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Stability of an Alternative Functional Equation Related to the Quadratic Equation

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Abstract. Given a rational number $\alpha \neq \pm 2$, a criterion is established for the existence of the general solution to the alternative quadratic functional equation of the form

$$f(xy) + f(xy^{-1}) = 2(f(x) + f(y)) \quad \text{or} \quad f(xy) + f(xy^{-1}) = \alpha(f(x) + f(y)),$$

where f is a mapping from an abelian group $(G, \cdot)$ to a uniquely divisible abelian group $(H, +)$. Subsequently, the Hyers-Ulam stability of this equation is proved for mappings from an abelian group to a Banach space, provided that $\alpha \notin \{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$ is a rational number.

1. Introduction

One of the most interesting problems in the theory of functional equations is solving *alternative functional equations*. These equations typically impose a disjunction of conditions, requiring the unknown function to satisfy one of several possible identities at each point in the domain. For instance, the alternative Cauchy functional equation

$$\big(f(x+y) - af(x) - bf(y)\big)\big(f(x+y) - f(x) - f(y)\big) = 0,$$

which is closely related to the classical Cauchy functional equation

$$f(x+y) = f(x) + f(y),$$

has been studied extensively by Kannappan [1] et al. Their work demonstrated that under certain regularity conditions, functions satisfying this alternative relation essentially reduce to solutions

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of the single classical equation. Forti [2] extended this work by determining the general solution in a more general setting of the form

$$(cf(x+y)-af(x)-bf(y)-d)\,(f(x+y)-f(x)-f(y))=0.$$

This generalization unified several previous results, extending the work of Ger [3] as well as that of Forti and Paganoni [4,5]. Moving from general solutions to stability problems, Batko [6] investigated the Hyers-Ulam stability of the alternative Cauchy equation

$$f(x+y)=\pm\big(f(x)+f(y)\big),$$

where f maps from an abelian group to a Banach space. Batko's result was significant because it showed that the stability of alternative equations often requires more delicate handling of the "sign" choices than standard stability problems.

While the Cauchy equation is fundamental, similar alternative structures arise in the study of quadratic mappings. Skof [7] proposed four alternative functional equations involving the absolute value:

$$\begin{aligned}|f(x+y)|&=|2f(x)+2f(y)-f(x-y)|,\\ |f(x-y)|&=|2f(x)+2f(y)-f(x+y)|,\\ |2f(y)|&=|f(x+y)+f(x-y)-2f(x)|,\\ \text{and}\quad |2f(x)|&=|f(x+y)+f(x-y)-2f(y)|.\end{aligned}$$

Skof proved that each of these functional equations is actually equivalent to the classical quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

where f is a function from a real linear space to the set of real numbers. This result illustrates that certain "norm-condition" alternative equations do not lead to new classes of functions, but rather characterize the standard quadratic form.

When the domain of f is a group (written multiplicatively), the quadratic functional equation (1.1) can be stated as

$$f(xy)+f(xy^{-1})=2f(x)+2f(y). \tag{1.2}$$

In this group-theoretic context, Nakmahachalasint [8,9] showed that the alternative quadratic functional equation of the form

$$f(xy)+f(xy^{-1})=\pm2\big(f(x)+f(y)\big) \tag{1.3}$$

is equivalent to the classical quadratic functional equation (1.2), where f is a function from a 2-divisible group to a uniquely divisible abelian group. Furthermore, Nakmahachalasint [10] investigated the Hyers-Ulam stability of (1.3) when f is a function from an abelian group to a Banach space. This established that the stability properties of the signed quadratic equation mirror those of the classical case.

In this paper, we generalize these investigations by considering a broader class of alternatives. Given a rational number $\alpha \neq \pm 2$, a criterion is established for the existence of the general solution to the alternative quadratic functional equation of the form

$$f(xy) + f(xy^{-1}) = 2(f(x) + f(y)) \quad \text{or} \quad f(xy) + f(xy^{-1}) = \alpha(f(x) + f(y)), \tag{1.4}$$

where f is a mapping from an abelian group to a uniquely divisible abelian group. Subsequently, the Hyers-Ulam stability of (1.4) is proved for mappings from an abelian group to a Banach space, provided that $\alpha \notin \{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$ is a rational number.

2. CRITERION FOR THE EXISTENCE OF THE GENERAL SOLUTION

In this section, we let $(G, \cdot)$ be an abelian group and $(H, +)$ be a uniquely divisible abelian group. Given a rational number $\alpha \neq \pm 2$ and a function $f : G \to H$, for every pair of $x, y \in G$, we denote the statement

$$\begin{aligned}\mathcal{P}f^{(\alpha)}(x,y) = \Big(f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \quad &\text{or} \\ f(xy) + f(xy^{-1}) = \alpha f(x) + \alpha f(y) \Big)&.\end{aligned}$$

The set of all solutions to the statement $\mathcal{P}f^{(\alpha)}(x,y)$ will be denoted by $\mathcal{A}^{(\alpha)}_{(G,H)}$, i.e.,

$$\mathcal{A}^{(\alpha)}_{(G,H)} = \{f : G \to H \mid \mathcal{P}f^{(\alpha)}(x,y) \ \text{ for all } x, y \in G\}.$$

First, we will prove some auxiliary lemmas that will lead to the proof of main results.

Lemma 2.1. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$. If $f(e) \neq 0$, then $\alpha = 1$ and f is constant.*

Proof. Assume that $f(e) \neq 0$. By the alternatives in $\mathcal{P}f^{(\alpha)}(e,e)$, we obtain that $\alpha = 1$. The alternatives in $\mathcal{P}f^{(\alpha)}(x,e)$ give $f(x) = f(e)$ for all $x \in G$. Hence we have the desired result. □

Lemma 2.2. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$ and $f(e) = 0$. Then $f(x^{-1}) = f(x)$ for all $x \in G$, i.e., f is even.*

Proof. Suppose that there exists $a \in G$ such that $f(a^{-1}) \neq f(a)$. The alternatives in $\mathcal{P}f^{(\alpha)}(e,a)$ give

$$f(a^{-1}) = (\alpha - 1)f(a). \tag{2.1}$$

It should be noted that if $f(a) = 0$, then $f(a^{-1}) = 0$, a contradiction to $f(a^{-1}) \neq f(a)$. By the alternatives in $\mathcal{P}f^{(\alpha)}(e,a^{-1})$, we get

$$f(a) = (\alpha - 1)f(a^{-1}). \tag{2.2}$$

By (2.1) and (2.2), we must have $\alpha = 0$ and then $f(a^{-1}) = -f(a)$. The alternatives in $\mathcal{P}f^{(0)}(a,a^{-1})$ give $f(a^2) = 0$. By the alternatives in $\mathcal{P}f^{(0)}(a^2,a)$, $\mathcal{P}f^{(0)}(a,a^2)$ and $\mathcal{P}f^{(0)}(a^2,a^{-1})$, we get

$$f(a^3) \in \{f(a), 0\}, f(a^3) \in \{3f(a), 0\} \text{ and } f(a^3) \in \{-3f(a), 0\},$$

respectively. Hence $f(a) = 0$, a contradiction. □

Lemma 2.3. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$ and $f(e) = 0$. If $\alpha \notin \{0, \pm\frac{1}{2}, -1\}$, then $f(x^2) = 4f(x)$ for all $x \in G$.*

Proof. Assume that $\alpha \notin \{0, \pm\frac{1}{2}, -1\}$ and there exists $a \in G$ such that $f(a^2) \neq 4f(a)$. By the alternatives in $\mathcal{P}f^{(\alpha)}(a,a)$, we get $f(a^2) = 2\alpha f(a)$ and then the alternatives in $\mathcal{P}f^{(\alpha)}(a^2,a^2)$ give

$$f(a^4) \in \{8\alpha f(a), 4\alpha^2 f(a)\}. \tag{2.3}$$

If $f(a) = 0$, then $f(a^2) = 0$, a contradiction to $f(a^2) \neq 4f(a)$. Hence $f(a) \neq 0$. By the alternatives in $\mathcal{P}f^{(\alpha)}(a^2,a)$, we obtain that

$$f(a^3) \in \{(1+4\alpha)f(a), (-1+\alpha+2\alpha^2)f(a)\}.$$

Case (i). Assume that $f(a^3) = (1+4\alpha)f(a)$. The alternatives in $\mathcal{P}f^{(\alpha)}(a^3,a)$ give

$$f(a^4) \in \{(4+6\alpha)f(a), 4\alpha^2 f(a)\}. \tag{2.4}$$

By (2.3) and (2.4), we get $f(a^4) = 4\alpha^2 f(a)$. The alternatives in $\mathcal{P}f^{(\alpha)}(a^4,a)$ give

$$f(a^5) \in \{(1-4\alpha+8\alpha^2)f(a), (-1-3\alpha+4\alpha^3)f(a)\}, \tag{2.5}$$

while the alternatives in $\mathcal{P}f^{(\alpha)}(a^3,a^2)$ give

$$f(a^5) \in \{(1+12\alpha)f(a), (-1+\alpha+6\alpha^2)f(a)\}. \tag{2.6}$$

By (2.5) and (2.6), we get a contradiction.

Case (ii). Assume that $f(a^3) = (-1+\alpha+2\alpha^2)f(a)$. The alternatives in $\mathcal{P}f^{(\alpha)}(a^3,a)$ give

$$f(a^4) \in \{4\alpha^2 f(a), (-2\alpha+\alpha^2+2\alpha^3)f(a)\}. \tag{2.7}$$

By (2.3) and (2.7), we get

$$f(a^4) = 4\alpha^2 f(a) \quad \text{or} \quad \alpha = -\frac{5}{2}.$$

Suppose $f(a^4) \neq 4\alpha^2 f(a)$. We obtain that $\alpha = -\frac{5}{2}$ and $f(a^4) = (-2\alpha+\alpha^2+2\alpha^3)f(a)$. Thus $f(a^2) = -5f(a), f(a^3) = 9f(a)$ and $f(a^4) = -20f(a)$. The alternatives in $\mathcal{P}f^{(-5/2)}(a^4,a)$ give $f(a^5) \in \{-47f(a), \frac{77}{2}f(a)\}$, while the alternatives in $\mathcal{P}f^{(-5/2)}(a^3,a^2)$ give $f(a^5) \in \{7f(a), -11f(a)\}$. We get $f(a) = 0$, a contradiction. Hence $f(a^4) = 4\alpha^2 f(a)$. The alternatives in $\mathcal{P}f^{(\alpha)}(a^4,a)$ give

$$f(a^5) \in \{(3-\alpha+6\alpha^2)f(a), (1-2\alpha^2+4\alpha^3)f(a)\}, \tag{2.8}$$

while the alternatives in $\mathcal{P}f^{(\alpha)}(a^3,a^2)$ give

$$f(a^5) \in \{(-3+6\alpha+4\alpha^2)f(a), (-1-\alpha+3\alpha^2+2\alpha^3)f(a)\}. \tag{2.9}$$

Case (ii.1). Assume that $f(a^5) = (1-2\alpha^2+4\alpha^3)f(a) = (-1-\alpha+3\alpha^2+2\alpha^3)f(a)$. We get $\alpha = 1$ and then $f(a^2) = f(a^3) = 2f(a), f(a^4) = 4f(a), f(a^5) = 3f(a)$. The alternatives in $\mathcal{P}f^{(1)}(a^5,a)$ give $f(a^6) \in \{4f(a), 0\}$, while the alternatives in $\mathcal{P}f^{(1)}(a^4,a^2)$ give $f(a^6) \in \{10f(a), 4f(a)\}$. Hence $f(a^6) = 4f(a)$. The alternatives in $\mathcal{P}f^{(1)}(a^6,a)$ give $f(a^7) \in \{7f(a), 2f(a)\}$, while the alternatives in $\mathcal{P}f^{(1)}(a^5,a^2)$ give $f(a^7) \in \{8f(a), 3f(a)\}$. Thus $f(a) = 0$, a contradiction.

Case (ii.2). Assume that $f(a^5) \neq (1-2\alpha^2+4\alpha^3)f(a)$ or $f(a^5) \neq (-1-\alpha+3\alpha^2+2\alpha^3)f(a)$. By (2.8) and (2.9), we conclude that $\alpha = \frac{3}{2}$. Hence

$$f(a^2) = f(a), f(a^3) = 5f(a), f(a^4) = 9f(a) \text{ and } f(a^5) = 15f(a).$$

The alternatives in $\mathcal{P}f^{(3/2)}(a^3,a^3)$ give $f(a^6) \in \{20f(a), 15f(a)\}$, while the alternatives in $\mathcal{P}f^{(3/2)}(a^4,a^2)$ give $f(a^6) \in \{21f(a), 15f(a)\}$. Thus $f(a^6) = 15f(a)$. The alternatives in $\mathcal{P}f^{(3/2)}(a^6,a)$ give $f(a^7) \in \{17f(a), 9f(a)\}$, while the alternatives in $\mathcal{P}f^{(3/2)}(a^5,a^2)$ give $f(a^7) \in \{31f(a), 22f(a)\}$. Thus $f(a) = 0$, a contradiction. □

Lemma 2.4. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$ and $x \in G$. If $f(x^2) = 4f(x)$, then $f(x^n) = n^2 f(x)$ for all non-negative integers n.*

Proof. Assume that $f(x^2) = 4f(x)$. If $f(e) \neq 0$, then by Lemma 2.1, we obtain that f is constant, i.e., $f(x) = f(e)$. Hence $f(x) = f(x^2) = 4f(x)$ and then $f(x) = 0$, a contradiction. We must have $f(e) = 0$. Suppose $f(x^3) \neq 9f(x)$. By the alternatives in $\mathcal{P}f^{(\alpha)}(x^2,x)$, we get $f(x^3) = (5\alpha - 1)f(x)$. Note that if $f(x) = 0$, then so is $f(x^3)$, a contradiction to $f(x^3) \neq 9f(x)$. The alternatives in $\mathcal{P}f^{(\alpha)}(x^3,x)$ give

$$f(x^4) \in \{(10\alpha - 4)f(x), (5\alpha^2 - 4)f(x)\}.$$

If $f(x) = (10\alpha - 4)f(x)$, then the alternatives in $\mathcal{P}f^{(\alpha)}(x^2,x^2)$ give $f(x) = 0$, a contradiction. We must get $f(x^4) = (5\alpha^2 - 4)f(x)$. The alternatives in $\mathcal{P}f^{(\alpha)}(x^2,x^2)$ give $\alpha = -\frac{2}{5}$ and then we have

$$f(x^3) = -3f(x) \text{ and } f(x^4) = -\tfrac{16}{5}f(x).$$

The alternatives in $\mathcal{P}f^{(2/5)}(x^4,x^2)$ give $f(x^6) \in \{-\frac{12}{5}f(x), -\frac{108}{25}f(x)\}$, while the alternatives in $\mathcal{P}f^{(2/5)}(x^3,x^3)$ give $f(x^6) \in \{-12f(x), \frac{12}{5}f(x)\}$. Hence $f(x) = 0$, a contradiction. Thus we must get $f(x^3) = 9f(x)$. Now we obtain that

$$f(x^n) = n^2 f(x) \tag{2.10}$$

holds for $n = 0, 1, 2, 3$. Let $k \geq 3$ be an integer. Assume that (2.10) holds for $n = 1, 2, \ldots, k$. Suppose $f(x^{k+1}) \neq (k+1)^2 f(x)$. The alternatives in $\mathcal{P}f^{(\alpha)}(x^k,x)$ give

$$f(x^{k+1}) = \Big(\alpha(k^2+1) - (k-1)^2\Big)f(x),$$

while the alternatives in $\mathcal{P}f^{(\alpha)}(x^{k-1},x^2)$ give

$$f(x^{k+1}) = \Big(\alpha(k^2-2k+5) - (k-3)^2\Big)f(x).$$

We get $f(x^{k+1}) = f(x) = 0$, a contradiction to $f(x^{k+1}) \neq (k+1)^2 f(x)$. Therefore, (2.10) must hold for $n = k+1$. By induction, we conclude that (2.10) holds for all non-negative integers n as desired. □

Lemma 2.5. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$ and $f(e) = 0$. If there exists $a \in G$ such that $f(a^2) \neq 4f(a)$, then $f(a) \neq 0$ and one of the following properties holds.*

(1) $\alpha = -1$ *and*

$$f(a^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ |n| f(a) & \text{if } 3 \mid (|n| - 1), \\ -|n| f(a) & \text{otherwise} \end{cases} \tag{2.11}$$

for all integers n.

(2) $\alpha = -\frac{1}{2}$ *and*

$$f(a^n) = \begin{cases} 0 & \textit{if } 5 \mid n, \\ f(a) & \textit{if } 5 \mid (n \pm 1), \\ -f(a) & \textit{otherwise} \end{cases} \tag{2.12}$$

for all integers n.

(3) $\alpha = 0$ *and*

$$f(a^n) = \begin{cases} 0 & \textit{if } n \textit{ is even}, \\ f(a) & \textit{otherwise} \end{cases} \tag{2.13}$$

for all integers n.

(4) $\alpha = \frac{1}{2}$ *and*

(iv.1)

$$f(a^n) = \begin{cases} 0 & \textit{if } 3 \mid n, \\ f(a) & \textit{otherwise} \end{cases} \tag{2.14}$$

for all integers n, or

(iv.2)

$$f(a^n) = \begin{cases} 0 & \textit{if } 9 \mid n, \\ 3f(a) & \textit{if } 9 \mid (n \pm 3), \\ f(a) & \textit{otherwise} \end{cases} \tag{2.15}$$

for all integers n.

Proof. It should be noted that by Lemma 2.2, f is even. Assume that there exists $a \in G$ such that $f(a^2) \neq 4f(a)$. By Lemma 2.3, we get $\alpha \notin \{0, \pm\frac{1}{2}, -1\}$. By the alternatives in $\mathcal{P}f^{(\alpha)}(a,a)$, we have $f(a^2) = 2\alpha f(a)$. If $f(a) = 0$, then $f(a^2) = 0$, a contradiction to $f(a^2) \neq 4f(a)$. Thus we must get $f(a) \neq 0$.

Case (i). Assume that $\alpha = -1$. We get $f(a^2) = -2f(a)$. The alternatives in $\mathcal{P}f^{(-1)}(a^2,a)$ give $f(a^3) \in \{-3f(a), 0\}$ Suppose $f(a^3) \neq 0$. Hence $f(a^3) = -3f(a)$. The alternatives in $\mathcal{P}f^{(-1)}(a^3,a)$ give $f(a^4) \in \{-2f(a), 4f(a)\}$, while the alternatives in $\mathcal{P}f^{(-1)}(a^2,a^2)$ give

$$f(a^4) \in \{-4f(a), 4f(a)\}. \tag{2.16}$$

Hence $f(a^4) = 4f(a)$. The alternatives in $\mathcal{P}f^{(-1)}(a^4,a)$ give $f(a^5) \in \{13f(a), -2f(a)\}$, while the alternatives in $\mathcal{P}f^{(-1)}(a^3,a^2)$ give $f(a^4) \in \{-11f(a), 4f(a)\}$. It is a contradiction. Therefore, we must get $f(a^3) = 0$. By (2.16) and the alternatives in $\mathcal{P}f^{(-1)}(a^3,a)$, we get $f(a^4) = 4f(a)$. The alternatives in $\mathcal{P}f^{(-1)}(a^4,a)$ give $f(a^5) \in \{10f(a), -5f(a)\}$, while the alternatives in $\mathcal{P}f^{(-1)}(a^3,a^2)$

give $f(a^5) \in \{-5f(a), f(a)\}$. Hence $f(a^5) = -5f(a)$. Now we have

$$f(a^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ |n|f(a) & \text{if } 3 \mid (|n|-1), \\ -|n|f(a) & \text{otherwise} \end{cases} \tag{2.17}$$

for $n = 0, 1 \ldots, 5$. For each positive integer k, we consider as the following process:

1. We use the alternatives in $\mathcal{P}f^{(-1)}(a^{3k}, a^3)$ to get $f(a^{3k+3}) = 0$.
2. We use the alternatives in $\mathcal{P}f^{(-1)}(a^{3k+3}, a)$ and $\mathcal{P}f^{(-1)}(a^{3k+2}, a^2)$ to get $f(a^{3k+4}) = (3k+4)f(a)$.
3. We use the alternatives in $\mathcal{P}f^{(-1)}(a^{3k+4}, a)$ and $\mathcal{P}f^{(-1)}(a^{3k+3}, a^2)$ to get $f(a^{3k+5}) = -(3k+5)f(a)$.

By induction and the evenness of f, we conclude that (2.17) holds for all integers n.

Case (ii). Assume that $\alpha = -\frac{1}{2}$. We get $f(a^2) = -f(a)$. The alternatives in $\mathcal{P}f^{(-1/2)}(a^2, a)$ give $f(a^3) = -f(a)$. The alternatives in $\mathcal{P}f^{(-1/2)}(a^3, a)$ give $f(a^4) = f(a)$. The alternatives in $\mathcal{P}f^{(-1/2)}(a^4, a)$ give $f(a^5) \in \{5f(a), 0\}$, while the alternatives in $\mathcal{P}f^{(-1/2)}(a^3, a^2)$ give $f(a^5) \in \{-5f(a), 0\}$. Hence $f(a^5) = 0$. Now we have

$$f(a^n) = \begin{cases} 0 & \text{if } 5 \mid n, \\ f(a) & \text{if } 5 \mid (n \pm 1), \\ -f(a) & \text{otherwise} \end{cases} \tag{2.18}$$

for $n = 0, 1, \ldots, 5$. For each positive integer k, we consider as the following process:

1. We use the alternatives in $\mathcal{P}f^{(-1/2)}(a^{5k-1}, a^2)$ to obtain $f(a^{5k+1}) = f(a)$.
2. We use the alternatives in $\mathcal{P}f^{(-1/2)}(a^{5k+1}, a)$ and $\mathcal{P}f^{(-1/2)}(a^{5k}, a^2)$ to get $f(a^{5k+2}) = -f(a)$.
3. We use the alternatives in $\mathcal{P}f^{(-1/2)}(a^{5k+2}, a)$ to get $f(a^{5k+3}) = -f(a)$.
4. We use the alternatives in $\mathcal{P}f^{(-1/2)}(a^{5k+3}, a)$ to get $f(a^{5k+2}) = f(a)$.

By induction and the evenness of f, we conclude that (2.18) holds for all integers n.

Case (iii). Assume that $\alpha = 0$. We get $f(a^2) = 0$ For each positive integer k, the alternatives in $\mathcal{P}f^{(0)}(a^{2k}, a^2)$ give $f(a^{2k+2}) = 0$. By induction and the evenness of f, we conclude that $f(a^{2n}) = 0$ for all integers n. The alternatives in $\mathcal{P}f^{(0)}(a^2, a)$ give $f(a^3) \in \{f(a), -f(a)\}$. Suppose $f(a^3) \neq f(a)$. Then we get $f(a^3) = -f(a)$. The alternatives in $\mathcal{P}f^{(0)}(a^4, a)$ give $f(a^5) \in \{3f(a), f(a)\}$, while the alternatives in $\mathcal{P}f^{(0)}(a^3, a^2)$ give $f(a^5) \in \{-3f(a), -f(a)\}$. Hence $f(a) = 0$ and then $f(a^3) = 0$, a contradiction to $f(a^3) \neq f(a)$. Thus $f(a^3) = f(a)$. Let k be a positive integer. Assume that $f(a^{2k+1}) = f(a)$ but $f(a^{2(k+1)+1}) \neq f(a)$. The alternatives in $\mathcal{P}f^{(0)}(a^{2(k+1)}, a)$ give $f(a^{2(k+1)+1}) = -f(a)$. The alternatives in $\mathcal{P}f^{(0)}(a^{2(k+1)+2}, a)$ give $f(a^{2(k+1)+3}) \in \{3f(a), f(a)\}$, while the alternatives in $\mathcal{P}f^{(0)}(a^{2(k+1)+1}, a^2)$ give $f(a^{2(k+1)+3}) \in \{-3f(a), -f(a)\}$. It is a contradiction. Therefore, we must have $f(a^{2(k+1)+1}) = f(a)$. By induction and the evenness of f, we get property (3) as desired.

Case (iv). Assume that $\alpha = \frac{1}{2}$. We get $f(a^2) = f(a)$.

Case (iv.1). Suppose $f(a^3) = 0$. For each positive integer k, the alternatives in $\mathcal{P}f^{(1/2)}(a^{3k}, a^3)$ give $f(a^{3k+3}) = 0$. By induction and the eveness of f, we get $f(a^{3n}) = 0$ for all integers n. The alternatives in $\mathcal{P}f^{(1/2)}(a^3, a)$ give $f(a^4) \in \{f(a), -\frac{1}{2}f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^2, a^2)$ give $f(a^4) \in \{4f(a), f(a)\}$. Hence $f(a^4) = f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^4, a)$ give $f(a^5) \in \{4f(a), f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^3, a^2)$ give $f(a^5) \in \{f(a), -\frac{1}{2}f(a)\}$. Hence $f(a^5) = f(a)$. For each positive integer $k \geq 2$, we use the alternatives in $\mathcal{P}f^{(1/2)}(a^{3k}, a)$ and $\mathcal{P}f^{(1/2)}(a^{3k-1}, a^2)$ to get $f(a^{3k+1}) = f(a)$, and we use the alternatives in $\mathcal{P}f^{(1/2)}(a^{3k+1}, a)$ and $\mathcal{P}f^{(1/2)}(a^{3k}, a^2)$ to get $f(a^{3k+2}) = f(a)$. By induction and the evenness of f, we get property (4.1) as desired.

Case (iv.2). Suppose $f(a^3) \neq 0$. The alternatives in $\mathcal{P}f^{(1/2)}(a^2, a)$ give $f(a^3) = 3f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^3, a)$ give $f(a^4) \in \{7f(a), f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^2, a^2)$ give $f(a^4) \in \{4f(a), f(a)\}$. Hence $f(a^4) = f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^4, a)$ give $f(a^5) \in \{f(a), -2f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^3, a^2)$ give $f(a^5) \in \{7f(a), f(a)\}$. Hence $f(a^5) = f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^3, a^3)$ give $f(a^6) \in \{12f(a), 3f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^4, a^2)$ give $f(a^6) \in \{3f(a), 0\}$. Hence $f(a^6) = 3f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^6, a)$ give $f(a^7) \in \{7f(a), f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^5, a^2)$ give $f(a^7) \in \{f(a), -2f(a)\}$. Hence $f(a^7) = f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^7, a)$ give $f(a^8) \in \{f(a), -2f(a)\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^6, a^2)$ give $f(a^8) \in \{7f(a), f(a)\}$. Hence $f(a^8) = f(a)$. The alternatives in $\mathcal{P}f^{(1/2)}(a^8, a)$ give $f(a^9) \in \{3f(a), 0\}$, while the alternatives in $\mathcal{P}f^{(1/2)}(a^6, a^3)$ give $f(a^9) \in \{9f(a), 0\}$. Hence $f(a^9) = 0$. Now we have

$$f(a^n) = \begin{cases} 0 & \text{if } 9 \mid n, \\ 3f(a) & \text{if } 9 \mid (n \pm 3), \\ f(a) & \text{otherwise}, \end{cases} \tag{2.19}$$

for $n = 0, 1, \ldots, 9$. For each positive integer k, we consider as the following process:

(1) We use $\mathcal{P}f^{(1/2)}(a^{9k}, a^9)$ to get $f(a^{9k+9}) = 0$.
(2) We use $\mathcal{P}f^{(1/2)}(a^{9k}, a^3)$ and $\mathcal{P}f^{(1/2)}(a^{9k-3}, a^6)$ to get $f(a^{9k+3}) = 3f(a)$.
(3) We use $\mathcal{P}f^{(1/2)}(a^{9k+3}, a^3)$ and $\mathcal{P}f^{(1/2)}(a^{9k}, a^6)$ to get $f(a^{9k+6}) = 3f(a)$.
(4) We use $\mathcal{P}f^{(1/2)}(a^{9k}, a)$ and $\mathcal{P}f^{(1/2)}(a^{9k-1}, a^2)$ to get $f(a^{9k+1}) = f(a)$.
(5) We use $\mathcal{P}f^{(1/2)}(a^{9k+1}, a)$ and $\mathcal{P}f^{(1/2)}(a^9, a^2)$ to get $f(a^{9k+2}) = f(a)$.
(6) We use $\mathcal{P}f^{(1/2)}(a^{9k+3}, a)$ and $\mathcal{P}f^{(1/2)}(a^{9k+2}, a^2)$ to get $f(a^{9k+4}) = f(a)$.
(7) We use $\mathcal{P}f^{(1/2)}(a^{9k+4}, a)$ and $\mathcal{P}f^{(1/2)}(a^{9k+3}, a^2)$ to get $f(a^{9k+5}) = f(a)$.
(8) We use $\mathcal{P}f^{(1/2)}(a^{9k+6}, a)$ and $\mathcal{P}f^{(1/2)}(a^{9k+5}, a^2)$ to get $f(a^{9k+7}) = f(a)$.
(9) We use $\mathcal{P}f^{(1/2)}(a^{9k+7}, a)$ and $\mathcal{P}f^{(1/2)}(a^{9k+6}, a^2)$ to get $f(a^{9k+8}) = f(a)$.

By induction and the evenness of f, (2.19) must hold for all integers n. □

Lemma 2.6. *Let* $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$. *If* $f(x^2) = 4f(x)$ *for all* $x \in G$, *then* f *is quadratic.*

Proof. It can be proved by contradiction as follows. Assume that

$$f(x^2) = 4f(x) \tag{2.20}$$

for all $x \in G$ but f is not quadratic. There exists $x, y \in G$ such that

$$f(xy) + f(xy^{-1}) \neq 2f(x) + 2f(y). \tag{2.21}$$

Thus $\mathcal{P}f^{(\alpha)}(x, y)$ give

$$f(xy) + f(xy^{-1}) = \alpha(f(x) + f(y)). \tag{2.22}$$

By (2.20) and (2.21), $\mathcal{P}f^{(\alpha)}(xy, xy^{-1})$ give

$$4f(x) + 4f(y) = \alpha(f(xy) + f(xy^{-1})). \tag{2.23}$$

By (2.22) and (2.23), we get

$$f(x) + f(y) = 0 \quad \text{and} \quad f(xy) + f(xy^{-1}) = 0,$$

a contradiction to (2.21). □

We now use these lemmas to prove the main theorem.

Theorem 2.1. *Let $f \in \mathcal{A}^{(\alpha)}_{(G,H)}$. Then f is quadratic or one of the following properties holds.*

(1) $\alpha = -1$ *and there exists $a \in G$ such that (2.11).*
(2) $\alpha = -\frac{1}{2}$ *and there exists $a \in G$ such that (2.12).*
(3) $\alpha = 0$ *and there exists $a \in G$ such that (2.13).*
(4) $\alpha = \frac{1}{2}$ *and there exists $a \in G$ such that (2.14) or (2.15).*
(5) $\alpha = 1$ *and f is constant.*

Proof. We consider the possible cases of $f(e)$ as follows. If $f(e) \neq 0$, then Lemma 2.1 gives the property (5). Assume that $f(e) = 0$. If $f(x^2) = 4f(x)$ for all $x \in G$, then by Lemma 2.6, we obtain that f is quadratic. If there exists $a \in G$ such that $f(a^2) \neq 4f(a)$, then Lemma 2.5 gives properties (1)–(4). □

Theorem 2.1 shows that f is not quadratic in the case when $\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$.

3. HYERS-ULAM STABILITY

In this section, we let $(G, \cdot)$ be an abelian group and $(E, \|\cdot\|)$ be a Banach space. Given a rational number $\alpha \notin \{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$ and a function $f : G \to E$, for every pair of $x, y \in G$ and for $\delta \geq 0$, we denote the statement

$$\mathcal{S}f^{(\alpha)}(x, y) = \Big(\|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\| \leq \delta \quad \text{or} \\ \|f(xy) + f(xy^{-1}) - \alpha f(x) - \alpha f(y)\| \leq \delta\Big).$$

The set of all solutions to the statement $\mathcal{S}f^{(\alpha)}(x, y)$ will be denoted by $\mathcal{A}^{(\alpha)}_{(G,E)}$, i.e.,

$$\mathcal{A}^{(\alpha)}_{(G,E)} = \{f : G \to E \mid \mathcal{S}f^{(\alpha)}(x, y) \text{ for all } x, y \in G\},$$

For a rational number λ, we define

$$\mathcal{M}(\lambda) = \begin{cases} |\lambda|^{-1} & \text{if} \quad 0 < |\lambda| < 1, \\ |\lambda| & \text{if} \quad |\lambda| \geq 1, \\ 1 & \text{if} \quad \lambda = 0. \end{cases}$$

It should be noted that

(1) $1 \leq \mathcal{M}(\lambda)$;
(2) $|\lambda| \leq \mathcal{M}(\lambda)$;
(3) $|\lambda|^{-1} \leq \mathcal{M}(\lambda)$ if $\lambda \neq 0$.

We denote $\Lambda = \{0, \pm1, \pm2, \pm3, \pm2 \pm \sqrt{3}, 5\}$ and

$$\mathbf{M} = \max_{\sigma_1,\sigma_2 \in \Lambda} \{\mathcal{M}(\sigma_1 + \sigma_2\alpha)\}.$$

Lemma 3.1. *Let $f \in \mathcal{SA}^{(\alpha)}_{(G,E)}$. Then $\|f(e)\| \leq \mathbf{M}\delta$.*

Proof. The proof of this lemma is complete by the alternatives in $\mathcal{S}f^{(\alpha)}(e,e)$. □

Lemma 3.2. *Let $f \in \mathcal{SA}^{(\alpha)}_{(G,E)}$. Then $\|f(x^{-1}) - f(x)\| \leq 8\mathbf{M}^5\delta$ for all $x \in G$.*

Proof. It should be noted that Lemma 3.1 gives $\|f(e)\| \leq \mathbf{M}\delta$. Let $x \in G$. Without loss of generality, we suppose $\|f(x^{-1}) - f(x)\| > 3\mathbf{M}\delta$. The alternatives in $\mathcal{S}f^{(\alpha)}(e,x)$ give

$$\|f(x^{-1}) - (\alpha - 1)f(x)\| \leq 2\mathbf{M}^2\delta,$$

while the alternatives in $\mathcal{S}f^{(\alpha)}(x,e)$ give

$$\|(\alpha - 1)f(x^{-1}) - f(x)\| \leq 2\mathbf{M}^2\delta.$$

Hence

$$\begin{aligned} \|f(x^{-1}) - f(x)\| &\leq |\alpha^{-1}|\|f(x^{-1}) - (\alpha - 1)f(x)\| + \|(\alpha - 1)f(x^{-1}) - f(x)\| \\ &\leq 4\mathbf{M}^3\delta \end{aligned}$$

□

Lemma 3.3. *Let $f \in \mathcal{SA}^{(\alpha)}_{(G,E)}$. Then $\|4^{-1}f(x^2) - f(x)\| \leq 355\mathbf{M}^{13}\delta$ for all $x \in G$.*

Proof. It should be noted that, Lemma 3.1 gives $\|f(e)\| \leq \mathbf{M}\delta$. Let $x \in G$. Without loss of generality, we suppose $\|f(x^2) - 4f(x)\| > \delta$. The alternatives in $\mathcal{S}f^{(\alpha)}(x,x)$ give

$$\|f(x^2) - 2\alpha f(x)\| \leq 2\mathbf{M}\delta. \tag{3.1}$$

The alternatives in $\mathcal{S}f^{(\alpha)}(x^2,x^2)$ give

$$\|f(x^4) - 4f(x^2)\| \leq 2\mathbf{M}\delta \quad \text{or} \quad \|f(x^4) - 2\alpha f(x^2)\| \leq 2\mathbf{M}\delta. \tag{3.2}$$

By (3.1) and (3.2), we obtain that

$$\|f(x^4) - 8\alpha f(x)\| \leq 10\mathbf{M}\delta \quad \text{or} \quad \|f(x^4) - 4\alpha^2 f(x)\| \leq 6\mathbf{M}^2\delta. \tag{3.3}$$

Eliminating $f(x^2)$ from (3.1) and the alternatives in $\mathcal{S}f^{(\alpha)}(x^2,x)$, we have

$$\|f(x^3)-(1+4\alpha)f(x)\|\le 5\mathbf{M}\delta \quad \text{or} \quad \|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\|\le 3\mathbf{M}^2\delta. \tag{3.4}$$

Case (i). Assume that $\|f(x^3)-(1+4\alpha)f(x)\|\le 5\mathbf{M}\delta$. Eliminating $f(x^2)$ from (3.1) and the alternatives in $\mathcal{S}f^{(\alpha)}(x^3,x)$, we get

$$\|f(x^4)-2f(x^3)-(2-2\alpha)f(x)\|\le 3\mathbf{M}\delta \quad \text{or} \quad \|f(x^4)-\alpha f(x^3)+\alpha f(x)\|\le 3\mathbf{M}\delta. \tag{3.5}$$

By $\|f(x^3)-(1+4\alpha)f(x)\|\le 5\mathbf{M}\delta$ and (3.5), we obtain that

$$\|f(x^4)-(4+6\alpha)f(x)\|\le 13\mathbf{M}\delta \quad \text{or} \quad \|f(x^4)-4\alpha^2 f(x)\|\le 8\mathbf{M}^2\delta. \tag{3.6}$$

Suppose $\|f(x^4)-4\alpha^2 f(x)\|>8\mathbf{M}^2\delta$. Hence we eliminate $f(x^4)$ from (3.3) and (3.6) to get

$$\|f(x)\|\le 12\mathbf{M}^2\delta$$

and then we conclude that

$$\|f(x^4)-4\alpha^2 f(x)\|\le \|f(x^4)-8\alpha f(x)\|+\|(4\alpha^2-8\alpha)f(x)\|\le 58\mathbf{M}^4\delta. \tag{3.7}$$

Eliminating $f(x^3)$ from $\|f(x^3)-(1+4\alpha)f(x)\|\le 5\mathbf{M}\delta$ and the alternatives in $\mathcal{S}f^{(\alpha)}(x^4,x)$, we get

$$\begin{aligned}&\|f(x^5)-2f(x^4)-(1-4\alpha)f(x)\|\le 6\mathbf{M}\delta \quad \text{or}\\ &\|f(x^5)-\alpha f(x^4)+(1+3\alpha)f(x)\|\le 6\mathbf{M}\delta.\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned}&\|f(x^5)-(1-4\alpha+8\alpha^2)f(x)\|\le 122\mathbf{M}^4\delta \quad \text{or}\\ &\|f(x^5)-(-1-3\alpha+4\alpha^3)f(x)\|\le 64\mathbf{M}^5\delta.\end{aligned} \tag{3.9}$$

Eliminating $f(x^3)$ from $\|f(x^3)-(1+4\alpha)f(x)\|\le 5\mathbf{M}\delta$ and the alternatives in $\mathcal{S}f^{(\alpha)}(x^3,x^2)$, we get

$$\begin{aligned}&\|f(x^5)-2f(x^2)-(1+8\alpha)f(x)\|\le 11\mathbf{M}\delta \quad \text{or}\\ &\|f(x^5)-\alpha f(x^2)-(-1+\alpha+4\alpha^2)f(x)\|\le 6\mathbf{M}^2\delta.\end{aligned} \tag{3.10}$$

By (3.1) and (3.10), we obtain that

$$\|f(x^5)-(1+12\alpha)f(x)\|\le 15\mathbf{M}\delta \quad \text{or} \quad \|f(x^5)-(-1+\alpha+6\alpha^2)f(x)\|\le 8\mathbf{M}^2\delta. \tag{3.11}$$

By (3.9) and (3.11), we get

$$|m|\|f(x)\|\le 137\mathbf{M}^6\delta,$$

when $m\in\left\{\alpha,(1-2\alpha),\left(3-4(1+\alpha)^2\right),\alpha(1+2\alpha)\right\}$. Hence $\|f(x)\|\le 236\mathbf{M}^8\delta$ and then by (3.1), we have $\|f(x^2)\|\le 474\mathbf{M}^7\delta$. Therefore, we get

$$\|4^{-1}f(x^2)-f(x)\|\le 355\mathbf{M}^7\delta.$$

Case (ii). Assume that $\|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\| \le 3\mathbf{M}^2\delta$. Eliminating $f(x^2)$ from (3.1) and the alternatives in $\mathcal{S}f^{(\alpha)}(x^3,x)$, we get (3.5). By $\|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\| \le 3\mathbf{M}^2\delta$ and (3.5), we obtain that

$$\|f(x^4)-4\alpha^2 f(x)\| \le 9\mathbf{M}^2\delta \quad \text{or} \quad \|f(x^4)-(-2\alpha+\alpha^2+2\alpha^3)f(x)\| \le 6\mathbf{M}^3\delta. \tag{3.12}$$

Suppose $\|f(x^4)-4\alpha^2 f(x)\| > 9\mathbf{M}^2\delta$. Eliminating $f(x^4)$ from (3.3) and (3.12), we get

$$|\alpha(2-\alpha)(5+2\alpha)|\|f(x)\| \le 16\mathbf{M}^3\delta.$$

Hence $|5+2\alpha|\|f(x)\| \le 16\mathbf{M}^5\delta$. If $\alpha \neq -\frac{5}{2}$, then $\|f(x)\| \le 16\mathbf{M}^6\delta$. Suppose $\alpha = -\frac{5}{2}$. From (3.1), $\|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\| \le 3\mathbf{M}^2\delta$ and (3.12), we get

$$\begin{aligned}&\|f(x^2)+5f(x)\| \le 2\mathbf{M}\delta, \quad \|f(x^3)-9f(x)\| \le 3\mathbf{M}^2\delta \quad \text{and}\\ &\|f(x^4)+20f(x)\| \le 6\mathbf{M}^3\delta,\end{aligned} \tag{3.13}$$

respectively. Eliminating $f(x^3)$ and $f(x^4)$ from the alternatives in $\mathcal{S}f^{(-5/2)}(x^4,x)$ and (3.13), we get

$$\|f(x^5)+47f(x)\| \le 16\mathbf{M}^3\delta \quad \text{or} \quad \left\|f(x^5)-\frac{77}{2}f(x)\right\| \le 19\mathbf{M}^3\delta. \tag{3.14}$$

Eliminating $f(x^2)$ and $f(x^3)$ from the alternatives in $\mathcal{S}f^{(-5/2)}(x^3,x^2)$ and (3.13), we get

$$\|f(x^5)-7f(x)\| \le 11\mathbf{M}^2\delta \quad \text{or} \quad \|f(x^5)+11f(x)\| \le 14\mathbf{M}^2\delta. \tag{3.15}$$

By (3.14) and (3.15), we conclude that $\|f(x)\| \le \mathbf{M}^3\delta \le 16\mathbf{M}^6\delta$. By (3.3), we get

$$\|f(x^4)-4\alpha^2 f(x)\| \le \|f(x^4)-8\alpha f(x)\| + \|(4\alpha^2-8\alpha)f(x)\| \le 74\mathbf{M}^8\delta. \tag{3.16}$$

Eliminating $f(x^3)$ from $\|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\| \le 3\mathbf{M}^2\delta$ and the alternatives in $\mathcal{P}f^{(\alpha)}(x^4,x)$, we have

$$\begin{aligned}&\|f(x^5)-2f(x^4)+(-3+\alpha+2\alpha^2)f(x)\| \le 4\mathbf{M}^2\delta \text{ or}\\ &\|f(x^5)-\alpha f(x^4)+(-1+2\alpha^2)f(x)\| \le 4\mathbf{M}^2\delta.\end{aligned} \tag{3.17}$$

By (3.16) and (3.17), we get

$$\begin{aligned}&\|f(x^5)+(-3+\alpha-6\alpha^2)f(x)\| \le 152\mathbf{M}^8\delta \text{ or}\\ &\|f(x^5)+(-1+2\alpha^2-4\alpha^3)f(x)\| \le 78\mathbf{M}^9\delta.\end{aligned} \tag{3.18}$$

Eliminating $f(x^3)$ from $\|f(x^3)-(-1+\alpha+2\alpha^2)f(x)\| \le 3\mathbf{M}^2\delta$ and the alternatives in $\mathcal{P}f^{(\alpha)}(x^3,x^2)$, we have

$$\begin{aligned}&\|f(x^5)-2f(x^4)+(3-2\alpha-4\alpha^2)f(x)\| \le 7\mathbf{M}^2\delta\\ &\text{or } \|f(x^5)-\alpha f(x^4)+(1+\alpha-\alpha^2-2\alpha^3)f(x)\| \le 4\mathbf{M}^3\delta.\end{aligned} \tag{3.19}$$

By (3.1) and (3.19), we get

$$\begin{aligned}&\|f(x^5)+(3-6\alpha-4\alpha^2)f(x)\| \le 11\mathbf{M}^2\delta \text{ or}\\ &\|f(x^5)+(1+\alpha-3\alpha^2-2\alpha^3)f(x)\| \le 6\mathbf{M}^3\delta.\end{aligned} \tag{3.20}$$

By (3.18) and (3.20), we conclude that

$$|m|\|f(x)\| \leq 163\mathbf{M}^{10}\delta,$$

when $m \in \Big\{(3-2\alpha), (2+\alpha+2\alpha^2), (1+4\alpha^2), (1-\alpha)(1+2\alpha)\Big\}$. Hence $\|f(x)\| \leq 163\mathbf{M}^{12}\delta$ and then by (3.1), we have $\|f(x^2)\| \leq 165\mathbf{M}^{13}\delta$. Therefore, we get

$$\|4^{-1}f(x^2) - f(x)\| \leq 205\mathbf{M}^{13}\delta.$$

□

Theorem 3.1. *If $f \in \mathcal{SA}^{(\alpha)}_{(G,E)}$, then there exists a unique function $q : G \to E$ such that q satisfies the Quadratic functional equation (1.2) and*

$$\|f(x) - q(x)\| \leq \varepsilon$$

for all $x \in G$ when $\varepsilon = \frac{4}{3} \cdot 355\mathbf{M}^{13}\delta$.

Proof. Let $f \in \mathcal{SA}^{(\alpha)}_{(G,E)}$. By Lemma 3.3, we obtain that

$$\|4^{-1}f(x^2) - f(x)\| \leq \tfrac{3}{4}\varepsilon \quad \text{for all } x \in G.$$

Let $x \in G$. For a positive integer n, we get

$$\|4^{-n}f(x^{2^n}) - f(x)\| \leq \sum_{i=0}^{n-1} 4^{-i}\|4^{-1}f\Big(\big(x^{2^i}\big)^2\Big) - f\big(x^{2^i}\big)\| \leq \sum_{i=0}^{n-1} 4^{-i}\frac{3}{4}\varepsilon.$$

Since $\displaystyle\sum_{i=0}^{n-1} 4^{-i}\frac{3}{4}\varepsilon \leq \sum_{i=0}^{\infty} 4^{-i}\frac{3}{4}\varepsilon = \varepsilon$, we get

$$\|4^{-n}f(x^{2^n}) - f(x)\| \leq \varepsilon. \tag{3.21}$$

Consider the sequence $\Big\{4^{-n}f(x^{2^n})\Big\}$. For all positive integers m, n with $m < n$, we use (3.21) to get

$$\|4^{-n}f(x^{2^n}) - 4^{-m}f(x^{2^m})\| = 4^{-m}\Big\|4^{-(n-m)}f\Big(\big(x^{2^m}\big)^{2^{n-m}}\Big) - f(x^{2^m})\Big\| \leq 4^{-m}\varepsilon.$$

Hence $\Big\{4^{-n}f(x^{2^n})\Big\}$ is a Cauchy sequence for all $x \in G$. We can define a function $q : G \to E$ by

$$q(x) = \lim_{n\to\infty} 4^{-n}f(x^{2^n}) \quad \text{for all } x \in G.$$

Multiplying the alternatives in $\mathcal{S}f^{(\alpha)}(x^{2^n}, y^{2^n})$ by 4^{-n}, we get

$$\|4^{-n}f\Big((xy)^{2^n}\Big) + 4^{-n}f\Big((xy^{-1})^{2^n}\Big) - 2\cdot 4^{-n}f(x^{2^n}) - 2\cdot 4^{-n}f(y^{2^n})\| \leq 4^{-n}\delta \text{ or}$$
$$\|4^{-n}f\Big((xy)^{2^n}\Big) + 4^{-n}f\Big((xy^{-1})^{2^n}\Big) - \alpha 4^{-n}f(x^{2^n}) - \alpha 4^{-n}f(y^{2^n})\| \leq 4^{-n}\delta,$$

for all $x, y \in G$ and for all positive integers n. Taking the limit as $n \to \infty$, we get $q \in \mathcal{A}^{(\alpha)}_{(G,E)}$. Since $\alpha \notin \{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$, by Theorem 2.1, we obtain that q is quadratic. From (3.21), as $n \to \infty$, we have

$$\|f(x) - q(x)\| \leq \varepsilon \quad \text{for all } x \in G.$$

To prove the uniqueness of q, let $\tilde{q} : G \to E$ satisfy (1.2) and

$$\|f(x) - \tilde{q}(x)\| \leq \varepsilon \qquad \text{for all } x \in G.$$

Since q and $\tilde{q}$ are qudratic, we have

$$q(x^{2^n}) = 4^n q(x) \quad \text{ and } \quad \tilde{q}(x^{2^n}) = 4^n \tilde{q}(x)$$

for all $x \in G$ and for all positive integers n. Hence

$$\|q(x) - \tilde{q}(x)\| \leq 4^{-n}\Big(\|f(x^{2^n}) - q(x^{2^n})\| + \|f(x^{2^n}) - \tilde{q}(x^{2^n})\|\Big) \leq 4^{-n}2\varepsilon.$$

As $n \to \infty$, we get $q(x) = \tilde{q}(x)$ for all $x \in G$. The proof is complete as desired. □

From the results of Batko [6], Nakmahachalasint [10], and the above theorem, we now conclude that the stability of (1.4) is treated for all rational numbers $\alpha \notin \{0, \pm\frac{1}{2}, \pm 1\}$. It is still an interesting question whether the alternative quadratic equation (1.4) will be stable in the remaining cases.

4. Conclusion

The alternative quadratic functional equation (1.4) was investigated. A criterion was established for the existence of the general solution, demonstrating that for $\alpha \neq \pm 2$, the solutions correspond to those of the classical quadratic functional equation. Furthermore, the Hyers-Ulam stability of the equation was proved for mappings from an abelian group to a Banach space, provided that α is a suitable rational number.

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