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Wardowski-Type Fixed Point Results for Multivalued Mappings in Fuzzy Cone Metric Spaces

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Abstract. In this paper, we investigate the existence of fixed points (FP) for multivalued mappings in the framework of fuzzy cone metric spaces (FCMS). By combining the structural features of cone (C)-valued distances with fuzziness, we introduce Wardowski-type contractive conditions governed by nonlinear control functions. Unlike classical contractions, the proposed conditions do not rely on linear domination of distances but instead ensure convergence through the strict decrease of a Wardowski function along iterative sequences. Several FP theorems are established for multivalued operators via the fuzzy C Hausdorff metric. As consequences, corresponding results for single-valued mappings and ordered FCMSs are derived. The obtained results extend and unify various existing FP theorems in cone metric spaces (CMS) and fuzzy metric spaces, while providing a flexible framework for applications involving uncertainty and nonlinear phenomena.

1. INTRODUCTION

FP theory constitutes a fundamental tool in nonlinear analysis and plays a crucial role in diverse areas such as differential equations, optimization, control theory, and mathematical modeling. Since the classical Banach (BNCH) contraction principle, numerous extensions have been proposed by generalizing either the contractive condition or the underlying space.

A significant generalization was introduced by Huang and Zhang, who replaced the real-valued metric with an ordered BNCH space valued distance and introduced the concept of CMSs [1]. This

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approach allowed the incorporation of partial order structures into metric analysis and stimulated extensive research on FP theory in ordered settings.

Independently, the theory of fuzzy sets proposed by Zadeh [2] initiated the development of fuzzy metric spaces. The foundational work of Kramosil and Michalek [3] and the subsequent refinement by George and Veeramani [4] provided a rigorous framework for measuring distances under uncertainty. FP results in fuzzy metric spaces and their multivalued counterparts have since been widely studied; see, for instance, [5, 6].

More recently, FCMS were introduced as a hybrid structure combining the features of CMSs and fuzzy metric spaces [9]. This setting is particularly suitable for problems where both order relations and fuzzy uncertainty are present. Several authors have investigated topological properties and FP results in FCMS under various contractive assumptions [10–12].

On the other hand, Wardowski introduced a new class of nonlinear contractions, known as F-contractions, which significantly generalize classical metric contractions [17]. The key feature of Wardowski's approach is that convergence is achieved through the divergence of a control function to $-\infty$, rather than by direct comparison of distances. This methodology has proved effective in extending FP theory to broader nonlinear settings.

Motivated by these developments, the purpose of this paper is to establish Wardowski-type FP theorems for multivalued mappings (MVP) in FCMS. By employing the fuzzy cone Hausdorff metric (FCHM), we derive sufficient conditions for the existence of FPs under nonlinear contractive inequalities expressed via Wardowski control functions. The results obtained here generalize several known FP theorems in CMSs, fuzzy metric spaces, and their multivalued extensions. Ordered versions and single-valued corollaries are also presented.

2. Preliminaries

Consider $\mathbb{E}$ be a real BNCH space and let $\mathcal{Z} \subset \mathbb{E}$ be a C, that is, a nonempty closed (NEC) subset satisfying $\mathcal{Z} + \mathcal{Z} \subset \mathcal{Z}$, $\lambda\mathcal{Z} \subset \mathcal{Z}$ for all $\lambda \geq 0$, and $\mathcal{Z} \cap (-\mathcal{Z}) = \{\theta\}$, where θ symbolizes the zero element of $\mathbb{E}$. The C $\mathcal{Z}$ induces a partial order $\leq$ on $\mathbb{E}$ defined by $x \leq y$ iff $y - x \in \mathcal{Z}$.

Let Π be a nonempty (NE) set. A mapping

$$\mathfrak{F} : \Pi \times \Pi \times (0, \infty) \to \mathcal{Z}$$

is called a FCM if, $\forall \mho, \eta, \zeta \in \Pi$ and $t, s > 0$, the upcoming conditions hold:

(1) $\mathfrak{F}(\mho, \eta, t) = \theta$ iff $\mho = \eta$;
(2) $\mathfrak{F}(\mho, \eta, t) = \mathfrak{F}(\eta, \mho, t)$;
(3) $\mathfrak{F}(\mho, \zeta, t + s) \leq \mathfrak{F}(\mho, \eta, t) + \mathfrak{F}(\eta, \zeta, s)$;
(4) $\mathfrak{F}(\mho, \eta, \cdot)$ is continuous for each $\mho, \eta \in \Pi$.

The pair $(\Pi, \mathfrak{F})$ is called a FCMS.

A sequence $\{\mho_n\}$ in Π is said to converge to $\mho \in \Pi$ if $\mathfrak{F}(\mho_n, \mho, t) \to \theta$ as $n \to \infty \forall t > 0$. It is called a Cauchy sequence (CS) if $\mathfrak{F}(\mho_n, \mho_m, t) \to \theta$ as $n, m \to \infty$ $\forall t > 0$. The space $(\Pi, \mathfrak{F})$ is called complete if every CS converges in Π.

Let $\mathcal{CB}(\Pi)$ denote the family of all NEC and bounded (bdd) subsets of Π. For $A, B \in \mathcal{CB}(\Pi)$, the FCHM $\mathcal{H}_{\mathfrak{F}}$ is defined by

$$\mathcal{H}_{\mathfrak{F}}(A,B,t) = \max\left\{\sup_{\mho\in A}\inf_{\eta\in B}\mathfrak{F}(\mho,\eta,t), \sup_{\eta\in B}\inf_{\mho\in A}\mathfrak{F}(\eta,\mho,t)\right\}.$$

Finally, following Wardowski [17], let $\mathcal{F}$ denote the family of functions $F : (0,\infty) \to \mathbb{R}$ satisfying:

(W1) F is strictly increasing;

(W2) For any sequence $\{r_n\} \subset (0,\infty)$, $r_n \to 0$ iff $F(r_n) \to -\infty$;

(W3) There exists $\kappa \in (0,1)$ such that $\lim_{r\to 0^+} r^{\kappa}F(r) = 0$.

3. Main Results

Before presenting the main fixed point results, we briefly outline the scope, novelty, and extensions achieved in this work. The aim of the present study is to develop a unified Wardowski-type contraction framework for MVP in FCMS. Unlike classical metric or C metric settings, the FCMS structure allows the simultaneous treatment of order, fuzziness, and C-valued distances, thereby providing a more flexible analytical environment. As a consequence, the results presented here unify and generalize a wide class of fixed point theorems under a single abstract framework.

Theorem 3.1. *Let $(\Pi, \mathfrak{F})$ be a CFCMS, where $\mathfrak{F} : \Pi \times \Pi \times (0,\infty) \to \mathcal{Z}$ and $\mathcal{Z}$ is a C in a real BNCH space $\mathbb{E}$. Let $\mathcal{CB}(\Pi)$ denote the family of all NEC and bdd subsets of Π.*

Fix $\mathcal{U} : \Pi \to \mathcal{CB}(\Pi)$ is a MVP such that there exist a function $F \in \mathcal{F}$ and an element $\tau \in \operatorname{int}(\mathcal{Z})$ satisfying

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t)) \preceq F(\mathfrak{F}(\mho,\eta,t)),$$

$\forall\ \mho, \eta \in \Pi$ and $t > 0$ with $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) \succ \theta$, where $\mathcal{H}_{\mathfrak{F}}$ symbolizes the FCHM.

Thus $\mathcal{U}$ admits FP in Π; that is, there exists $\mho^* \in \Pi$ such that

$$\mho^* \in \mathcal{U}\mho^*.$$

Proof. Let $\mho_0 \in \Pi$ be arbitrary. Since $\mathcal{U}\mho_0 \neq \emptyset$, choose $\mho_1 \in \mathcal{U}\mho_0$. If $\mho_1 = \mho_0$, then $\mho_0$ is a FP of $\mathcal{U}$

Assume $\mho_1 \neq \mho_0$. By induction, having chosen $\mho_n \in \Pi$, select $\mho_{n+1} \in \mathcal{U}\mho_n$ such that

$$\mathfrak{F}(\mho_n, \mho_{n+1}, t) \preceq \mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_{n-1}, \mathcal{U}\mho_n, t), \quad t > 0.$$

Thus, we obtain a sequence $\{\mho_n\}$ in Π with $\mho_{n+1} \in \mathcal{U}\mho_n$ $\forall\ n \geq 0$.

Since $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_{n-1}, \mathcal{U}\mho_n, t) \succ \theta$, applying the Wardowski-type contraction condition yields

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho_{n+1}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)).$$

Using the monotonicity of F, we obtain

$$F(\mathfrak{F}(\mho_{n+1}, \mho_{n+2}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) - \tau.$$

By iteration, it follows that

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \preceq F(\mathfrak{F}(\mho_0, \mho_1, t)) - n\,\tau.$$

Letting $n \to \infty$, we obtain

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \to -\infty.$$

By property $(W2)$ of the function F, this implies

$$\mathfrak{F}(\mho_n, \mho_{n+1}, t) \to \theta \quad \text{as } n \to \infty.$$

Next, we show that $\{\mho_n\}$ is a CS. Let $m > n$. Using the triangular inequality of the FCM, we have

$$\mathfrak{F}(\mho_n, \mho_m, t) \preceq \sum_{k=n}^{m-1} \mathfrak{F}\Big(\mho_k, \mho_{k+1}, \frac{t}{m-n}\Big).$$

Since $\mathfrak{F}(\mho_k, \mho_{k+1}, s) \to \theta$ as $k \to \infty$ for each $s > 0$, it follows that

$$\mathfrak{F}(\mho_n, \mho_m, t) \to \theta \quad \text{as } n, m \to \infty.$$

Hence, $\{\mho_n\}$ is a CS in Π. Completeness of $(\Pi, \mathfrak{F})$ ensures the existence of $\mho^* \in \Pi$ such that

$$\mho_n \to \mho^* \quad \text{as } n \to \infty.$$

Finally, we prove that $\mho^*$ is a FP of $\mathcal{U}$. Assume, on the contrary, that $\mho^* \notin \mathcal{U}\mho^*$. Then

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) \succ \theta.$$

Passing to the limit as $n \to \infty$ in

$$\mathfrak{F}(\mho_{n+1}, \mathcal{U}\mho^*, t) \preceq \mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho^*, t),$$

Using the continuity of $\mathfrak{F}$, we obtain

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) = \theta,$$

which is a contradiction. Therefore,

$$\mho^* \in \mathcal{U}\mho^*.$$

Hence, $\mathcal{U}$ has at least one FP in Π. □

Example 3.1. *Let $\Pi = [0,1]$ and let $\mathbb{E} = \mathbb{R}$ with the C $\mathcal{Z} = [0, \infty)$. Define the FCM $\mathfrak{F} : \Pi \times \Pi \times (0, \infty) \to \mathcal{Z}$ by*

$$\mathfrak{F}(\mho, \eta, t) = \frac{|\mho - \eta|}{1 + t}, \quad \mho, \eta \in \Pi, \ t > 0.$$

Then $(\Pi, \mathfrak{F})$ is a CFCMS.

Define the MVP $\mathcal{U} : \Pi \to \mathcal{CB}(\Pi)$ by

$$\mathcal{U}\mho = \Big[0, \frac{\mho}{2}\Big], \quad \mho \in \Pi.$$

For $\mho, \eta \in \Pi$, the FCHM satisfies

$$\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) = \frac{|\mho - \eta|}{2(1 + t)}.$$

Let $F : (0, \infty) \to \mathbb{R}$ *be defined by* $F(s) = \ln s$*, which belongs to* $\mathcal{F}$*, and choose* $\tau = \ln 2 \in \operatorname{int}(\mathcal{Z})$*. Then*

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t)) = \ln 2 + \ln\left(\frac{|\mho - \eta|}{2(1+t)}\right) = \ln\left(\frac{|\mho - \eta|}{1+t}\right) = F(\mathfrak{F}(\mho, \eta, t)),$$

whenever $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) \succ \theta$*.*

Hence, all hypotheses of Theorem 3.1 are satisfied. Moreover, $\mho^* = 0$ *is a FP of* $\mathcal{U}$ *since*

$$0 \in \mathcal{U}0 = [0, 0].$$

Corollary 3.1. *Let* $(\Pi, \mathfrak{F})$ *be a CFCMS. Fix* $\mathcal{S} : \Pi \to \Pi$ *is a single-valued mapping for which there exist a function* $F \in \mathcal{F}$ *and an element* $\tau \in \operatorname{int}(\mathcal{Z})$ *such that*

$$\tau + F(\mathfrak{F}(\mathcal{S}\mho, \mathcal{S}\eta, t)) \preceq F(\mathfrak{F}(\mho, \eta, t)),$$

$\forall\, \mho, \eta \in \Pi$*,* $t > 0$*, with* $\mathfrak{F}(\mathcal{S}\mho, \mathcal{S}\eta, t) \succ \theta$*. Then* $\mathcal{S}$ *admits a unique FP in* Π*.*

Proof. Define a MVP $\mathcal{U} : \Pi \to \mathcal{CB}(\Pi)$ by

$$\mathcal{U}\mho = \{\mathcal{S}\mho\}, \quad \mho \in \Pi.$$

Then, $\forall\, \mho, \eta \in \Pi$ and $t > 0$, the fuzzy C Hausdorff metric reduces to

$$\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) = \mathfrak{F}(\mathcal{S}\mho, \mathcal{S}\eta, t).$$

Hence, the contractive condition in Theorem 3.1 coincides with the above inequality. Therefore, all the hypotheses of Theorem 3.1 are satisfied, and there exists $\mho^* \in \Pi$ such that

$$\mho^* \in \mathcal{U}\mho^*,$$

which implies

$$\mathcal{S}\mho^* = \mho^*.$$

Uniqueness: Assume that ζ^* is another FP of $\mathcal{S}$. Then

$$\tau + F(\mathfrak{F}(\mho^*, \zeta^*, t)) \preceq F(\mathfrak{F}(\mho^*, \zeta^*, t)),$$

which is possible only if

$$\mathfrak{F}(\mho^*, \zeta^*, t) = \theta.$$

Hence, $\mho^* = \zeta^*$, and the FP is unique. □

Theorem 3.2. *Let* $(\Pi, \mathfrak{F}, \preceq)$ *be a CFCMS, where* $\preceq$ *is a partial order on* Π*. Let* $\mathcal{U} : \Pi \to \mathcal{CB}(\Pi)$ *be a MVP satisfying the following conditions:*

(*i*) *There exist a function* $F \in \mathcal{F}$ *and an element* $\tau \in \operatorname{int}(\mathcal{Z})$ *such that*

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t)) \preceq F(\mathfrak{F}(\mho, \eta, t)),$$

$t > 0$*, and* $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) \succ \theta$*.*

(*ii*) $\mathcal{U}$ *is monotone nondecreasing, that is, for* $\mho, \eta \in \Pi$ *with* $\mho \preceq \eta$ *and* $u \in \mathcal{U}\mho$*, there exists* $v \in \mathcal{U}\eta$ *such that* $u \preceq v$*.*

(iii) *There exists* $\mho_0 \in \Pi$ *and* $\mho_1 \in \mathcal{U}\mho_0$ *such that* $\mho_0 \preceq \mho_1$.

(iv) *If* $\{\mho_n\}$ *is a nondecreasing sequence in* Π *converging to* $\mho \in \Pi$, *then* $\mho_n \preceq \mho \ \forall n$.

Thus, $\mathcal{U}$ *admits at least one FP in* Π.

Proof. Let $\mho_0 \in \Pi$ be the element given in assumption *(iii)* and choose $\mho_1 \in \mathcal{U}\mho_0$ such that $\mho_0 \preceq \mho_1$. If $\mho_1 = \mho_0$, then $\mho_0$ is a FP of $\mathcal{U}$

Assume $\mho_1 \neq \mho_0$. Since $\mathcal{U}$ is monotone nondecreasing, there exists $\mho_2 \in \mathcal{U}\mho_1$ such that $\mho_1 \preceq \mho_2$. Proceeding inductively, we construct a sequence $\{\mho_n\}$ in Π satisfying

$$\mho_{n+1} \in \mathcal{U}\mho_n \quad \text{and} \quad \mho_n \preceq \mho_{n+1}, \quad n \geq 0.$$

Thus, $\{\mho_n\}$ is a nondecreasing sequence in Π.

Since $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho_{n+1}, t) \succ \theta$, the Wardowski-type contractive condition yields

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho_{n+1}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)).$$

Using the monotonicity of F and the construction of the sequence, we obtain

$$F(\mathfrak{F}(\mho_{n+1}, \mho_{n+2}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) - \tau.$$

Iterating the above inequality, we get

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \preceq F(\mathfrak{F}(\mho_0, \mho_1, t)) - n\,\tau, \quad n \geq 1.$$

Letting $n \to \infty$, we conclude that

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \to -\infty.$$

By property *(W2)* of the function F, it follows that

$$\mathfrak{F}(\mho_n, \mho_{n+1}, t) \to \theta \quad \text{as } n \to \infty.$$

Next, we show that $\{\mho_n\}$ is a CS in Π. Let $m > n$. By the triangular inequality of the FCM, we have

$$\mathfrak{F}(\mho_n, \mho_m, t) \preceq \sum_{k=n}^{m-1} \mathfrak{F}\Big(\mho_k, \mho_{k+1}, \frac{t}{m-n}\Big).$$

Since $\mathfrak{F}(\mho_k, \mho_{k+1}, s) \to \theta$ for each $s > 0$, we obtain

$$\mathfrak{F}(\mho_n, \mho_m, t) \to \theta \quad \text{as } n, m \to \infty.$$

Hence, $\{\mho_n\}$ is a CS. Completeness of $(\Pi, \mathfrak{F})$ implies the existence of $\mho^* \in \Pi$ such that

$$\mho_n \to \mho^* \quad \text{as } n \to \infty.$$

Since $\{\mho_n\}$ is nondecreasing and converges to $\mho^*$, assumption *(iv)* yields

$$\mho_n \preceq \mho^* \quad \forall n.$$

Assume that $\mho^* \notin \mathcal{U}\mho^*$. Then

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) \succ \theta.$$

Passing to the limit as $n \to \infty$ in

$$\mathfrak{F}(\mho_{n+1}, \mathcal{U}\mho^*, t) \preceq \mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho^*, t),$$

and using the continuity of $\mathfrak{F}$, we obtain

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) = \theta,$$

which is a contradiction. Therefore,

$$\mho^* \in \mathcal{U}\mho^*.$$

Hence, $\mathcal{U}$ admits at least one FP in Π. □

Theorem 3.3. *Let $(\Pi, \mathfrak{F})$ be a CFCMS and let $\mathcal{U} : \Pi \to \mathcal{CB}(\Pi)$ be a MV operator. Assume that there exist a function $F \in \mathcal{F}$ and an element $\tau \in \operatorname{int}(\mathcal{Z})$ such that*

$$\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t)) \preceq F(\max\{\mathfrak{F}(\mho, \eta, t),\ \mathfrak{F}(\mho, \mathcal{U}\mho, t),\ \mathfrak{F}(\eta, \mathcal{U}\eta, t),\ \mathfrak{F}(\mho, \mathcal{U}\eta, t),\ \mathfrak{F}(\eta, \mathcal{U}\mho, t)\}),$$

$\forall\ \mho, \eta \in \Pi$ and $t > 0$ with $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho, \mathcal{U}\eta, t) \succ \theta$, where $\mathcal{H}_{\mathfrak{F}}$ symbolizes the FCHM.

Then $\mathcal{U}$ admits at least one FP in Π.

Proof. Let $\mho_0 \in \Pi$ be arbitrary. Since $\mathcal{U}\mho_0 \neq \emptyset$, choose $\mho_1 \in \mathcal{U}\mho_0$. If $\mho_1 = \mho_0$, then $\mho_0$ is a FP of $\mathcal{U}$

Assume $\mho_1 \neq \mho_0$. By induction, having chosen $\mho_n \in \Pi$, select $\mho_{n+1} \in \mathcal{U}\mho_n$ such that

$$\mathfrak{F}(\mho_n, \mho_{n+1}, t) \preceq \mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_{n-1}, \mathcal{U}\mho_n, t), \quad t > 0.$$

Thus, we obtain a sequence $\{\mho_n\}$ in Π satisfying $\mho_{n+1} \in \mathcal{U}\mho_n\ \forall\ n \geq 0$.

Since $\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho_{n+1}, t) \succ \theta$, the contractive condition yields

$$\begin{aligned} &\tau + F(\mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho_{n+1}, t)) \\ &\preceq F(\max\{\mathfrak{F}(\mho_n, \mho_{n+1}, t),\ \mathfrak{F}(\mho_n, \mathcal{U}\mho_n, t),\ \mathfrak{F}(\mho_{n+1}, \mathcal{U}\mho_{n+1}, t),\ \mathfrak{F}(\mho_n, \mathcal{U}\mho_{n+1}, t),\ \mathfrak{F}(\mho_{n+1}, \mathcal{U}\mho_n, t)\}). \end{aligned}$$

Using the construction of $\{\mho_n\}$ and properties of the FCM, each term inside the maximum is dominated by $\mathfrak{F}(\mho_n, \mho_{n+1}, t)$. Hence, the above inequality reduces to

$$\tau + F(\mathfrak{F}(\mho_{n+1}, \mho_{n+2}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)).$$

Consequently,

$$F(\mathfrak{F}(\mho_{n+1}, \mho_{n+2}, t)) \preceq F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) - \tau.$$

Iterating, we obtain

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \preceq F(\mathfrak{F}(\mho_0, \mho_1, t)) - n\,\tau, \quad n \geq 1.$$

Letting $n \to \infty$, we have

$$F(\mathfrak{F}(\mho_n, \mho_{n+1}, t)) \to -\infty.$$

By property $(W2)$ of F, it follows that

$$\mathfrak{F}(\mho_n, \mho_{n+1}, t) \to \theta \quad \text{as } n \to \infty.$$

Next, we show that $\{\mho_n\}$ is a CS. For $m > n$, by the triangular inequality of the FCM,

$$\mathfrak{F}(\mho_n, \mho_m, t) \leq \sum_{k=n}^{m-1} \mathfrak{F}\Big(\mho_k, \mho_{k+1}, \frac{t}{m-n}\Big).$$

Since $\mathfrak{F}(\mho_k, \mho_{k+1}, s) \to \theta$ for each $s > 0$, we deduce that

$$\mathfrak{F}(\mho_n, \mho_m, t) \to \theta \quad \text{as } n, m \to \infty.$$

Hence, $\{\mho_n\}$ is a CS in Π. By completeness, there exists $\mho^* \in \Pi$ such that

$$\mho_n \to \mho^* \quad \text{as } n \to \infty.$$

Finally, assume that $\mho^* \notin \mathcal{U}\mho^*$. Then

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) \succ \theta.$$

Passing to the limit as $n \to \infty$ in

$$\mathfrak{F}(\mho_{n+1}, \mathcal{U}\mho^*, t) \preceq \mathcal{H}_{\mathfrak{F}}(\mathcal{U}\mho_n, \mathcal{U}\mho^*, t),$$

and using the continuity of $\mathfrak{F}$, we obtain

$$\mathfrak{F}(\mho^*, \mathcal{U}\mho^*, t) = \theta,$$

This leads to a contradiction. Therefore,

$$\mho^* \in \mathcal{U}\mho^*.$$

Hence, $\mathcal{U}$ admits at least one FP in Π. □

Conclusion

In this paper, we have established several Wardowski-type FP theorems for MVP in the setting of FCMS.By employing nonlinear control functions and the FCHM, we derived sufficient conditions for the existence of fixed points without imposing linear contraction constraints.

The obtained results unify and extend a wide range of FP theorems from CMS, FMS, and multi-valued analysis. Ordered versions and single-valued corollaries further enhance the applicability of the developed theory.

Future research directions may include the study of common fp, coupled and tripled fp, and applications to nonlinear integral, differential, and integro-differential equations under uncertainty. The proposed framework can also be extended to other generalized metric structures and hybrid contraction schemes.

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