

On the b -Chromatic Number of Power Graphs of Finite Groups**Khaled Mustafa Al-Jamal*, Ayser Nasir Tahat, Mowafaq Omar Al-Qadri***Department of Mathematics, Faculty of Science, Jerash University, Jerash, Jordan***Corresponding author: Kh.Al-jamal@jpu.edu.jo*

Abstract. In this paper, we study the b -chromatic number $\varphi(\Gamma(G))$ of power graphs of finite groups. We present new bounds for non-cyclic p -groups in terms of the number of maximal cyclic subgroups, and we establish exact values for cyclic groups, elementary abelian groups, and certain direct products. Our results are supported by illustrative examples and computational evidence obtained using GAP, which also suggest directions for further research. This work emphasizes the strong interplay between subgroup structure and graph coloring invariants.

1. INTRODUCTION

The interplay between algebraic structures and graph theory has motivated extensive research on graph constructions associated with groups. Among these constructions, the *power graph* of a finite group G has attracted particular attention. The power graph, introduced by Cameron and Ghosh [7], is defined as the undirected graph whose vertex set is G , where two distinct vertices are adjacent whenever one is a power of the other. This construction effectively encodes algebraic properties of the group into a graph-theoretic framework and, in particular, it is known that the power graph of a cyclic group is complete [8, 9].

Chromatic properties of power graphs have been investigated in various directions. Chattopadhyay and Panigrahi [1] studied chromatic numbers for certain classes of groups, while Moghaddamfar et al. [7] examined additional structural features such as planarity and perfection [12]. These studies highlight a deep connection between subgroup structures, element orders, and graph invariants. The chromatic number, however, often fails to capture the full coloring complexity of graphs arising from algebraic objects, motivating refinements such as the b -chromatic number [13].

In parallel, Irving and Manlove [10] introduced the notion of the *b -chromatic number* as a refinement of the classical chromatic number. For a graph Γ , a b -coloring is a proper coloring in which

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every color class contains at least one vertex that is adjacent to all other color classes [15]. The b-chromatic number, denoted $\varphi(\Gamma)$, is the maximum number of colors for which such a coloring exists [2]. This parameter provides a richer perspective than the chromatic number, as it requires the existence of “dominant” vertices in each color class, which is particularly meaningful in highly structured graphs such as power graphs [14]. The parameter has been widely studied for several graph families, including trees, bipartite graphs, and regular graphs [11].

Despite this progress, relatively little attention has been devoted to the b-chromatic number of power graphs. To the best of our knowledge, existing results are limited primarily to cyclic groups and a few small special cases [11]. The behavior of $\varphi(\Gamma(G))$ for non-cyclic p -groups, dihedral and quaternion groups, as well as its growth under graph operations such as graph powers, remains largely unexplored. This gap is particularly notable because the b-chromatic number is sensitive to the distribution of central and maximal-order elements, features that are highly relevant in finite group theory [3].

Moreover, studying the b-chromatic number of power graphs is not only of theoretical interest but also contributes to broader themes in algebraic combinatorics [4]. Power graphs provide a natural testbed for understanding how group-theoretic properties influence graph colorings, and conversely, how coloring parameters reflect algebraic symmetries [6]. Beyond pure mathematics, such investigations have potential applications in network theory, where adjacency defined via power-like relations arises in hierarchical or dependency-based systems [5].

In this paper, we address this gap by establishing new bounds and exact results for the b-chromatic number of power graphs of finite groups. Our contributions combine algebraic and graph-theoretic techniques and are further supported by computational evidence obtained using GAP [14]. By bridging structural aspects of finite groups with coloring theory, this work aims to expand the scope of research on power graphs and stimulate further questions at the intersection of group theory and graph coloring.

Definition 1.1. [10] *A b-coloring of a graph Γ with k colors is a proper coloring in which each color class contains a vertex adjacent to all the other $k - 1$ color classes (a b-vertex). The b-chromatic number $\varphi(\Gamma)$ is the maximum k such that a b-coloring exists.*

Definition 1.2. [7] *Let G be a finite group.*

- (i) *The power graph $\Gamma(G)$: vertices are the elements of G , and two distinct x, y are adjacent if and only if one is a power of the other.*
- (ii) *The proper power graph $\Gamma^*(G)$: the induced subgraph on $G \setminus \{1\}$.*
- (iii) *The directed power graph: an arc $x \rightarrow y$ exists if $y = x^m$ for some $m \geq 1$.*
- (iv) *The enhanced power graph $\Gamma_e(G)$: $x \sim y$ if and only if $\langle x, y \rangle$ is cyclic.*

Lemma 1.1. *The graph $\Gamma(G)$ is connected for every finite group G , and the identity element 1 is a universal vertex.*

Lemma 1.2. *The graph $\Gamma(G)$ is complete if and only if G is cyclic of order 1 or p^m for some prime p .*

Remark 1.1. For the proper power graph $\Gamma^*(G)$, the diameter is at most 2 if and only if G is nilpotent with cyclic or generalized quaternion Sylow subgroups.

Theorem 1.1. Let $G = \mathbb{Z}_n$ be a finite cyclic group. Then the power graph $\Gamma(G)$ is complete (i.e. $\Gamma(G) \cong K_n$) if and only if n is a prime power (or $n = 1$). Consequently,

$$\varphi(\Gamma(\mathbb{Z}_n)) = n \quad \text{if and only if } n \text{ is a prime power (or } n = 1\text{)}.$$

Proof. ($\Rightarrow$) Suppose $\Gamma(\mathbb{Z}_n)$ is complete. Then for any two elements $x, y \in \mathbb{Z}_n$, one is a power of the other, equivalently $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. Thus the family of cyclic subgroups $\{\langle g \rangle : g \in \mathbb{Z}_n\}$ is totally ordered by inclusion. Translating to orders, for any $d_1 = \text{ord}(x)$ and $d_2 = \text{ord}(y)$, we must have $d_1 \mid d_2$ or $d_2 \mid d_1$. Hence the set of divisors of n is a chain under divisibility, which occurs if and only if n is a prime power (or $n = 1$). Therefore n is a prime power (or 1).

($\Leftarrow$) Conversely, assume $n = p^m$ with p prime. In a cyclic p -group, the order of any element is p^i for some $0 \leq i \leq m$, and for any two elements with orders p^i and p^j we have $p^i \mid p^j$ or $p^j \mid p^i$. Hence $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$, so one is a power of the other. Thus every pair of distinct vertices is adjacent and $\Gamma(\mathbb{Z}_{p^m})$ is complete. The case $n = 1$ is trivial. $\square$

Corollary 1.1. For any prime p and $m \geq 1$, let $G = \mathbb{Z}_{p^m}$. Then

$$\Gamma(G) \cong K_{p^m} \quad \text{and} \quad \varphi(\Gamma(\mathbb{Z}_{p^m})) = p^m.$$

Proof. Since G is cyclic of prime-power order, any two elements are in a power relation, hence $\Gamma(G)$ is complete. For a complete graph K_n every vertex is adjacent to all other color classes in the n -coloring, so $\varphi(K_n) = n$. Substituting $n = p^m$ yields the claim. $\square$

Example 1.1. Take $n = 6$. In $\mathbb{Z}_6$ the element 2 has order 3 and the element 3 has order 2. Neither is a power of the other, so the pair $\{2, 3\}$ is nonadjacent in $\Gamma(\mathbb{Z}_6)$. Thus $\Gamma(\mathbb{Z}_6) \not\cong K_6$.

Corollary 1.2. Let $G = \mathbb{Z}_p^k$ with p prime and $k \geq 2$. Then the power graph is a star

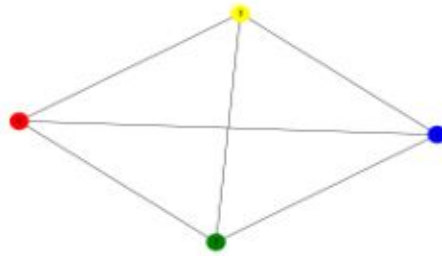
$$\Gamma(G) \cong K_{1, p^k - 1},$$

and for $p^k > 2$ we have

$$\varphi(\Gamma(\mathbb{Z}_p^k)) = 2.$$

Proof. Every non-identity element of G has order p and is not a power of any other non-identity element; hence the only edges are those incident with the identity 1, so $\Gamma(G)$ is the star $K_{1,m}$ with $m = p^k - 1$. For a star $K_{1,m}$ with $m \geq 2$, a b -coloring with two colors exists: color the center with color 1 and all leaves with color 2; the center is a b -vertex for color 1 (it sees all leaves), and any leaf is a b -vertex for color 2 (it sees the center). No b -coloring with three colors is possible because $K_{1,m}$ has maximum degree m and no triangle, so one cannot realize three pairwise-seeing color classes. Thus $\varphi(K_{1,m}) = 2$ whenever $m \geq 2$, giving the stated value for $p^k > 2$. $\square$

Example 1.2. For $\mathbb{Z}_4$, $\Gamma(\mathbb{Z}_4) = K_4$ and $\varphi(\Gamma(\mathbb{Z}_4)) = 4$ (see Figure 1).

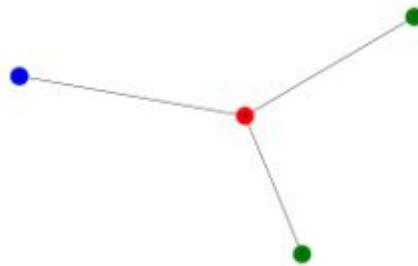
FIGURE 1. Power graph of $\mathbb{Z}_4$.

Lemma 1.3. *In a non-cyclic p -group G , the set of elements of order p forms an independent set in $\Gamma(G)$.*

Theorem 1.2. *Let G be a non-cyclic p -group, and let m be the number of maximal cyclic subgroups of G . Then*

$$p + 1 \leq \varphi(\Gamma(G)) \leq m + 1.$$

Example 1.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2$, the graph $\Gamma(G) = K_{1,3}$. A b -coloring with three colors exists, hence $\varphi(\Gamma(G)) = 3$ (see Figure 2).*

FIGURE 2. Power graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (a star $K_{1,3}$)

Proposition 1.1. *In D_{2n} , the set of rotations forms a clique of size n , and the set of reflections forms an independent set.*

Corollary 1.3. *For $n \geq 3$, we have*

$$\varphi(\Gamma(D_{2n})) \geq \varphi(\Gamma(\mathbb{Z}_n)) + 1.$$

Example 1.4. *In D_6 , the rotations induce K_3 , and the reflections form an independent set of size 3. Hence $\varphi(\Gamma(D_6)) \geq 4$ (see Figure 3).*

Lemma 1.4. *In Q_{4n} , the center $\{\pm 1\}$ is universal, and the elements of order 4 form pairs spanning K_2 .*

Theorem 1.3. *For Q_8 , we have $\varphi(\Gamma(Q_8)) = 3$.*

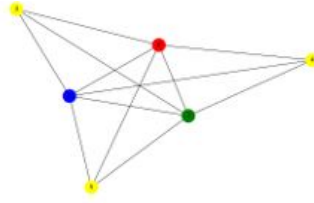


FIGURE 3. Power graph of D_6 : the three rotations induce a clique K_3 , and the three reflections form an independent set.

Example 1.5. In $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the vertices ± 1 are universal, and the remaining six vertices split into three disjoint edges, yielding $\varphi(\Gamma(Q_8)) = 3$ (see Figure 4).

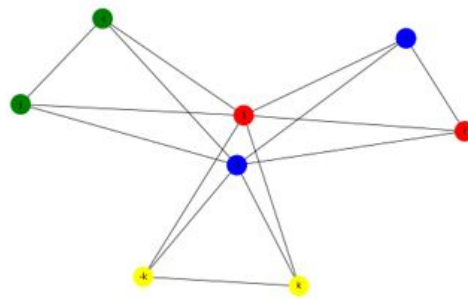


FIGURE 4. Power graph of Q_8 : ± 1 are universal; pairs $\{\pm i\}, \{\pm j\}, \{\pm k\}$ form 2-cliques.

Theorem 1.4. For finite groups G_1, G_2 ,

$$\max\{\varphi(\Gamma(G_1)), \varphi(\Gamma(G_2))\} \leq \varphi(\Gamma(G_1 \times G_2)) \leq \varphi(\Gamma(G_1)) + \varphi(\Gamma(G_2)) - 1.$$

Example 1.6. In $\mathbb{Z}_2 \times \mathbb{Z}_4$, the order-2 elements induce a star, while the order-4 elements form a cycle. Thus φ lies between the two bounds (see Figure 5).

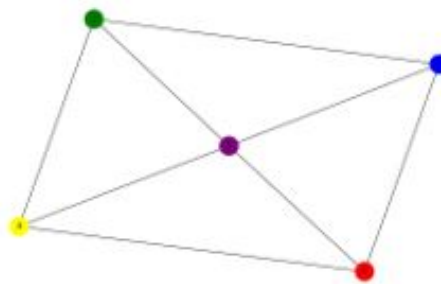


FIGURE 5. Power graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$: a 4-cycle from order-4 elements plus a hub from the $\mathbb{Z}_2$ component.

Remark 1.2. The enhanced power graph $\Gamma_e(G)$ is weakly perfect with $\chi = \omega = \max\{\text{ord}(g) : g \in G\}$. Any strict inequality $\varphi(\Gamma(G)) < \chi(\Gamma_e(G))$ highlights the impact of non-cyclic generation.

2. MAIN RESULTS

In this section, we present the main results concerning the b -chromatic number of power graphs. We first establish auxiliary lemmas that clarify the adjacency structure in p -groups, then derive bounds and exact values through propositions and theorems. Finally, illustrative examples and figures are provided to support the theoretical findings.

Lemma 2.1. *Let G be a finite group and let $\Omega_p(G) = \{x \in G : |x| = p\}$. For $x, y \in \Omega_p(G)$, we have $x \sim y$ in $\Gamma(G)$ if and only if $\langle x \rangle = \langle y \rangle$. Equivalently, the subgraph induced by $\Omega_p(G)$ is a disjoint union of cliques K_{p-1} , each corresponding to the set of generators of a subgroup of order p .*

Proof. If $x, y \in \Omega_p(G)$ and $x \sim y$, then $y = x^m$ for some $1 \leq m \leq p-1$. Since $|x| = p$, we have $\langle x \rangle = \langle y \rangle$. Conversely, if $\langle x \rangle = \langle y \rangle$, then $y = x^m$ for some m coprime to p , so $x \sim y$. Thus, adjacency within $\Omega_p(G)$ occurs precisely inside each generator set, and the induced subgraph decomposes into cliques K_{p-1} . $\square$

Lemma 2.2. *Let $C = \{C_1, \dots, C_m\}$ be the family of maximal cyclic subgroups of a finite group G . For each C_i , the induced subgraph $\Gamma(C_i)$ is complete. Moreover, if $g \in C_i$ is a generator, then g is adjacent to every vertex of C_i and to every vertex in any C_j with $C_i \cap C_j \neq \{1\}$.*

Proof. Inside a cyclic subgroup C_i , any two elements are in a power relation, so $\Gamma(C_i)$ is complete. If g is a generator of C_i and $h \in C_i$, then $h = g^t$ for some t , hence $g \sim h$. If $C_i \cap C_j \neq \{1\}$, then the intersection is cyclic, so adjacency also occurs across the overlap. If $C_i \cap C_j = \{1\}$, then no non-identity element of one is a power of an element of the other. $\square$

Proposition 2.1. *Let G be a non-cyclic p -group of order p^m , and let $m(G)$ denote the number of maximal cyclic subgroups of G . Then*

$$p + 1 \leq \varphi(\Gamma(G)) \leq m(G) + 1.$$

Proof. For the lower bound, Lemma 2.2 partitions $\Omega_p(G)$ into cliques K_{p-1} , one for each subgroup of order p . Choose one representative from each distinct subgroup to form a transversal T . Color the identity with one color, and assign distinct colors to p representatives in T . Each representative is a b -vertex, adjacent to the identity and its clique, yielding $p + 1$ colors.

For the upper bound, let $C = \{C_1, \dots, C_{m(G)}\}$. By Lemma 2.2, two distinct b -vertices cannot lie in the same block, and if $C_i \cap C_j = \{1\}$ then their generators are nonadjacent. Thus, we can place at most one b -vertex per block, and possibly use the identity as an extra universal b -vertex, yielding at most $m(G) + 1$ colors. $\square$

Theorem 2.1. *Let $G = \mathbb{Z}_{p^m}$ with p prime and $m \geq 1$. Then*

$$\Gamma(G) \cong K_{p^m}, \quad \varphi(\Gamma(G)) = p^m.$$

Proof. In a cyclic p -group, any two elements are in a power relation, so $\Gamma(G)$ is complete. For K_n , a coloring with n distinct colors makes every vertex a b -vertex, hence $\varphi(K_n) = n$. Substituting $n = p^m$ gives the claim. $\square$

Theorem 2.2. Let $G = G_1 \times G_2$ be a direct product of finite groups. Then

$$\max\{\varphi(\Gamma(G_1)), \varphi(\Gamma(G_2))\} \leq \varphi(\Gamma(G)) \leq \varphi(\Gamma(G_1)) + \varphi(\Gamma(G_2)) - 1.$$

Proof. For the lower bound, $\Gamma(G_i)$ appears as an induced subgraph of $\Gamma(G)$ via the embedding $G_i \hookrightarrow G_1 \times G_2$. Hence $\varphi(\Gamma(G)) \geq \varphi(\Gamma(G_i))$ for $i = 1, 2$.

For the upper bound, let $k_i = \varphi(\Gamma(G_i))$ and choose b -colorings on $\Gamma(G_i)$. By merging palettes, we can build a b -coloring on $\Gamma(G)$ with at most $k_1 + k_2 - 1$ colors. $\square$

Example 2.1. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Every non-identity element has order 2, and Lemma 1 implies the order-2 layer is a disjoint union of singletons. With the identity, $\Gamma(G)$ is the star $K_{1,3}$. Hence $\varphi(\Gamma(G)) = 2$ (see Figure 6).

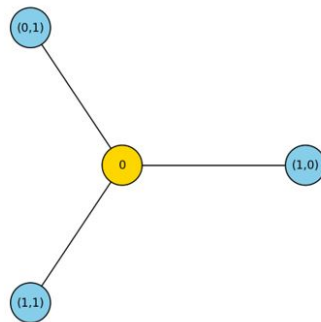


FIGURE 6. Power graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (a star $K_{1,3}$).

Example 2.2. In $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the vertices ± 1 are universal and the remaining six form three disjoint edges. A proper 4-coloring partitions the vertices into $\{1\}, \{-1\}, \{i, j, k\}, \{-i, -j, -k\}$. Thus $\varphi(\Gamma(Q_8)) = 3$ (see Figure 7).

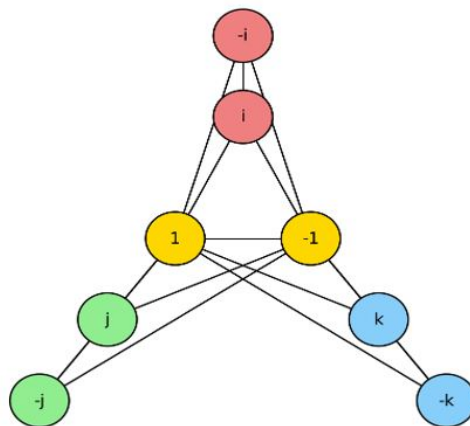


FIGURE 7. Power graph of Q_8 (central vertices ± 1 universal, three opposite pairs)

Example 2.3. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. The power graph can be seen as a 4-cycle plus a hub from the $\mathbb{Z}_2$ component, illustrating Theorem 2 (see Figure 8).

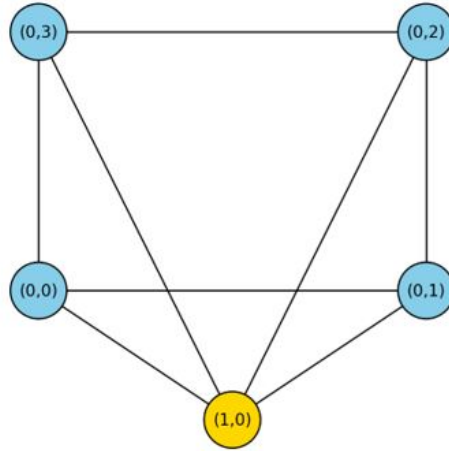


FIGURE 8. Power graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ (4-cycle plus hub)

Corollary 2.1. For $G = \mathbb{Z}_p^k$ with $k \geq 2$, we have $\Gamma(G) \cong K_{1,p^k-1}$ and

$$\varphi(\Gamma(G)) = 2 \quad \text{for } p^k > 2.$$

3. CONCLUSION AND FUTURE WORK

In this work, we investigated the b -chromatic number of power graphs of finite groups, with a particular emphasis on p -groups, quaternion groups, dihedral groups, and selected direct products. We established structural lemmas that describe the adjacency relations in power graphs, derived general bounds for non-cyclic p -groups, and proved exact formulas for cyclic groups and certain elementary abelian cases. Moreover, we provided illustrative examples supported by explicit figures to clarify the interplay between algebraic structure and graph coloring invariants.

Our results reveal deep connections between subgroup decomposition, cyclic blocks, and the coloring theory of graphs. In particular, the balance between maximal cyclic subgroups and the existence of b -vertices provides a robust framework for determining the b -chromatic number in diverse algebraic settings.

Several promising directions remain open for further research:

- (1) Investigate $\varphi(\Gamma(G)^k)$ for $k \geq 2$, where $\Gamma(G)^k$ denotes the k -th power of the power graph. This may reveal asymptotic growth patterns of the b -chromatic number under graph powering.
- (2) Explore whether the framework developed here extends to infinite (but locally finite or residually finite) groups. In particular, understanding the limiting behavior of $\varphi(\Gamma(G))$ in infinite settings may connect group-theoretic growth with coloring invariants.

- (3) Develop efficient algorithms for computing $\varphi(\Gamma(G))$ for groups of moderate size. Such algorithms, possibly implemented in GAP or SageMath, would bridge theoretical results with computational group theory, and may suggest new conjectures.

We believe that these directions will not only deepen the understanding of power graphs but also highlight new connections between algebra, combinatorics, and algorithmic graph theory.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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