open sets* $A, B \in \mathcal{T}_X$ *such that*

$$x \in A, \quad y \in B, \quad A \cap B = \{1\}.$$

Example 3.4. *From example 4,*

$$\mathcal{T}_X = \{\{1\}, \{1, a\}, \{1, b\}, \{1, c\}, \{1, a, b\}, \{1, a, c\}, \{1, b, c\}, X\}.$$

Then we have that $(X, \mathcal{T}_X)$ *is a* BET_0*-space. Consider* $x, y \in X$ *with* $x \neq y$*,*

- *If* $x = 1, y = a$*, there is* $\{1\} \in T_X$ *is an open set of* x*, and* $\{1, a\} \in T_X$ *is an open set of* y*, where* $y = a \notin \{1\}$.
- *If* $x = 1, y = b$*, there is* $\{1\} \in T_X$ *is an open set of* x*, and* $\{1, b\} \in T_X$ *is an open set of* y*, where* $y = b \notin \{1\}$.
- *If* $x = 1, y = c$*, there is* $\{1\} \in T_X$ *is an open set of* x*, and* $\{1, c\} \in T_X$ *is an open set of* y*, where* $c \notin \{1\}$.
- *If* $x = a, y = b$*, there is* $\{1, a\} \in T_X$ *is an open set of* x *and* $\{1, b\} \in T_X$ *is an open set of* y*, where* $b \notin \{1, a\}$.
- *If* $x = a, y = c$*, there is* $\{1, a\} \in \mathcal{T}_X$ *is an open set of* x *and* $\{1, c\} \in \mathcal{T}_X$ *is an open set of* y*, where* $c \notin \{1, a\}$.
- *If* $x = b, y = c$*, there is* $\{1, b\} \in \mathcal{T}_X$ *is an open set of* x *and* $\{1, c\} \in \mathcal{T}_X$ *is an open set of* y*, where* $c \notin \{1, b\}$.

Thus, for every $x, y \in X$ *with* $x \neq y$*, there exist open sets* A *and* B *such that* $x \notin B$ *or* $y \notin A$*. Therefore,* $(X, \mathcal{T}_X)$ *is a* BET_0*-space.*

Example 3.5. *From example 4,* $X = \{1, a, b, c\}$.

$$\mathcal{T}_X = \{\{1\}, \{1, a\}, \{1, b\}, \{1, c\}, \{1, a, b\}, \{1, a, c\}, \{1, b, c\}, X\}.$$

Consider $x, y \in X^*$ *with* $x \neq y$*, where* $X^* = X \setminus \{1\} = \{a, b, c\}$.

Case 1: *If* $x = a$*,* $y = b$*, there is* $A = \{1, a\} \in \mathcal{T}_X$ *is an open set of* x *and* $B = \{1, b\} \in \mathcal{T}_X$ *is an open set of* y *such that* $x \notin B$*,* $y \notin A$ *and* $A \cap B = \{1\}$.

Case 2: *If* $x = a$*,* $y = c$*, there is* $A = \{1, a\} \in \mathcal{T}_X$ *is an open set of* x *and* $B = \{1, c\} \in \mathcal{T}_X$ *is an open set of* y *such that* $x \notin B$*,* $y \notin A$ *and* $A \cap B = \{1\}$.

Case 3: *If* $x = b$*,* $y = c$*, there is* $A = \{1, b\} \in \mathcal{T}_X$ *is an open set of* x *and* $B = \{1, c\} \in \mathcal{T}_X$ *is an open set of* y *such that* $x \notin B$*,* $y \notin A$ *and* $A \cap B = \{1\}$.

Hence, for every $x, y \in X^*$ with $x \neq y$, there exist open sets A containing x and B containing y such that $x \notin B$ and $y \notin A$. Thus, $(X^*, \mathcal{T}_X)$ is a BET_1-space an so is a BET_2-space.

Lemma 3.1. *Let $(X, *, 1)$ be a BE-algebra, then $A(1, x) = \{1, x\}$ for all $x \in X$*

Proof. Clearly by theorem 1, $\{1, x\} \subset A(1, x)$. Next, let $a \in A(1, x)$, then $1 * (x * a) = 1$, by $BE1$ and $BE3$ we have $x * a = 1$ and hence $a = x \in \{1, x\}$.

□

Theorem 3.2. *Let $(X, *, 1)$ be a BE-algebra. Define $\mathcal{T}_X = \{A(x, y) \mid x, y \in X\} \cup \{X\}$. Then $(X^*, \mathcal{T}_X)$ is a BET_1-space and so is BET_2.*

Proof. Let $x, y \in X$. Choose the open set $A = A(1, x) = \{1, x\} \in \mathcal{T}_X$ containing x and open set $B = A(1, y) = \{1, y\} \in \mathcal{T}_X$ containing y. It follows that $x \notin B$ and $y \notin A$ and $A \cap B = \{1\}$. Thus, $(X^*, \mathcal{T}_X)$ is a BET_1-space and so is BET_2.

□

Theorem 3.3. *If $(X, \mathcal{T}_X)$ is a BET_2-space, then it is also a BET_1-space and consequently a BET_0-space.*

$$BET_2 \Rightarrow BET_1 \Rightarrow BET_0,$$

Proof. Suppose that (X, τ) is a BET_2-space. For any pair of distinct elements $x, y \in X$, there exist disjoint open sets $A, B \in \mathcal{T}_X$ such that $x \in A$ and $y \in B$, implying that each point can be separated from the other by an open set not containing the other. Therefore, $(X, \mathcal{T}_X)$ satisfies the BET_1 condition.

Furthermore, since $(X, \mathcal{T}_X)$ is a BET_1-space, for every $x, y \in X$ with $x \neq y$, there exists an open set $A \in \mathcal{T}_X$ such that either $x \in A$ and $y \notin A$, or vice versa. This directly satisfies the definition of a BET_0-space.

Hence, every BET_2-space is also a BET_1-space and a BET_0-space.

□

Theorem 3.4. *Let $(X, \mathcal{A})$ be a BE-space and $\emptyset \neq Y \subseteq X$. Define $\mathcal{T}_Y = \{A \cap Y \mid A \in \mathcal{T}_X\} \cup \{\{1\}\}$. Then $(Y, \mathcal{T}_Y)$ is a BE-subspace of $(X, \mathcal{T}_X)$.*

Proof. (1) By the definition of $\mathcal{T}_Y$, we clearly have $\{1\} \in \mathcal{T}_Y$.
Since $X \in \mathcal{T}_X$, it follows that $Y = Y \cap X \in \mathcal{T}_Y$.

(2) Let $A, B \in \mathcal{T}_Y$. Then there exist $A_1, A_2 \in \mathcal{T}_X$ such that

$$A = Y \cap A_1 \quad \text{and} \quad B = Y \cap A_2.$$

Therefore,

$$A \cap B = (Y \cap A_1) \cap (Y \cap A_2) = Y \cap (A_1 \cap A_2) = Y \cap A_3 \quad \text{where } A_3 = A_1 \cap A_2 \in \mathcal{T}_X$$

$$Thus\ A \cap B = (Y \cap A_1) \cap (Y \cap A_2) = Y \cap A_3 \in \mathcal{T}_Y$$

Hence, from (1) and (2), $\mathcal{T}_Y$ is a *BE-structure* on X. Thus, we conclude that $(Y, \mathcal{T}_Y)$ is a *BE-subspace* of $(X, \mathcal{T}_X)$.

□

Conclusion

In this paper, we have introduced and investigated a new class of topological structures on BE-algebras, namely BET_0, BET_1, and BET_2 spaces, which serve as algebraic generalizations of the classical separation axioms T_0, T_1, and T_2 in topology. By employing the concept of upper sets induced by the algebraic operation of BE-algebras, we formulated new topologies based on families of subsets of the form $A(x, y) = \{z \in X \mid x * (y * z) = 1\}$.

Our investigation yielded several important results. We provided rigorous definitions and characterizations for each type of BET_i-space and examined their fundamental properties. Moreover, we established logical implications between BET_i-spaces, showing that the hierarchy $BET_2 \Rightarrow BET_1 \Rightarrow BET_0$ holds in the framework of BE-algebras, analogous to the classical topological hierarchy.

The separation axioms developed here deepen the interplay between algebraic structures and topological notions, offering new avenues for research in fuzzy logic, ideal theory, and categorical approaches to algebra. The findings are expected to facilitate further studies on morphisms between BE-topological spaces, topological congruences, and the development of categorical topological structures within algebraic logic.

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