

Discrete Potential Theory on Schrödinger Products

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Abstract. This article develops a discrete potential-theoretic framework for Cartesian products of infinite Schrödinger networks. We introduce the notion of product Schrödinger networks and define separately q -supermean and separately q -potential functions, establishing their structural and comparison properties. Integral representation theorems and a Riesz-type decomposition are proved for non-negative separately q -supermean functions on product networks. Furthermore, we present a classification of product Schrödinger networks based on bounded and unbounded mean behavior, showing how the q -mean (q -harmonic) and potential-theoretic nature of the product network are determined by the corresponding properties of the individual factor networks.

1. INTRODUCTION

Discrete potential theory has emerged as a central area of analysis, connecting classical mean (harmonic) analysis, discrete mathematics, and probability theory [2, 7]. In particular, Schrödinger operators on finite and infinite networks provide a powerful framework for studying mean and supermean functions in discrete settings [6, 18]. Originating in classical analysis and mathematical physics, potential theory was adapted to discrete finite networks by Bendito, Carmona, and Encinas [9], leading to discrete analogues of the Dirichlet problem, capacities, and Green's functions [22]. The introduction of discrete Schrödinger operators of the form $\Delta u = qu$, with non-negative potential q , further extends this framework [17]. Understanding potential theory on Cartesian product networks is essential for the analysis of higher-dimensional discrete models arising in random walks, statistical physics, and network-based Schrödinger operators.

When networks are combined through Cartesian products, the resulting structure introduces new analytical challenges, since function-theoretic properties must be examined both on the individual factor networks and through their interactions. Despite these developments, a systematic

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potential-theoretic treatment of Schrödinger operators on Cartesian product networks, particularly at the level of supermean and potential functions, remains largely unexplored. Such product networks have been studied in the contexts of random walks [14, 19] and mean function theory. In the product setting, supermean functions with respect to each variable separately provide a natural generalization of the classical notion.

The main novelty of this work lies in extending discrete potential theory to Cartesian products of infinite Schrödinger networks and in establishing representation and decomposition results in this setting. This paper develops a unified potential-theoretic framework for discrete Schrödinger operators on Cartesian product networks [1]. We extend earlier work on supermean functions [5, 15] by establishing integral representation results for separately supermean functions, building on recent characterizations by Nathiya and Premalatha [20]. We also introduce separately q -potentials associated with the discrete Schrödinger equation $\Delta u - qu = 0$, revealing new structural properties and extending known results to the product setting [9, 17]. A central contribution is the derivation of integral representation formulas that extend the Riesz decomposition theorem to Cartesian products and relate global behavior to component-wise structure [15, 20].

We further classify bounded and unbounded mean behavior for supermean and q -potential functions, linking growth properties to the geometry and spectral characteristics of product networks [10, 16]. After reviewing preliminaries from discrete potential theory [13, 21, 23], we develop the main results for separately q -supermean functions and separately q -potentials, treating both bounded and unbounded cases. We also derive integral representation formulas and discuss their connection with Green's functions [8, 12]. Our approach emphasizes constructive methods adapted to the discrete product structure, building on existing developments in potential theory and Schrödinger operators on product networks [5, 8].

2. PRELIMINARIES

Let $\{X, t, q\}$ be an infinite Schrödinger network defined on a locally finite, connected infinite graph X without self-loops. If x and y are neighbouring vertices, we write $x \sim y$. If $E \subset X$, a vertex x is said to be an interior vertex of E if x and all its neighbours belong to E . The set of all interior vertices of E is denoted by $\overset{\circ}{E}$, and the boundary of E is defined by $\partial E = E \setminus \overset{\circ}{E}$.

A graph X is regarded as an *infinite network* when its vertex set X is infinite and it is endowed with a transition index $t(x, y)$ for each ordered pair of vertices $x, y \in X$. These indices satisfy $t(x, y) \geq 0$ for all x, y , and $t(x, y) > 0$ if and only if $x \sim y$. Symmetry is not imposed; hence it is possible that $t(x, y) \neq t(y, x)$. For every vertex $x \in X$, the total transition weight

$$t(x) = \sum_{y \in X} t(x, y)$$

is finite and strictly positive.

A Schrödinger network $\{X, t, q\}$ is an infinite network $\{X, t\}$ equipped with a real-valued function q on X satisfying $q(x) \geq 0, x \in X$.

Let f be a real-valued function on the Schrödinger network $\{X, t, q\}$. The Schrödinger Laplacian [4] Δ_q is defined by

$$\Delta_q f(x) = \sum_{y \sim x} t(x, y) [f(y) - f(x)] - q(x)f(x), \quad x \in X.$$

A function f defined on a subset $E \subset X$ is said to be q -supermean, q -mean, or q -submean on E if

$$\Delta_q f(x) \leq 0, \quad \Delta_q f(x) = 0, \quad \Delta_q f(x) \geq 0,$$

respectively, for all $x \in E$.

Remark 2.1. In this paper, the term mean is used interchangeably with harmonic in the discrete Schrödinger setting.

Let $\{X_1, t_1, q_1\}$ and $\{X_2, t_2, q_2\}$ be two infinite Schrödinger networks. Their product Schrödinger network is the triple $\{X, t, q\}$, where $X = X_1 \times X_2$, $t = t_1 \times t_2$, and $q(x, y) = q_1(x) + q_2(y)$, $(x, y) \in X_1 \times X_2$.

The neighbours of $(x, y) \in X_1 \times X_2$ are the vertices (x_i, y) with $x_i \sim x$ in X_1 and (x, y_j) with $y_j \sim y$ in X_2 . The transition function t is given by

$$t((x, y), (x_i, y)) = t_1(x, x_i), \quad t((x, y), (x, y_j)) = t_2(y, y_j).$$

We assume that $q(x, y) \geq 0$ for all (x, y) and that there exists at least one vertex (x_0, y_0) such that $q(x_0, y_0) > 0$.

Let f be a function defined on a subset $E \subset X_1 \times X_2$. For fixed $y \in X_2$, define

$$E_y = \{x \in X_1 : (x, y) \in E\},$$

and for fixed $x \in X_1$, define

$$E^x = \{y \in X_2 : (x, y) \in E\}.$$

The function $f_y(x) = f(x, y)$ is defined on E_y , and $f^x(y) = f(x, y)$ is defined on E^x . Then

$$\Delta_q f(x, y) = \sum_{(x, y) \sim (a, b)} t((x, y), (a, b)) [f(a, b) - f(x, y)] - q(x, y)f(x, y),$$

Then for any $f(x, y)$ on X_1, X_2

$$\begin{aligned} \Delta_q f(x, y) &= \Delta_1 f_y(x) + \Delta_2 f^x(y) - q(x, y)f(x, y) \\ &= \Delta_1 f_y(x) + \Delta_2 f^x(y) - [q_1(x) + q_2(y)]f(x, y) \\ &= [\Delta_1 f_y(x) - q_1 f_y(x)] + [\Delta_2 f^x(y) - q_2 f^x(y)] \end{aligned}$$

and hence

$$\Delta_q f(x, y) = \Delta_{q_1} f_y(x) + \Delta_{q_2} f^x(y).$$

Definition 2.1 (*q-Dirichlet Problem*). Let $E \subset X_1 \times X_2$ be a domain, and let ∂E denote its boundary, The *q-Dirichlet problem* on the product network is to find a function $u(x, y)$ such that

$$\Delta_q u(x, y) = 0 \quad \text{for all } (x, y) \in \overset{\circ}{E}$$

and the function u must satisfy the boundary condition

$$u(x, y) = f(x, y) \quad \text{for all } (x, y) \in \partial E.$$

A function u satisfying both conditions is called the *q-mean solution* of the Dirichlet problem on the product network.

Definition 2.2 (*q-Poisson Modification*). Let $\{X, t, q\}$ be a product Schrödinger network, where $F \subset X$ be a finite set, and let u be any real valued function. The *q-Poisson modification* of u on F is defined by

$$P_F^q u(x) = \begin{cases} u(x), & x \notin F, \\ h(x), & x \in F, \end{cases}$$

where h is the unique *q-mean* function on F satisfying

$$h = u \quad \text{on } \partial F.$$

Definition 2.3 (*q-Perron Family*). Let $E \subset X_1 \times X_2, F \subset \overset{\circ}{E}$. A non-empty family P of real-valued functions on E is called a *lower-directed Δ_q -Perron family* on (E, F) if the following hold:

(i) For any $u_1, u_2 \in P$, there exists $u \in P$ such that

$$u \leq \inf(u_1, u_2) \quad \text{on } E.$$

(ii) For every $u \in P$ and every $a \in F$, the *q-Poisson modification* $P_a u$ belongs to P , and satisfies

$$P_a u \leq u \quad \text{on } E.$$

(iii) There exists a function $\varphi : E \rightarrow \mathbb{R}$ such that

$$u \geq \varphi \quad \text{on } E \quad \text{for all } u \in P.$$

3. SEPARATELY *q*-SUPERMEAN FUNCTIONS

Definition 3.1 (*Separately q-supermean*). For any $v(x, y)$ v is *separately q-supermean* iff $\Delta_{q_1}[v_y(x)] \leq 0$ and $\Delta_{q_2}[v^x(y)] \leq 0$.

3.1. Properties of Separately *q*-supermean functions.

Property 3.1. A *separately q-supermean* function in E is Δ_q -supermean function in E .

Proof. E is the subset of $X_1 \times X_2$. $(x, y) \in \overset{\circ}{E}$, $x \in \overset{\circ}{E}_y$ and $y \in \overset{\circ}{E}^x$, where E_y and E^x is in E . Now *separately q-supermean* implies

- $f_y(x) \in E_y$ is *q-supermean* on $E_y \in E \Rightarrow \Delta_{q_1} f_y(x) \leq 0$.
- $f^x(y) \in E^x$ is *q-supermean* on $E^x \in E \Rightarrow \Delta_{q_2} f^x(y) \leq 0$.

$$\begin{aligned}\Delta_q f(x, y) &= \Delta_{q_1} f_y(x) + \Delta_{q_2} f^x(y) \\ \Delta_q f(x, y) &\leq 0\end{aligned}$$

□

Lemma 3.1. Let $\{X_1, t_1, q_1\}$ and $\{X_2, t_2, q_2\}$ be Schrödinger networks. If ξ_1 is a non-negative q_1 -supermean function on X_1 and ξ_2 is a non-negative q_2 -supermean function on X_2 , then the function

$$v(x, y) = \xi_1(x) + \xi_2(y)$$

is a q -supermean function on $X_1 \times X_2$.

Proof. Let

$$v(x, y) = \xi_1(x) + \xi_2(y),$$

where ξ_1 and ξ_2 are non-negative q_1 -supermean and q_2 -supermean functions on X_1 and X_2 , respectively.

For fixed $y \in X_2$,

$$v_y(x) = \xi_1(x) + \xi_2(y),$$

so

$$\Delta_{q_1} v_y(x) = \Delta_{q_1} \xi_1(x) - q_1(x) \xi_2(y) \leq 0,$$

since $\Delta_{q_1} \xi_1(x) \leq 0$, $q_1(x) \geq 0$, and $\xi_2(y) \geq 0$.

Similarly, for fixed $x \in X_1$,

$$\Delta_{q_2} v^x(y) = \Delta_{q_2} \xi_2(y) - q_2(y) \xi_1(x) \leq 0.$$

Hence v is separately q -supermean on $X_1 \times X_2$. Therefore,

$$\Delta_q v(x, y) = \Delta_{q_1} v_y(x) + \Delta_{q_2} v^x(y) \leq 0,$$

and so v is q -supermean on $X_1 \times X_2$. □

Note: We do not say that $s(x, y)$ is a separately q -supermean function if $s_y(x)$ and $s^x(y)$ are q -potentials in X_1 and X_2 .

Property 3.2. If f, g are separately q -supermean on E of $X_1 \times X_2$, then for any non negative numbers α, β , the function $\alpha f + \beta g$ and $\inf(f, g)$ are separately q -supermean on E .

Proof. Consider for a fixed y , $f_y(x)$ and $g_y(x)$ is separately q -supermean on E , (i.e.), $\Delta_q f_y(x) \leq 0$ and $\Delta_q g_y(x) \leq 0$.

$$\begin{aligned}\Delta_q(\alpha f_y(x) + \beta g_y(x)) &= \alpha \Delta_q f_y(x) + \beta \Delta_q g_y(x) \\ &\leq 0\end{aligned}$$

which proves $\alpha f + \beta g$ is separately q -supermean on E . Suppose $f_y(x) = \inf(f, g)$, for a fixed y , then by definition $\inf(f, g)$ is separately q -supermean. □

Property 3.3. Let $f_n(x, y)$ be a sequence of q -supermean functions in E , such that $\lim_{n \rightarrow \infty} f_n(x, y) = f(x, y)$ is finite for each $(x, y) \in E$. Then $f(x, y)$ is separately q -supermean in E .

Proof. Since $f_n(x, y)$ is a sequence of q -supermean function in E , $f_n^x(y)$ for a fixed $x \in X_1$ and $f_{n_y}(x)$ for a fixed $y \in X_2$ is q -supermean in E .

By definition,

- for a fixed $x \in X_1$, $\lim_{n \rightarrow \infty} f_n^x(y) = f^x(y)$, where $y \in E^x$. Then $\Delta_{q_2} f^x(y) \leq 0$
- for a fixed $y \in X_2$, $\lim_{n \rightarrow \infty} f_{n_y}(x) = f_y(x)$, where $x \in E_y$. Then $\Delta_{q_1} f_y(x) \leq 0$

Now $\Delta_{q_1} f_y(x) + \Delta_{q_2} f^x(y) \leq 0$.

$$\Delta_{q_1} f_y(x) + \Delta_{q_2} f^x(y) = \Delta_q f(x, y)$$

$$\Delta_q f(x, y) \leq 0$$

□

Property 3.4. If $f \geq 0$ is q_1 -supermean in X_1 and $g \geq 0$ is q_2 -supermean in X_2 then $k(x, y) = f(x)g(y)$ is a separately q -supermean function in $X_1 \times X_2$.

Proof. For a fixed $y \in X_2$, $\Delta_{q_1} k(x, y) = g(y) \Delta_{q_1} f_y(x)$, since $f \geq 0$ is q_1 -supermean in X_1 , then $\Delta_{q_1} f_y(x) \leq 0$.

For a fixed $x \in X_1$, $\Delta_{q_2} k(x, y) = f(x) \Delta_{q_2} g^x(y)$, since $g \geq 0$ is q_2 -supermean in X_2 , then $\Delta_{q_2} g^x(y) \leq 0$. Thus $k(x, y)$ is separately q -supermean in the product network $X_1 \times X_2$. □

Definition 3.2. Let $s(x, y) \geq 0$ be a function on $X_1 \times X_2$. A function $h(x, y)$ is called the greatest separately q -mean minorant of s if $h(x, y) \leq s(x, y)$ for all (x, y) and $h_y(x)$ is q_1 -mean on X_1 , $h^x(y)$ is q_2 -mean on X_2 , and whenever $k(x, y)$ is any other separately q -mean function satisfying $k(x, y) \leq s(x, y)$, we have $k(x, y) \leq h(x, y)$ on $X_1 \times X_2$.

Property 3.5. If a separately q -supermean $s(x, y)$ is q -submean, then $s(x, y)$ is separately q -mean.

Proof. $\Delta_q [s(x, y)] = \Delta_{q_1} [s_y(x)] + \Delta_{q_2} [s^x(y)]$. Hence the left side is non negative and each term in the sum is non-positive. Hence $\Delta_{q_1} [s_y(x)] = 0 = \Delta_{q_2} [s^x(y)]$. That is, $s(x, y)$ is separately q -mean. □

Theorem 3.1. Let $h(x, y)$ be a real valued function on $X_1 \times X_2$ be a function. If for every fixed $y \in X_2$ the function $h_y(x)$ is q_1 -mean on X_1 , and for every fixed $x \in X_1$ the function $h^x(y)$ is q_2 -mean on X_2 (i.e. h is separately q -mean), then h is q -mean on the product:

$$\Delta_q h(x, y) = \Delta_{q_1} h_y(x) + \Delta_{q_2} h^x(y) = 0$$

for all $(x, y) \in X_1 \times X_2$.

Proof. By hypothesis, for every fixed $y \in X_2$ we have

$$\Delta_{q_1} h_y(x) = 0 \quad \text{for all } x \in X_1,$$

and for every fixed $x \in X_1$ we have

$$\Delta_{q_2} h^x(y) = 0 \quad \text{for all } y \in X_2.$$

But the product Schrödinger operator splits as

$$\Delta_q = \Delta_{q_1} + \Delta_{q_2},$$

so for every $(x, y) \in X_1 \times X_2$,

$$\Delta_q h(x, y) = \Delta_{q_1} h_y(x) + \Delta_{q_2} h^x(y) = 0 + 0 = 0.$$

Hence h is q -mean on the product. $\square$

Theorem 3.2. *If $h(x, y)$ is non negative separately q -mean, then $h(x, y)$ is non negative q -mean on $X_1 \times X_2$.*

Proof. Let $h_1(x)$, $h_2(y)$ be non negative q_1 -mean, q_2 -mean functions on X_1, X_2 respectively. Note $\Delta_{q_1}[h_y(x)] = \Delta_{q_1}[h_2(y)h_1(x)]$ for fixed y , since $h(x, y) = h_1(x)h_2(y)$,

$$\begin{aligned} \Delta_{q_1}[h_y(x)] &= \sum_{z \in X_1} t_1(x, z) [h_2(y)h_1(z) - h_2(y)h_1(x)] - q_1(x)h_1(x)h_2(y) \\ &= h_2(y)\Delta_1[h_1(x)] - q_1(x)h_1(x)h_2(y) \\ &= h_2(y)[\Delta_1[h_1(x)]] - q_1(x)h_1(x)h_2(y) \\ &= h_2(y)[\Delta_{q_1}[h_1(x)]] = 0. \end{aligned}$$

Similarly, $\Delta_{q_2}[h^x(y)] = 0$. Hence $h(x, y)$ is separately q -mean, so is q -mean on $\{X_1 \times X_2, t, q\}$. Then there exist non negative q -mean functions on $\{X_1 \times X_2, t, q\}$. $\square$

Definition 3.3. *A Schrödinger network $\{X, t, q\}$ is said to be bounded q -mean if there exists a bounded, nonzero function $h : X \rightarrow [0, \infty)$ such that*

$$\Delta_q h = 0 \quad \text{on } X.$$

Consequence: Suppose $\{X_1, t_1, q_1\}$ and $\{X_2, t_2, q_2\}$ are bounded q -mean. Then if we can take $h_1(x)$ and $h_2(y)$ as bounded q -mean functions so that $h(x, y)$ is a non negative bounded (non-zero) q -mean function on $\{X_1 \times X_2, t, q\}$. That is, $X_1 \times X_2$ is a bounded q -mean Schrödinger network.

4. SEPARATELY q -POTENTIALS

Definition 4.1 (Separately q -potentials). *A non-negative function $s(x, y)$ is called a separately q -potential if it is separately q -supermean and if, whenever $u(x, y)$ is a separately q -submean function satisfying*

$$u(x, y) \leq s(x, y),$$

then

$$u(x, y) \leq 0.$$

4.1. Properties of Separately q -potentials.

Proposition 4.1. *If $s_1(x)$ and $s_2(y)$ are q -potentials in X_1 and X_2 , then $s(x, y) = s_1(x) + s_2(y)$ is a separately q -potential functions.*

Proof. Let $u(x, y)$ be q -submean and $u(x, y) \leq s(x, y)$. Then for fixed y in X_2 , $u_y(x) \leq s_1(x) + s_2(y)$. Note $u_y(x)$ is q -submean and $s_2(y)$ is constant. Since the function $u_y(x) - s_2(y)$ is q -submean in X_1 , $u_y(x) - s_2(y) \leq s_1(x)$. We get $u_y(x) - s_2(y) \leq 0$ for fixed $y \in X_2$ and any $x \in X_1$. Now $u(x, y) \leq s_2(y)$ for fixed y . $u^x(y) \leq s_2(y)$ for any x . Hence the left side is q -submean and the right side potential in X_2 . $u^x(y) \leq 0$ for any $y \in X_2$. Consequently for any $(x_0, y_0) \in X_1 \times X_2$, $u(x_0, y_0) = u^{x_0}(y_0) \leq 0$

$$u(x, y) \leq 0.$$

Thus if $u(x, y)$ is separately q -submean and if $u(x, y) \leq s(x, y)$, then $u(x, y) \leq 0$. Hence $s(x, y)$ is a separately q -potential function. $\square$

Consequence: Suppose X_1, X_2 are hyperbolic. Let $G_1(a, b)$, $G_2(c, d)$ be q -Green's functions on X_1, X_2 . Then $G[(a, b), (c, d)] = G_1(a, b) + G_2(c, d)$.

In order to analyze potentials on the product Schrödinger network $\{X_1 \times X_2, t, q\}$, it is essential to understand how potential-theoretic objects from the factor networks interact under the product structure. In particular, if each coordinate network admits a potential that dominates all nonnegative q -submean functions, one expects such domination to extend to the product. Let s_1 and s_2 be q_1 - and q_2 -potentials on X_1 and X_2 , respectively, and define the product function $s(x, y) = s_1(x)s_2(y)$. The following proposition shows that this product potential retains the characteristic vanishing property that defines potentials.

Proposition 4.2. *If $u \geq 0$ is separately q -submean on $X_1 \times X_2$ and $u(x, y) \leq s_1(x)s_2(y)$ for all (x, y) , then $u \equiv 0$. Hence $s(x, y) = s_1(x)s_2(y)$ is a q -potential on the product network.*

Proof. To show, $s(x, y)$ is a q -potential on $\{X_1 \times X_2, t, q\}$. Let $0 \leq u(x, y)$ be separately q -submean and $u(x, y) \leq \Delta s(x, y)$. Then for any vertex (x_0, y_0) in X_1, X_2 .

$$u_{y_0}(x) = u(x, y_0) \leq s(x, y_0) = s_1(x)s_2(y_0)$$

Since $s_1(x)s_2(y_0)$ is q_1 potential in X_1 and $u_{y_0}(x)$ is q -submean in X_1 and $0 \leq u_{y_0}(x) \leq s_2(y_0)s_1(x)$.

$$u_{y_0}(x) = 0$$

for any $x \in X_1$. In particular, for $x = x_0$, $u(x_0, y_0) = u_{y_0}(x_0) = 0$. Then for any (x_0, y_0) in $X_1 \times X_2$, $u(x_0, y_0) = 0$.

$$u(x, y) = 0.$$

Hence $s(x, y)$ is a separately q -potential functions. $\square$

Proposition 4.3. *Suppose $s(x, y) \geq 0$ on $X_1 \times X_2$ is such that $s_y(x)$ is a q_1 -potential and $s^x(y)$ is q_2 -supermean on X_2 . Then $s(x, y)$ is a separately q -potential function.*

Proof. Let $0 \leq u(x, y)$ be a separately q -submean function, $u(x, y) \leq s(x, y)$ which is separately q -supermean. Then $u_y(x) = u(x, y) \leq s(x, y) = s_y(x)$. Since $u_y(x)$ is q -submean and $s_y(x)$ is q -supermean at $\{X_1, t_1, q_1\}$, $u_y(x) \leq 0$. Let (x_0, y_0) be a vertex in $X_1 \times X_2$. Then $0 \leq u(x_0, y_0) = u_{y_0}(x) \leq 0$

$$u_{y_0}(x) = 0$$

for all $x \in X_1$. In particular $u_{y_0}(x_0) = 0$, $u(x_0, y_0) = 0$. Since (x_0, y_0) is an arbitrary vertex in $X_1 \times X_2$, $u \equiv 0$. Hence $s(x, y)$ is a separately q -potential function. $\square$

Let $s_1(x) \geq 0$ be a q_1 -supermean function on X_1 and let $s_2(y) \geq 0$ be a q_2 -submean function on X_2 . Define $s(x, y) = s_1(x)s_2(y)$, where $(x, y) \in X_1 \times X_2$. Then $\Delta_q s(x, y) = s_2(y) \Delta_{q_1} s_1(x) + s_1(x) \Delta_{q_2} s_2(y)$, so the sign of $\Delta_q s(x, y)$ may vary with (x, y) ; hence $s(x, y)$ need not be q -supermean or q -submean.

Proposition 4.4. Let s_1 be a q_1 -potential on X_1 and let s_2 be a q_2 -mean function on X_2 . Define $s(x, y) = s_1(x)s_2(y)$, $(x, y) \in X_1 \times X_2$. Then s is a separately q -potential function on the product Schrödinger network $\{X_1 \times X_2, t, q\}$ (that is, for each fixed y , $s_y(x)$ is q_1 -supermean on X_1 , and for each fixed x , $s^x(y)$ is q_2 -mean on X_2).

Proof. Since s_1 is a q_1 -potential on X_1 , we have

$$\Delta_{q_1} s_1(x) \leq 0 \quad \text{for all } x \in X_1,$$

and since s_2 is q_2 -mean on X_2 , it satisfies

$$\Delta_{q_2} s_2(y) = 0 \quad \text{for all } y \in X_2.$$

Fix $y \in X_2$. As $s_2(y)$ is constant with respect to x , we obtain

$$\Delta_{q_1} s(x, y) = \Delta_{q_1} (s_1(x)s_2(y)) = s_2(y) \Delta_{q_1} s_1(x).$$

Since $s_2(y) \geq 0$ and $\Delta_{q_1} s_1(x) \leq 0$, it follows that

$$\Delta_{q_1} s(x, y) \leq 0.$$

Hence, for each fixed y , the function $s_y(x)$ is q_1 -supermean on X_1 . Next, fix $x \in X_1$. As $s_1(x)$ is constant with respect to y , we have

$$\Delta_{q_2} s(x, y) = \Delta_{q_2} (s_1(x)s_2(y)) = s_1(x) \Delta_{q_2} s_2(y) = 0.$$

Thus, for each fixed x , the function $s^x(y)$ is q_2 -mean on X_2 .

Therefore, $s(x, y) = s_1(x)s_2(y)$ is a separately q -potential function on $X_1 \times X_2$. $\square$

Theorem 4.1. If s_1 is a q_1 -potential and s_2 is q_2 -mean, then the function $s(x, y) = s_1(x)s_2(y)$ is a separately q -potential function, i.e. it is q_1 -supermean in X_1 and q_2 -mean in X_2 .

Proof. Let $s(x, y) = s_1(x)s_2(y)$. Using the product decomposition of the Schrödinger Laplacian on $X_1 \times X_2$, we obtain

$$\Delta_q [s(x, y)] = s_2(y) \Delta_{q_1} [s_1(x)] + s_1(x) \Delta_{q_2} [s_2(y)].$$

Since s_1 is q_1 -supermean, we have $\Delta_{q_1}s_1(x) \leq 0$, and since s_2 is q_2 -submean, we have $\Delta_{q_2}s_2(y) \geq 0$. Thus the two terms in the expression for $\Delta_q[s(x, y)]$ may have opposite signs, depending on (x, y) . Therefore $s(x, y)$ may be neither q -supermean nor q -submean. Now suppose that s_1 is a q_1 -potential and s_2 is q_2 -mean. Then

$$\Delta_{q_2}s_2(y) = 0,$$

and the expression simplifies to

$$\Delta_q[s(x, y)] = s_2(y) \Delta_{q_1}s_1(x).$$

Since $s_2(y) \geq 0$ and $\Delta_{q_1}s_1(x) \leq 0$, we have $\Delta_q[s(x, y)] \leq 0$. For fixed y ,

$$s_y(x) = s(x, y) = s_1(x)s_2(y)$$

is a positive scalar multiple of the q_1 -potential $s_1(x)$. Hence it is q_1 -supermean. Similarly, for fixed x ,

$$s^x(y) = s(x, y) = s_1(x)s_2(y)$$

is a positive multiple of the q_2 -mean function $s_2(y)$, and hence is q_2 -mean.

Therefore $s(x, y)$ is separately q -supermean (q -potential) in the X_1 and q -mean in the X_2 . $\square$

Theorem 4.2 (q -Riesz Decomposition theorem). *Let $s(x, y) \geq 0$ be a separately q -supermean function on $X_1 \times X_2$. Assume there exists a separately q -submean function $u(x, y) \leq s(x, y)$. Then s admits the q -Riesz decomposition*

$$s(x, y) = p(x, y) + h(x, y),$$

where p is a q -potential and h is the greatest separately q -mean minorant of s . In particular, if $h \equiv 0$, then s is a q -potential.

Proof. The family

$$\mathcal{F} = \{u(x, y) \leq s(x, y) : u \text{ is separately } q\text{-submean}\}$$

forms a q -Perron family. Applying q -Poisson modification on increasing finite subsets of $X_1 \times X_2$, we obtain a decreasing sequence of separately q -mean functions dominated by s . Their limit defines the greatest separately q -mean minorant $h(x, y)$ of s .

Set $p = s - h$. Since h is q -mean,

$$\Delta_q p = \Delta_q s \leq 0, \quad p \geq 0,$$

and hence p is a q -potential. If $h \equiv 0$, then $s = p$ and is itself a q -potential. $\square$

Proposition 4.5. *Let $s \geq 0$ be a function on the product Schrödinger network $X = X_1 \times X_2$. Assume s is a separately q -potential, then s is a q -potential on X .*

Proof. Since s is separately q -supermean (potentials are supermean), the Riesz decomposition(4.2) for separately q -supermean functions on X yields

$$s = p + h,$$

where p is a q -potential on X and h is the greatest separately q -mean minorant of s (so h is separately q -mean and $0 \leq h \leq s$).

We claim $h \equiv 0$. Fix any $y_0 \in X_2$ and consider for fixed y_0 , $h_{y_0}(x) := h(x, y_0)$. Because h is separately q -mean, h_{y_0} is q_1 -mean on X_1 . Also, since $0 \leq h \leq s$ and $s_{y_0}(x) := s(x, y_0)$ is a q_1 -potential by hypothesis, we have

$$0 \leq h_{y_0}(x) \leq s_{y_0}(x) \quad \text{for all } x \in X_1.$$

But a q_1 -potential on X_1 admits no nontrivial nonnegative q_1 -mean minorant (potentials have no positive harmonic part). Hence $h_{y_0} \equiv 0$ on X_1 .

Since y_0 was arbitrary, $h(x, y) \equiv 0$ on $X_1 \times X_2$. Therefore $s = p$ and s is a q -potential on X . $\square$

Theorem 4.3. Assume that on X_2 the constant function 1 is a q_2 -potential (equivalently, X_2 admits no nontrivial bounded q_2 -mean functions). Then any bounded function $h(x, y)$ which is separately q -mean (i.e. for each fixed y , $h_y(x)$ is q_1 -mean on X_1 , and for each fixed x , $h^x(y)$ is q_2 -mean on X_2) must be identically zero.

Proof. Let h be a bounded separately q -mean function on $X_1 \times X_2$. Fix $x \in X_1$ and consider $h^x(y) := h(x, y)$ for each fixed y , $y \in X_2$. Because h is separately q -mean, h^x is a bounded q_2 -mean function on X_2 . By hypothesis X_2 admits no nontrivial bounded q_2 -mean functions, so $h^x \equiv 0$ on X_2 .

Since this holds for every $x \in X_1$, we conclude $h \equiv 0$ on $X_1 \times X_2$. This proves the claim. $\square$

Theorem 4.4. Let $\{X_1, t_1, q_1\}$ and $\{X_2, t_2, q_2\}$ be Schrödinger networks and let s_1 and s_2 be q_1 -potentials and q_2 -potentials on X_1 and X_2 , respectively. Define

$$s(x, y) = s_1(x)s_2(y), \quad (x, y) \in X_1 \times X_2,$$

and let

$$q(x, y) = q_1(x) + q_2(y).$$

Then s is a q -potential on the product network $X_1 \times X_2$.

Proof. Since s_1 and s_2 are q_1 -potentials and q_2 -potentials, respectively, they are q_1 -supermean and q_2 -supermean. Hence the function

$$s(x, y) = s_1(x)s_2(y)$$

is separately q -supermean on $X_1 \times X_2$.

Assume that s is not a q -potential. By the q -Riesz decomposition theorem for product Schrödinger networks, there exist a non-zero separately q -mean function $h \geq 0$ and a q -potential p such that

$$s(x, y) = p(x, y) + h(x, y).$$

Fix a vertex $y_0 \in X_2$. Then

$$s(x, y_0) = s_1(x)s_2(y_0) = p(x, y_0) + h(x, y_0).$$

Since $h_{y_0}(x)$ is a non-negative q_1 -mean function on X_1 , this gives a non-trivial q_1 -mean minorant of the q_1 -potential $s_1(x)s_2(y_0)$, which contradicts the fact that a q_1 -potential admits no non-zero q_1 -mean minorant.

Therefore $h \equiv 0$ and

$$s(x, y) = p(x, y),$$

so s is a q -potential on $X_1 \times X_2$. □

5. INTEGRAL REPRESENTATION OF q -POTENTIAL FUNCTIONS FOR PRODUCT NETWORKS

Theorem 5.1. *Let $p(x_1, x_2)$ be a q -potential on X . Then*

$$p(x_1, x_2) = \sum_{(y_1, y_2) \in X} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q p(y_1, y_2)].$$

Proof. For $n \in \mathbb{N}$ define the truncated q -potential

$$p_n(x_1, x_2) = \sum_{(y_1, y_2) \in X_n} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q p(y_1, y_2)].$$

Since p_n is a finite sum of q -potentials, it is itself a q -potential. Moreover,

$$-\Delta_q p_n(x_1, x_2) = \sum_{(y_1, y_2) \in X_n} [-\Delta_q p(y_1, y_2)] \delta_{(y_1, y_2)}(x_1, x_2) \leq -\Delta_q p(x_1, x_2).$$

Thus

$$-\Delta_q(p - p_n) \geq 0,$$

which shows that $p - p_n$ is q -supermean. Hence there exists a q -supermean function s such that

$$p(x_1, x_2) - p_n(x_1, x_2) = s(x_1, x_2).$$

Since p_n is a q -potential, the function $-s$ is a q -submean function dominated by p_n . By the domination principle, $-s \leq 0$, and hence $p \geq p_n$.

Letting $n \rightarrow \infty$ we obtain

$$\sum_{(y_1, y_2) \in X} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q p(y_1, y_2)] = \lim_{n \rightarrow \infty} p_n(x_1, x_2) \leq p(x_1, x_2).$$

Now the infinite sum

$$u(x_1, x_2) = \sum_{(y_1, y_2) \in X} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q p(y_1, y_2)]$$

is the limit of q -potentials, hence is itself a q -supermean function. Furthermore,

$$-\Delta_q u(x_1, x_2) = \sum_{(y_1, y_2) \in X} [-\Delta_q p(y_1, y_2)] \delta_{(y_1, y_2)}(x_1, x_2) = -\Delta_q p(x_1, x_2).$$

Thus the difference

$$h = p - u$$

satisfies

$$\Delta_q h = 0,$$

so h is q -mean on X . By uniqueness of the representation of q -potentials on the product network, the only q -mean function dominated by a q -potential is the zero function. Hence $h \equiv 0$, and therefore

$$p = u.$$

Thus

$$p(x_1, x_2) = \sum_{(y_1, y_2) \in X} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q p(y_1, y_2)].$$

□

6. INTEGRAL REPRESENTATION OF POSITIVE q -MEAN AND POSITIVE q -SUPERMEAN FUNCTIONS FOR PRODUCT SCHRÖDINGER NETWORKS

Let $\{X_1 \times X_2, t, q\}$ be a hyperbolic product Schrödinger network, where $q(x_1, x_2) = q_1(x_1) + q_2(x_2)$ and $t = t_1 \otimes t_2$. In this section, we derive an integral representation of positive q -mean functions on the product network, using the Choquet Integral Representation Theorem as in [20]. Subsequently, we obtain a canonical representation for non-negative q -supermeans functions on $X_1 \times X_2$.

Let

$$H_q^+ = \{h \geq 0 : \Delta_q h = 0 \text{ on } X_1 \times X_2\}$$

denote the cone of all non-negative q -mean functions on the product network. This cone is convex and forms a lattice under the natural pointwise order.

Fix a vertex $(e_1, e_2) \in X_1 \times X_2$. We define the base

$$B = \{h \in H_q^+ : h(e_1, e_2) = 1\}.$$

Definition 6.1 (Harnack Inequality). Let $\mathcal{H}_q^+$ denote the class of positive separately q -supermean functions on $X_1 \times X_2$. For any two vertices (a_1, a_2) and (b_1, b_2) in $X_1 \times X_2$, there exist positive constants A and B such that

$$A s(b_1, b_2) \leq s(a_1, a_2) \leq B s(b_1, b_2),$$

for all $s \in \mathcal{H}_q^+$.

Using the product Harnack inequality (inherited from the Harnack inequalities on each factor, (see [3])), we conclude that the base B is compact. The extremal elements $E \subseteq B$ are precisely the minimal q -mean functions on the product network.

Let

$$H = H_q^+ - H_q^+.$$

For $f, g \in H$, we introduce the metric

$$\|f - g\| = \sup_{(x_1, x_2) \in X_1 \times X_2} \frac{|f(x_1, x_2) - g(x_1, x_2)|}{1 + |f(x_1, x_2) - g(x_1, x_2)|}.$$

Under this metric, H_q^+ is a closed convex lattice cone with compact base B . Thus, by the Choquet integral representation theorem [11], we obtain the following representation theorem.

Theorem 6.1. Let $X_1 \times X_2$ be a product Schrödinger network. Then for any $f \in H_q^+$, there exists a unique unitary measure $\nu \geq 0$ with $\text{supp}(\nu) \subseteq E$, where E is the set of minimal q -mean functions, such that

$$f(x_1, x_2) = \int_{h \in E} h(x_1, x_2) d\nu(h), \quad (x_1, x_2) \in X_1 \times X_2.$$

We have therefore obtained a complete integral representation for positive q -mean functions on the product Schrödinger network $\{X_1 \times X_2, t, q\}$. Using representation of separately q -potentials and separately q -mean, we can now establish integral representation of separately q -supermean.

Theorem 6.2. Let $s(x_1, x_2)$ be a nonnegative separately q -supermean function on a product Schrödinger network $X = X_1 \times X_2$. Then there exists a unique unitary Radon measure ν with supported on the set E of minimal separately q -mean functions such that

$$s(x_1, x_2) = \sum_{(y_1, y_2) \in X} \mathcal{G}_{(y_1, y_2)}^{(q)}(x_1, x_2) [-\Delta_q s(y_1, y_2)] + \int_E h(x_1, x_2) d\nu(h), \quad h \in E.$$

7. CLASSIFICATION OF PRODUCT NETWORKS

Proposition 7.1. Assume the constant function 1 is a q_1 -potential on X_1 and also a q_2 -potential on X_2 . Then the constant function 1 is a q -potential on $X_1 \times X_2$.

Proof. By hypothesis the functions $s_1 \equiv 1$ on X_1 and $s_2 \equiv 1$ on X_2 are q_1 -potential and q_2 -potentials respectively. Consider their product $s(x, y) := s_1(x)s_2(y) \equiv 1$ on $X_1 \times X_2$.

We show s is a q -potential, if $u \geq 0$ is separately q -submean on X and $u \leq s$ everywhere, then $u \equiv 0$.

For fixed $y_0 \in X_2$, $u_{y_0}(x) := u(x, y_0)$. Since u is separately q -submean, u_{y_0} is q_1 -submean on X_1 and by the domination hypothesis

$$0 \leq u_{y_0}(x) \leq s(x, y_0) = s_1(x) s_2(y_0) = 1 \cdot 1 = 1,$$

so $u_{y_0} \leq 1$ on X_1 . But $s_1 \equiv 1$ is a q_1 -potential, and a q_1 -potential admits no nonzero nonnegative q_1 -mean minorant; hence the only q_1 -submean function dominated by s_1 is the zero function. Therefore $u_{y_0} \equiv 0$ on X_1 .

Since y_0 was arbitrary, $u \equiv 0$ on $X_1 \times X_2$. This proves that the constant function 1 is a q -potential on $X_1 \times X_2$. $\square$

Theorem 7.1. Assume the constant function 1 is q_1 -mean on X_1 (i.e. $\Delta_{q_1} 1 = 0$) and the constant function 1 is a q_2 -potential on X_2 (i.e. $\Delta_{q_2} 1 < 0$). Then the constant function 1 on $X_1 \times X_2$ is q -supermean but is neither q -mean nor a q -potential on the product.

Proof. Compute the Schrödinger equation of the constant function 1:

$$\Delta_q 1 = \Delta_{q_1} 1 + \Delta_{q_2} 1 = 0 + \Delta_{q_2} 1.$$

By hypothesis $\Delta_{q_2} 1 < 0$ on X_2 , hence $\Delta_q 1 < 0$ on $X_1 \times X_2$. Therefore 1 is q -supermean but not q -mean on the product.

Suppose for contradiction that 1 were a q -potential on X . Then, for each fixed variable, the corresponding section of this q -potential would be a q -potential on the corresponding factor. In particular, for any fixed $y_0 \in X_2$ the function $1_{y_0}(x) \equiv 1$ would be a q_1 -potential on X_1 , contradicting $\Delta_{q_1}1 = 0$ (the constant 1 is q_1 -mean, hence cannot be a nonzero q_1 -potential). Thus 1 is not a q -potential on the product. This completes the proof. $\square$

Theorem 7.2. *If both X_1 and X_2 are bounded q -mean (constant 1 is q -mean on each factor), then the product is bounded q -mean.*

Proof. By definition the product Schrödinger operator splits as

$$\Delta_q = \Delta_{q_1}^{(1)} + \Delta_{q_2}^{(2)}.$$

Applying Δ_q to the constant function 1 gives

$$\Delta_q 1 = \Delta_{q_1} 1 + \Delta_{q_2} 1.$$

By hypothesis $\Delta_{q_1} 1 = 0$ on X_1 and $\Delta_{q_2} 1 = 0$ on X_2 , hence

$$\Delta_q 1 = 0 + 0 = 0$$

on $X_1 \times X_2$. Therefore the constant function 1 is q -mean on the product network, so the product is bounded q -mean. $\square$

APPLICATIONS

- **Random Walks on Product Networks:** The potential-theoretic framework developed in this paper is closely related to random walks on Cartesian products of infinite networks, where q -mean and q -supermean functions correspond to harmonic and superharmonic functions associated with transition operators. The classification of product networks into bounded and unbounded q -mean types provides criteria for recurrence and transience phenomena on product graphs, while the Green-function representations describe expected occupation behavior of random walks [24].
- **Discrete Schrödinger Operators on Graphs:** The results obtained in this work apply to discrete Schrödinger operators on graphs, where q -mean and q -supermean functions arise as solutions and supersolutions of discrete Schrödinger equations. The representation and decomposition results provide structural insight into the behavior of supersolutions and potentials on product networks, complementing existing studies on Schrödinger operators on graphs [16].
- **Green's Functions and Boundary Value Problems on Product Networks:** The integral representation results for q -potentials can be used to analyze Green's functions and boundary value problems on Cartesian product networks. In particular, the product structure allows Green-type representations on the product to be expressed in terms of the factor networks, which is useful in the study of Dirichlet and Poisson problems on infinite graphs [8].

CONCLUSION

In this article, we developed a discrete potential-theoretic framework for Cartesian products of infinite Schrödinger networks. By introducing separately q -supermean, q -mean, and q -potential functions, we extended classical discrete potential theory to product networks and clarified how separate behavior in each coordinate governs global behavior. We derived Green-type integral representation formulas for q -potentials and a Choquet-type representation for positive q -mean functions, leading to a Riesz-type decomposition for non-negative separately q -supermean functions. We also classified product Schrödinger networks according to bounded and unbounded q -mean behavior, showing that the potential-theoretic nature of the product is determined by the properties of the factor networks. These results unify and extend existing work in discrete potential theory and provide a foundation for further studies of Schrödinger operators and related discrete models.

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