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STRONG AND Δ-CONVERGENCE OF MODIFIED TWO-STEP ITERATIONS FOR NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. The aim of this article is to establish a Δ-convergence and some strong convergence theorems of modified two-step iterations for two nearly asymptotically nonexpansive mappings in the setting of hyperbolic spaces. Our results extend and generalize the previous work from the current existing literature.

1. Introduction

The class of asymptotically nonexpansive mapping, introduced by Goebel and Kirk [7] in 1972, is an important generalization of the class of nonexpansive mapping. They proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of C has a fixed point.

There are number of papers dealing with the approximation of fixed points / common fixed points of asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces using modified Mannn and Ishikawa iteration processes and have been studied by many authors (see, e.g., [17, 18, 24, 28, 29, 31, 34, 35]).

The concept of Δ-convergence in a general metric space was introduced by Lim [16]. In 2008, Kirk and Panyanak [14] used the notion of Δ-convergence introduced by Lim [16] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [6] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the Δ-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, nearly asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping in the intermediate sense, total asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping through Picard, Mann [19], Ishikawa[10], modified Agarwal et al. [2] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this

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direction (see, e.g., [1, 5, 6, 11, 20, 25]).

The purpose of this paper is to establish some strong convergence theorems of modified two-step iteration process for two nearly asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces which include both uniformly convex Banach spaces and CAT(0) spaces. Our results extend and improve the previous work from the current existing literature.

2. Preliminaries

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed points of the mapping T. We begin with the following definitions.

Definition 2.1. Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ said to be

(1) nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in K$;

(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \le (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \ge 1$;

(3) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \le (1 + u_n)d(x, p)$ for all $x \in K$, $p \in F(T)$ and $n \ge 1$;

(4) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \le L\, d(x, y)$ for all $x, y \in K$ and $n \ge 1$;

(5) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as $k \to \infty$.

(6) a sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu [26] (see, also [27]).

Definition 2.2. Let K be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping $T\colon K \to K$ said to be nearly Lipschitzian with respect to $\{a_n\}$ if for all $n \ge 1$, there exists a constant $k_n \ge 0$ such that $d(T^n x, T^n y) \le k_n[d(x, y) + a_n]$ for all x, $y \in K$.

The infimum of the constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant of T^n.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be:

(i) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \ge 1$;

(i) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} \eta(T^n) = 1$.

(ii) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \geq 1$.

Throughout this paper, we work in the setting of hyperbolic space introduced by Kohlenbach [15]. It is worth noting that they are different from Gromov hyperbolic space [4] or from other notions of hyperbolic space that can be found in the literature (see for example [8, 13, 23].

A hyperbolic space [15] is a triple (X, d, W) where (X, d) is a metric space and $W\colon X^2 \times [0,1] \to X$ is such that

(i) $d(u, W(x, y, \alpha)) \leq \alpha\, d(u, x) + (1-\alpha) d(u, y)$

(ii) $d\Big(W(x, y, \alpha), W(x, y, \beta)\Big) = |\alpha - \beta|\, d(x, y)$

(iii) $W(x, y, \alpha) = W(x, y, (1-\alpha))$

(iv) $d\Big(W(x, z, \alpha), W(y, w, \beta)\Big) \leq \alpha\, d(x, y) + (1-\alpha)\, d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [15] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov [9]. An important example of a hyperbolic space is the open unit ball B_H in a real Hilbert space H is as follows.

Let B_H be the open unit ball in H. Then

$$k_{B_H}(x, y) = arg\, tanh(1 - \sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{\Big(1 - \|x\|^2\Big)\Big(1 - \|y\|^2\Big)}{|1 - \langle\, x, y\,\rangle|^2}$$

for all x, $y \in B_H$, defines a metric on B_H (also known as Kobayashi distance).

A metric space (X, d) is called a convex metric space introduced by Takahashi in [33] if it satisfies only (i). A subset K of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

A hyperbolic space (X, d, W) is said to be uniformly convex [32] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d(W(x, y, \frac{1}{2}), u) \leq (1-\delta) r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon\, r$.

A mapping $\eta\colon (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Let K be a nonempty subset of hyperbolic space X. Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\}) \colon X \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \big\{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\})\big\} \quad \text{for any } y \in K.$$

The set of all asymptotic center of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [32] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [14]. In this case, we write Δ-$\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that Δ-convergence coincides with weak convergence in Banach space with Opial's property [21].

In the sequel we need the following lemmas.

Lemma 2.3. [12] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 2.4. [12] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 2.5. *(See [34]) Let $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of non-negative numbers satisfying the inequality*

$$p_{n+1} \le (1 + q_n)p_n + r_n, \ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

First, we define the modified two-step iteration scheme in hyperbolic space as follows.

Let K be a nonempty closed convex subset of a hyperbolic space X and $S, T \colon K \to K$ be two nearly asymptotically nonexpansive mappings. Then, for an arbitrary chosen $x_1 \in K$, we construct the sequence $\{x_n\}$ in K such that

$$\begin{cases} x_{n+1} = W(T^n x_n, S^n y_n, \alpha_n), \\ \quad y_n = W(S^n x_n, T^n x_n, \beta_n), \quad n \ge 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in (0,1) is called modified two-step iteration scheme. Iteration scheme (2.1) is independent of modified Ishikawa iteration and modified Mann iteration schemes.

If $\beta_n = 0$ for all $n \geq 1$ and $S = I$, where I is the identity mapping, then iteration scheme (2.1) reduces to the following.

$$\left\{ x_{n+1} = W(T^n x_n, x_n, \alpha_n), \quad n \geq 1, \right. \tag{2.2}$$

where $\{\alpha_n\}$ is an appropriate sequence in (0,1) is called modified Mann iteration scheme in hyperbolic space.

3. **Main Results**

Lemma 3.1. *Let K be a nonempty convex subset of a hyperbolic space X and let $S, T\colon K \to K$ be two nearly asymptotically nonexpansive mappings with sequences $\{(a'_n, \eta(S^n))\}$ and $\{(a''_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F = F(S) \cap F(T)$.*

Proof. Let $p \in F = F(S) \cap F(T)$, $\rho = \sup_{n\in\mathbb{N}} \eta(S^n) \vee \sup_{n\in\mathbb{N}} \eta(T^n)$ and $a_n = \max\{a'_n, a''_n\}$ for all n. From (2.1), we have

$$\begin{aligned}
d(y_n, p) &= d(W(S^n x_n, T^n x_n, \beta_n), p) \\
&\leq (1-\beta_n) d(S^n x_n, p) + \beta_n d(T^n x_n, p) \\
&\leq (1-\beta_n)[\eta(S^n)(d(x_n, p) + a'_n)] + \beta_n[\eta(T^n)(d(x_n, p) + a''_n)] \\
&\leq (1-\beta_n)[\eta(S^n)(d(x_n, p) + a_n)] + \beta_n[\eta(T^n)(d(x_n, p) + a_n)] \\
&= (1-\beta_n)\eta(S^n)\, d(x_n, p) + \beta_n \eta(T^n)\, d(x_n, p) + \big(\eta(S^n) + \eta(T^n)\big) a_n \\
&\leq \eta(S^n)\eta(T^n)[(1-\beta_n) d(x_n, p) + \beta_n d(x_n, p)] + 2\rho\, a_n \\
&= \eta(S^n)\eta(T^n) d(x_n, p) + 2\rho\, a_n. \qquad (3.1)
\end{aligned}$$

Again, using (2.1) and (3.1), we get

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(T^n x_n, S^n y_n, \alpha_n), p) \\
&\leq (1-\alpha_n) d(T^n x_n, p) + \alpha_n d(S^n y_n, p) \\
&\leq (1-\alpha_n)[\eta(T^n)(d(x_n, p) + a''_n)] + \alpha_n[\eta(S^n)(d(y_n, p) + a'_n)] \\
&\leq (1-\alpha_n)[\eta(T^n)(d(x_n, p) + a_n)] + \alpha_n[\eta(S^n)(d(y_n, p) + a_n)] \\
&= (1-\alpha_n)\eta(T^n) d(x_n, p) + \alpha_n \eta(S^n) d(y_n, p) + \big(\eta(S^n) + \eta(T^n)\big) a_n \\
&\leq (1-\alpha_n)\eta(T^n) d(x_n, p) + 2\rho\, a_n \\
&\quad + \alpha_n \eta(S^n)[\eta(S^n)\eta(T^n) d(x_n, p) + 2\rho\, a_n] \\
&\leq \eta(S^n)^2 \eta(T^n)^2 d(x_n, p) + (1 + \eta(S^n)) 2\rho\, a_n \\
&\leq \eta(S^n)^2 \eta(T^n)^2 d(x_n, p) + 2\rho(1+\rho) a_n \\
&= (1+\mu_n) d(x_n, p) + \nu_n \qquad (3.2)
\end{aligned}$$

where $\mu_n = \Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big)$ and $\nu_n = 2\rho(1+\rho)a_n$. Since $\sum_{n=1}^{\infty} \Big(\eta(S^n)^2 \eta(T^n)^2 - 1\Big) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, it follows that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n <$

∞. Hence by Lemma 2.5, we get that $\lim_{n\to\infty} d(x_n, p)$ exists. This completes the proof. □

Lemma 3.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $S, T\colon K \to K$ be two uniformly continuous nearly asymptotically nonexpansive mappings with sequences $\{(a'_n, \eta(S^n))\}$ and $\{(a''_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. If $d(x, T^n x) \leq d(S^n x, T^n x)$ and $d(x, S^n x) \leq d(T^n x, S^n x)$ for all $x \in K$, then $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. From Lemma 3.1, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Suppose that $\lim_{n\to\infty} d(x_n, p) = r \geq 0$. Since

$$d(S^n x_n, p) \leq \eta(S^n)(d(x_n, p) + a_n) \quad \text{for all } n \geq 1,$$

we have

$$\limsup_{n\to\infty} d(S^n x_n, p) \leq r.$$

Also, since

$$d(T^n x_n, p) \leq \eta(T^n)(d(x_n, p) + a_n) \quad \text{for all } n \geq 1,$$

we have

$$\limsup_{n\to\infty} d(T^n x_n, p) \leq r.$$

Also (3.1) yields

$$\limsup_{n\to\infty} d(y_n, p) \leq r. \tag{3.3}$$

Hence

$$\limsup_{n\to\infty} d(S^n y_n, p) \leq \limsup_{n\to\infty} \eta(S^n)(d(y_n, p) + a_n) \leq r. \tag{3.4}$$

Since

$$r = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T^n x_n, S^n y_n, \alpha_n), p),$$

it follows from Lemma 2.3 that

$$\lim_{n\to\infty} d(T^n x_n, S^n y_n) = 0. \tag{3.5}$$

From (2.1) and (3.5), we have

$$\begin{aligned} d(x_{n+1}, T^n x_n) &= d(W(T^n x_n, S^n y_n, \alpha_n), T^n x_n) \\ &\leq \alpha_n\, d(T^n x_n, S^n y_n) \\ &\leq d(T^n x_n, S^n y_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.6}$$

Hence from (3.5) and (3.6), we have

$$\begin{aligned} d(x_{n+1}, S^n y_n) &\leq d(x_{n+1}, T^n x_n) + d(T^n x_n, S^n y_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.7}$$

Now using (3.7), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, S^n y_n) + d(S^n y_n, p) \\ &\leq d(x_{n+1}, S^n y_n) + \eta(S^n)(d(y_n, p) + a_n). \end{aligned} \tag{3.8}$$

The inequality (3.8) gives

$$r \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.9}$$

From (3.3) and (3.9), we get

$$r = \lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(W(S^n x_n, T^n x_n, \beta_n), p). \tag{3.10}$$

Applying Lemma 2.3 in (3.10), we obtain

$$\lim_{n\to\infty} d(S^n x_n, T^n x_n) = 0. \tag{3.11}$$

Now using (3.11) and hypothesis of the theorem $d(x, T^n x) \leq d(S^n x, T^n x)$ for all $x \in K$, we get

$$\begin{aligned} d(x_n, S^n x_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, S^n x_n) \\ &\leq d(S^n x_n, T^n x_n) + d(T^n x_n, S^n x_n) \\ &= 2\, d(S^n x_n, T^n x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.12}$$

Again using (3.11) and hypothesis of the theorem $d(x, S^n x) \leq d(T^n x, S^n x)$ for all $x \in K$, we get

$$\begin{aligned} d(x_n, T^n x_n) &\leq d(x_n, S^n x_n) + d(S^n x_n, T^n x_n) \\ &\leq d(T^n x_n, S^n x_n) + d(S^n x_n, T^n x_n) \\ &= 2\, d(S^n x_n, T^n x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.13}$$

By uniform continuity of S and T, $\lim_{n\to\infty} d(x_n, S^n x_n) = 0$ implies that $\lim_{n\to\infty} d(Sx_n, S^{n+1} x_n) = 0$ and $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$ implies that $\lim_{n\to\infty} d(Tx_n, T^{n+1} x_n) = 0$. Note that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(T^n x_n, S^n y_n, \alpha_n), x_n) \\ &\leq (1-\alpha_n) d(x_n, T^n x_n) + \alpha_n\, d(S^n y_n, x_n) \\ &\leq (1-\alpha_n) d(x_n, T^n x_n) + \alpha_n\, d(S^n y_n, x_{n+1}) + \alpha_n\, d(x_{n+1}, x_n) \\ &\leq (1-\alpha_n) d(x_n, T^n x_n) + \alpha_n\, d(S^n y_n, x_{n+1}) + m\, d(x_{n+1}, x_n) \end{aligned}$$

This implies that

$$(1-m)\, d(x_{n+1}, x_n) \leq (1-\alpha_n) d(x_n, T^n x_n) + \alpha_n\, d(S^n y_n, x_{n+1}). \tag{3.14}$$

Since $(1-m) > 0$, using (3.7) and (3.13) in (3.14), we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.15}$$

Also

$$\begin{aligned} d(x_n, Sx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1} x_{n+1}) \\ &\quad + d(S^{n+1} x_{n+1}, S^{n+1} x_n) + d(S^{n+1} x_n, Sx_n) \\ &\leq \Big(1 + \eta(S^{n+1})\Big) d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1} x_{n+1}) \\ &\quad + d(S^{n+1} x_n, Sx_n) + a_{n+1}. \end{aligned} \tag{3.16}$$

Using (3.12), (3.15) and uniform continuity of S, equation (3.16) gives

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{3.17}$$

Similarly

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq \Big(1 + \eta(T^{n+1})\Big) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) + a_{n+1}. \end{aligned} \tag{3.18}$$

The above inequality gives

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.19}$$

This completes the proof. □

We now establish a Δ-convergence and some strong convergence theorems of modified two-step iteration scheme for non-Lipschitzian mappings in the framework of uniformly convex hyperbolic spaces.

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $S, T\colon K \to K$ be two uniformly continuous nearly asymptotically nonexpansive mappings with sequences $\{(a'_n, \eta(S^n))\}$ and $\{(a''_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. Then $\{x_n\}$ is Δ-convergent to an element of F.*

Proof. It follows from Lemma 3.1 that $\{x_n\}$ is bounded, therefore $\{x_n\}$ has a unique asymptotic center (see, [32]), that is, $A(\{x_n\}) = \{x\}$ (say). Let $A(\{y_n\}) = \{v\}$. Then by Lemma 3.2, $\lim_{n\to\infty} d(y_n, Sy_n) = 0$ and $\lim_{n\to\infty} d(y_n, Ty_n) = 0$. S and T are nearly asymptotically nonexpansive mappings with sequences $\{(a'_n, \eta(S^n))\}$ and $\{(a''_n, \eta(T^n))\}$. By uniform continuity of S and T, we have

$$\lim_{n\to\infty} d(S^i y_n, S^{i+1}y_n) = 0 \quad \text{for } i = 1, 2, \ldots. \tag{3.20}$$

and

$$\lim_{n\to\infty} d(T^j y_n, T^{j+1}y_n) = 0 \quad \text{for } j = 1, 2, \ldots. \tag{3.21}$$

Now we claim that v is a common fixed point of S and T. For this, we define a sequence $\{z_n\}$ in K by $z_m = S^m v$ and $z_m = T^m v$, $m \in \mathbb{N}$. for integers $m, n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_m, y_n) &\leq d(S^m v, S^m y_n) + d(S^m y_n, S^{m-1}y_n) + \cdots + d(Sy_n, y_n) \\ &\leq \eta(S^n)(d(v, y_n) + a'_m) + \sum_{i=0}^{m-1} d(S^i y_n, S^{i+1}y_n). \end{aligned} \tag{3.22}$$

Then from (3.20) and (3.22), we have

$$\begin{aligned} r(z_m, \{y_n\}) &= \limsup_{m\to\infty} d(z_m, y_n) \\ &\leq \eta(S^m)[r(v, \{y_n\}) + a'_m]. \end{aligned}$$

Hence

$$\limsup_{m\to\infty} r(z_m, \{y_n\}) \leq r(v, \{y_n\}). \tag{3.23}$$

Since $A_K(\{y_n\}) = \{v\}$, by definition of asymptotic center $A_K(\{y_n\})$ of a bounded sequence $\{y_n\}$ with respect to $K \subset X$, we have

$$r(v, \{y_n\}) \quad \leq \quad r(y, \{y_n\}), \quad \forall\, y \in K.$$

This implies that

$$\liminf_{m\to\infty} r(z_m, \{y_n\}) \quad \geq \quad r(v, \{y_n\}), \tag{3.24}$$

therefore, from (3.23) and (3.24), we have

$$\lim_{m\to\infty} r(z_m, \{y_n\}) \quad = \quad r(v, \{y_n\}).$$

It follows from Lemma 2.4 that $S^m v \to v$. By uniform continuity of S, we have

$$Sv = S(\lim_{m\to\infty} S^m v) = S^{m+1} v = v,$$

which implies that v is a fixed point of S, that is, $v \in F(S)$. Similarly, we can show that $v \in F(T)$. Thus $v \in F = F(S) \cap F(T)$.

Next, we claim that v is the unique asymptotic center for each subsequence $\{y_n\}$ of $\{x_n\}$. Assume contrarily, that is, $x \neq v$. Since $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.1, therefore, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(y_n, v) &< \limsup_{n\to\infty} d(y_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(y_n, v),
\end{aligned}$$

a contradiction and hence $x = v$. Since $\{y_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore, $A_K(\{y_n\}) = \{v\}$ for all subsequence $\{y_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to an element of F. This completes the proof. □

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $S, T\colon K \to K$ be two uniformly continuous nearly asymptotically nonexpansive mappings with sequences $\{(a'_n, \eta(S^n))\}$ and $\{(a''_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F = F(S) \cap F(T) \neq \emptyset$ is a closed set. Then $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 3.1, for all $p \in F$, $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, we have from (3.2)

$$\begin{aligned}
d(x_{n+m}, p) &\leq (1 + \mu_{n+m-1})d(x_{n+m-1}, p) + \nu_{n+m-1} \\
&\leq e^{\mu_{n+m-1}} d(x_{n+m-1}, p) + \nu_{n+m-1} \\
&\leq e^{\mu_{n+m-1}}[e^{\mu_{n+m-2}} d(x_{n+m-2}, p) + \nu_{n+m-2}] \\
&\quad +\nu_{n+m-1} \\
&\leq e^{(\mu_{n+m-1}+\mu_{n+m-2})} d(x_{n+m-2}, p) + e^{(\mu_{n+m-1}+\mu_{n+m-2})} \times \\
&\quad [\nu_{n+m-1} + \nu_{n+m-2}] \\
&\leq \dots \\
&\leq \left(e^{\sum_{k=n}^{n+m-1} \mu_k}\right) d(x_n, p) + \left(e^{\sum_{k=n}^{n+m-1} \mu_k}\right) \sum_{k=n}^{n+m-1} \nu_k \\
&= W\, d(x_n, p) + W \sum_{k=n}^{n+m-1} \nu_k, \qquad (3.25)
\end{aligned}$$

where $W = e^{\sum_{n=1}^{\infty} \mu_n} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F$ such that $d(x_{n_k}, p_{n_k}) \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{4W} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} \nu_k < \frac{\varepsilon}{4W}, \qquad (3.26)$$

for all $k \geq k_\varepsilon$.

For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by (3.25) and (3.26), we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\
&\leq W\, d(x_n, p_{n_k}) + W \sum_{k=n_{k_\varepsilon}}^{\infty} \nu_k \\
&\quad +W\, d(x_n, p_{n_k}) + W \sum_{k=n_{k_\varepsilon}}^{\infty} \nu_k \\
&= 2W\, d(x_n, p_{n_k}) + 2W \sum_{k=n_{k_\varepsilon}}^{\infty} \nu_k \\
&< 2W.\frac{\varepsilon}{4W} + 2W.\frac{\varepsilon}{4W} = \varepsilon. \qquad (3.27)
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in closed subset K of a complete hyperbolic space X and so it must converge to a point z in K, that is, $\lim_{n\to\infty} x_n = z$. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives $d(z, F) = 0$. Since F is closed, we have $z \in F$. Thus $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $S, T\colon K \to K$ be two uniformly continuous nearly asymptotically nonexpansive* *mappings with sequences* $\{(a'_n, \eta(S^n))\}$ *and* $\{(a''_n, \eta(T^n))\}$ *such that* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty}\Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. *Let* $\{x_n\}$ *be a sequence in* K *defined by (2.1). Assume that* $F = F(S) \cap F(T) \neq \emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequence in* $[l, m]$ *for some* $l, m \in (0,1)$. *If either* S^m *or* T^m *for some* $m \geq 1$ *is semi-compact, then* $\{x_n\}$ *converges strongly to a point in* F.

Proof. Suppose T^m for some $m \geq 1$ is semi-compact. By Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By the uniform continuity of T, we get

$$d(x_n, Tx_n) \to 0 \quad \Rightarrow \quad d(Tx_n, T^2x_n) \to 0 \quad \Rightarrow \\ \cdots \Rightarrow \quad d(T^ix_n, T^{i+1}x_n) \to 0$$

for all $i = 1, 2, 3, \dots$, it follows that

$$d(x_n, T^m x_n) \leq \sum_{i=0}^{m-1} d(T^i x_n, T^{i+1}x_n) \to 0 \quad \text{as } n \to \infty.$$

Since $d(x_n, T^m x_n) \to 0$ and T^m is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} T^m x_{n_j} = x \in K$.

Note that

$$d(x_{n_j}, x) \leq d(x_{n_j}, T^m x_{n_j}) + d(T^m x_{n_j}, x) \to 0 \quad \text{as } j \to \infty.$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we get $x \in F(T)$. Similarly, we can show that $x \in F(S)$. Thus $x \in F = F(S) \cap F(T)$. Since $\lim_{n\to\infty} d(x_n, x)$ exists by Lemma 3.1 and $\lim_{j\to\infty} d(x_{n_j}, x) = 0$, we conclude that $x_n \to x \in F$. This shows that the sequence $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Senter and Dotson [30] introduced the concept of condition (A) as follows.

Definition 3.6. (See [30]) A mapping $T\colon K \to K$ is said to satisfy condition (A) if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$.

We modify this definition for two mappings.

Definition 3.7. Two mappings $S, T\colon K \to K$, where K is a subset of a metric space (X, d), is said to satisfy condition (A') if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $a\, d(x, Sx) + b\, d(x, Tx) \geq f(d(x, F))$ for all $x \in K$ where $d(x, F) = \inf\{d(x, p) : p \in F = F(S) \cap F(T) \neq \emptyset\}$ and a and b are two nonnegative real numbers such that $a + b = 1$. It is to be noted that condition (A') is weaker than compactness of the domain K.

Remark 3.8. Condition (A') reduces to condition (A) when $S = T$.

As an application of Theorem 3.3, we establish another strong convergence result employing condition (A').

Theorem 3.9. *Let* K *be a nonempty closed convex subset of a complete uniformly convex hyperbolic space* X *with monotone modulus of uniform convexity* η *and let* $S, T\colon K \to K$ *be two uniformly continuous nearly asymptotically nonexpansive mappings with sequences* $\{(a'_n, \eta(S^n))\}$ *and* $\{(a''_n, \eta(T^n))\}$ *such that* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty}\Big(\eta(S^n)^2\eta(T^n)^2 - 1\Big) < \infty$. *Let* $\{x_n\}$ *be a sequence in* K *defined by (2.1).*

Assume that $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. Suppose that S and T satisfy the condition (A'). Then $\{x_n\}$ converges strongly to a point in F.

Proof. By Lemma 3.2, we know that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.28}$$

From condition (A') and (3.28), we get

$$\lim_{n\to\infty} f(d(x_n, F)) \leq a. \lim_{n\to\infty} d(x_n, Sx_n) + b. \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

that is,

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

The conclusion now follows from Theorem 3.4. This completes the proof. □

Example 3.10. (See [26]) Let $E = \mathbb{R}$, $K = [0, 1]$ and $T\colon K \to K$ be a mapping defined by

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Here $F(T) = \{\frac{1}{2}\}$. Clearly, T is discontinuous and a non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\} = \{\frac{1}{2^n}, 1\}$. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$d(Tx, Ty) \leq d(x, y) + a_1 \text{ for all } x,\ y \in K$$

and

$$d(T^n x, T^n y) \leq d(x, y) + a_n \text{ for all } x,\ y \in K \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \text{ for all } x \in [0, 1] \text{ and } n \geq 2.$$

4. **Conclusion**

1. We prove a Δ-convergence and some strong convergence theorems of modified two-step iteration process which contains modified Mann iteration process in the framework of uniformly convex hyperbolic spaces.

2. Lemma 3.2 extends Theorem 3.8 of Agarwal et al. [2] to the case of modified two-step iteration scheme for two mappings and from uniformly convex Banach space to a uniformly convex hyperbolic space considered in this paper.

3. Our results also extend and generalize the corresponding results of [3, 22, 24, 34] to the case of more general class of nonexpansive and asymptotically nonexpansive mappings, modified two-step iteration scheme for two mappings and from uniformly convex metric space and Banach space to a uniformly convex hyperbolic space considered in this paper.

References

[1] M. Abbas, Z. Kadelburg and D.R. Sahu, Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces, Math. Comput. Model. 55 (2012), 1418-1427.

[2] R.P. Agarwal, Donal O'Regan and D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, Nonlinear Convex Anal. 8(1) (2007), 61-79.

[3] I. Beg, An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces, Nonlinear Anal. Forum, 6 (2001), 27-34.

[4] M.R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Vol. 319 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 1999.

[5] S.S. Chang, L. Wang, H.W. Joesph Lee, C.K. Chan, L. Yang, Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces, Appl. Math. Comput. 219(5) (2012), 2611-2617.

[6] S. Dhompongsa and B. Panyanak, On $\triangle$-convergence theorem in CAT(0) s-paces, Comput. Math. Appl. 56 (2008), 2572-2579.

[7] K. Goebel and W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171-174.

[8] K. Goebel and W.A. Kirk, Iterations processes for nonexpansive mappings, Contemp. Math. 21 (1983), 115-123.

[9] M. Gromov, Hyperbolic groups. Essays in group theory (S. M. Gersten, ed). Springer Verlag, MSRI Publ. 8 (1987), 75-263.

[10] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[11] S.H. Khan and M. Abbas, Strong and $\triangle$-convergence of some iterative schemes in CAT(0) spaces, Comput. Math. Appl. 61 (2011), 109-116.

[12] A.R. Khan, H. Fukhar-ud-din and M.A.A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 54.

[13] W.A. Kirk, Krasnoselskii's iteration process in hyperbolic space, Numer. Funct. Anal. Optim 4 (1982), 371-381.

[14] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008), 3689-3696.

[15] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, Trans. Amer. Math. Soc. 357 (2005), 89-128.

[16] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976), 179-182.

[17] Q.H. Liu, Iterative sequences for asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl. 259 (2001), 1-7.

[18] Q.H. Liu, Iterative sequences for asymptotically quasi-nonexpansive mappings with error member, J. Math. Anal. Appl. 259 (2001), 18-24.

[19] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.

[20] B. Nanjaras and B. Panyanak, Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl. 2010 (2010), Art. ID 268780.

[21] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73(1967), 591-597.

[22] M.O. Osilike, S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. and Computer Modelling 32(2000), 1181-1191.

[23] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, Nonlinear Anal.: TMA, Series A, Theory Methods, 15(6)(1990), 537-558.

[24] B.E. Rhoades, Fixed point iteration for certain nonlinear mappings, J. Math. Anal. Appl. 183(1994), 118-120.

[25] A. Şahin and M. Başarir, On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space, Fixed Point Theory Appl. 2013 (2013), Article ID 12.

[26] D.R. Sahu, Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces, Comment. Math. Univ. Carolinae 46(4) (2005), 653-666.

[27] D.R. Sahu and I. Beg, Weak and strong convergence for fixed points of nearly asymptotically nonexpansive mappings, Int. J. Mod. Math. 3 (2008), 135-151.

[28] G.S. Saluja, Strong convergence theorem for two asymptotically quasi-nonexpansive mappings with errors in Banach space, Tamkang J. Math. 38(1) (2007), 85-92.

[29] G.S. Saluja, Convergence result of (L,α)-uniformly Lipschitz asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces, J$\tilde{n}\bar{a}$n$\bar{a}$bha 38 (2008), 41-48.

[30] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974), 375-380.

[31] N. Shahzad, A. Udomene, Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, 2006 (2006), Article ID 18909.

[32] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal. 8(1) (1996), 197-203.

[33] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, Kodai Math. Semin. Rep. 22 (1970), 142-149.

[34] K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178, 301-308, 1993.

[35] K.K. Tan and H.K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 122(1994), 733-739.

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