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CONVERGENCE OF HYBRID FIXED POINT FOR A PAIR OF NONLINEAR MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study hybrid fixed point of a modified two-step iteration process with errors for a pair of asymptotically quasi-nonexpansive mapping and asymptotically quasi-nonexpansive mapping in the intermediate sense in the framework of Banach spaces. Also we establish some strong convergence theorems and a weak convergence theorem for the iteration scheme and mappings. The results presented in this paper extend, improve and generalize some previous work from the existing literature.

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$;

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$;

(3) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all x, $y \in K$ and $n \geq 1$;

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(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \quad \leq \quad k_n\|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(5) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\|x - y\|$$

for all x, $y \in K$ and $n \geq 1$;

(6) uniformly quasi-Lipschitzian if there exists $L \in [1, +\infty)$ such that

$$\|T^n x - p\| \quad \leq \quad L\|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

Remark 1.1. It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive is uniformly Lipschitzian. Also, if $F(T) \neq \emptyset$, then a nonexpansive mapping is a quasi-nonexpansive mapping, an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian mapping with $L = \sup_{n\geq 1}\{k_n\} \geq 1$ but the converse is not true in general.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as a generalization of the class of nonexpansive mappings. Recall also that a mapping $T\colon K \to K$ is said to be asymptotically quasi-nonexpansive in the intermediate sense [18] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in K,\, p\in F(T)} \Big(\|T^n x - p\| - \|x - p\|\Big) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example:

Example 1.2. *Let $X = \mathbb{R}$ be a normed linear space and $K = [0, 1]$. For each $x \in K$, we define*

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n|x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned} \limsup_{n\to\infty}\{|T^n x - T^n y| - |x - y|\} \quad &= \quad \limsup_{n\to\infty}\{k^n\,\|x - y\| - \|x - y\|\} \\ &\leq \quad 0, \end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ *as* $0 < k < 1$*, for all* $x, y \in K$*,* $n \in \mathbb{N}$ *and* T *is continuous. Hence* T *is an asymptotically nonexpansive mapping in the intermediate sense.*

Example 1.3. *Let* $X = \mathbb{R}$*,* $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ *and* $|k| < 1$*. For each* $x \in K$*, define*

$$T(x) = \begin{cases} k\,x\,sin(1/x), & if\; x \neq 0, \\ 0, & if\; x = 0. \end{cases}$$

Then T *is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.*

Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see, e.g., [5, 6, 8, 10, 11, 12, 13, 17] and references therein).

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debate [3] gave and studied a two mappings process. Later on, many authors, for example Khan and Takahashi [8], Shahzad and Udomene [14] and Takahashi and Tamura [16] have studied the two mappings case of iterative schemes for different types of mappings.

Recently, Khan et al. [7] studied the modified two-step iteration process for two mappings as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Inspired and motivated by [1, 7] and many others, in this paper we introduce the following iteration scheme for a pair of asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. The proposed iteration scheme is as follows.

Definition 1.4. Let $S\colon K \to K$ be an asymptotically quasi-nonexpansive mapping and $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense on a closed convex subset K of a real Banach space E with $K + K \subset K$.

Let $\{x_n\}$ be the sequence defined as follows:

$$
\begin{aligned}
x_1 &= x \in K, \\
x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n + u_n, \\
y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \ n \geq 1,
\end{aligned} \tag{1.3}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{u_n\}$ and $\{v_n\}$ are two sequences in K. The iteration scheme (1.3) is called modified two-step iteration process with errors for a pair of above said mappings.

The aim of this paper is to establish some strong convergence theorems and a weak convergence theorem for newly proposed iteration scheme (1.3) in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize some previous work from the existing literature.

2. **Preliminaries**

For the sake of convenience, we restate the following concepts.

A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

We say that a Banach space E satisfies the *Opial's condition* [9] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Now, we state the following useful lemma to prove our main results.

Lemma 2.1. *(See* [15]*) Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of non-negative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

3. **Main Results**

In this section, we prove some strong convergence theorems and a weak convergence theorem of the iteration scheme (1.3) for a pair of asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense in the framework of real Banach spaces.

Theorem 3.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $S\colon K \to K$ be a asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \in [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $T\colon K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}(k_n-1)\alpha_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$. Put*

$$D_n = \max\Big\{ \sup_{x\in K,\, q\in F} \Big(\|T^n x - q\| - \|x-q\|\Big) \vee 0 : n \geq 1\Big\} \tag{3.1}$$

such that $\sum_{n=1}^{\infty} D_n < \infty$. Then $\{x_n\}$ converges to a hybrid fixed point of the mappings S and T if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$, where $d(x,F) = \inf_{p\in F} d(x,p)$.

Proof. The necessity is obvious. Thus we only prove the sufficiency. Let $q \in F$. Then from (1.3) and (3.1), we have

$$\begin{aligned}
\|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n + v_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n\|T^n x_n - q\| + \|v_n\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n[\|x_n - q\| + D_n] + \|v_n\| \\
&= (1-\beta_n)\|x_n - q\| + \beta_n\|x_n - q\| + \beta_n D_n + \|v_n\| \\
&\leq \|x_n - q\| + D_n + \|v_n\|. \qquad (3.2)
\end{aligned}$$

Again using (1.3), (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n + u_n - q\| \\
&\leq (1-\alpha_n)\|T^n x_n - q\| + \alpha_n\|S^n y_n - q\| + \|u_n\| \\
&\leq (1-\alpha_n)[\|x_n - q\| + D_n] + \alpha_n k_n\|y_n - q\| + \|u_n\| \\
&\leq (1-\alpha_n)\|x_n - q\| + \alpha_n k_n[\|x_n - q\| + D_n + \|v_n\|] \\
&\quad +(1-\alpha_n)D_n + \|u_n\| \\
&\leq [1 + (k_n-1)\alpha_n]\|x_n - q\| + [1 + (k_n-1)\alpha_n]D_n \\
&\quad +\|u_n\| + k_n\|v_n\| \\
&= (1+t_n)\|x_n - q\| + \theta_n \qquad (3.3)
\end{aligned}$$

where $t_n = (k_n-1)\alpha_n$ and $\theta_n = [1 + (k_n-1)\alpha_n]D_n + \|u_n\| + k_n\|v_n\|$. Since by hypothesis of the theorem $\sum_{n=1}^{\infty}(k_n-1)\alpha_n < \infty$, $\sum_{n=1}^{\infty} D_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$. Hence by Lemma 2.1, we know that the limit $\lim_{n\to\infty}\|x_n - q\|$ exists. Also from (3.3), we obtain

$$d(x_{n+1},F) \leq (1+t_n)\, d(x_n,F) + \theta_n \tag{3.4}$$

for all $n \geq 1$. From Lemma 2.1 and (3.4), we know that $\lim_{n\to\infty} d(x_n,F)$ exists. Since $\liminf_{n\to\infty} d(x_n,F) = 0$, we have that $\lim_{n\to\infty} d(x_n,F) = 0$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $1+x \leq e^x$ for $x \geq 0$, therefore, for any $m, n \geq 1$ and for given $q \in F$, from (3.3), we have

$$
\begin{aligned}
\|x_{n+m}-q\| &\leq (1+t_{n+m-1})\|x_{n+m-1}-q\| + \theta_{n+m-1} \\
&\leq e^{t_{n+m-1}}\|x_{n+m-1}-q\| + \theta_{n+m-1} \\
&\leq e^{t_{n+m-1}}[e^{t_{n+m-2}}\|x_{n+m-2}-q\| + \theta_{n+m-2}] + \theta_{n+m-1} \\
&\leq e^{(t_{n+m-1}+t_{n+m-2})}\|x_{n+m-2}-q\| \\
&\quad + e^{(t_{n+m-1}+t_{n+m-2})}[\theta_{n+m-2}+\theta_{n+m-1}] \\
&\leq \dots \\
&\leq e^{\left(\sum_{k=n}^{n+m-1} t_k\right)}\|x_n-q\| + e^{\left(\sum_{k=n}^{n+m-1} t_k\right)} \sum_{k=n}^{n+m-1} \theta_k \\
&\leq e^{\left(\sum_{n=1}^{\infty} t_k\right)}\|x_n-q\| + e^{\left(\sum_{n=1}^{\infty} t_k\right)} \sum_{k=n}^{n+m-1} \theta_k \\
&= R\,\|x_n-q\| + R \sum_{k=n}^{n+m-1} \theta_k \qquad (3.5)
\end{aligned}
$$

where $R = e^{\left(\sum_{n=1}^{\infty} t_k\right)} < \infty$. Since

$$\lim_{n\to\infty} d(x_n, F) = 0, \quad \sum_{n=1}^{\infty} \theta_n < \infty \tag{3.6}$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n, F) < \frac{\varepsilon}{4(R+1)}, \quad \sum_{k=n}^{\infty} \theta_k < \frac{\varepsilon}{2R} \quad \forall\, n \geq n_1. \tag{3.7}$$

Hence, there exists $q_1 \in F$ such that

$$\|x_n - q_1\| < \frac{\varepsilon}{2(R+1)} \;\forall\, n \geq n_1. \tag{3.8}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (3.5), we have

$$
\begin{aligned}
\|x_{n+m}-x_n\| &\leq \|x_{n+m}-q_1\| + \|x_n-q_1\| \\
&\leq R\,\|x_n-q_1\| + R \sum_{k=n}^{n+m-1} \theta_k + \|x_n-q_1\| \\
&\leq (R+1)\|x_n-q_1\| + R \sum_{k=n}^{n+m-1} \theta_k \\
&< (R+1).\, \frac{\varepsilon}{2(R+1)} + R.\, \frac{\varepsilon}{2R} = \varepsilon. \qquad (3.9)
\end{aligned}
$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that

$$\|x_n - q^*\| \;<\; \frac{\varepsilon_1}{2(L+1)}, \;\forall\, n \geq n_2. \tag{3.10}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, there must exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$, we have

$$d(x_n, F) < \frac{\varepsilon_1}{3(L+1)} \tag{3.11}$$

and in particular, we have

$$d(x_{n_3}, F) < \frac{\varepsilon_1}{3(L+1)}. \tag{3.12}$$

Therefore, there exists $z^* \in F$ such that

$$\|x_{n_3} - z^*\| < \frac{\varepsilon_1}{2(L+1)}. \tag{3.13}$$

Consequently, we have

$$\begin{aligned}
\|Tq^* - q^*\| &= \|Tq^* - z^* + z^* - x_{n_3} + x_{n_3} - q^*\| \\
&\leq \|Tq^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\leq L\|q^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\leq L\|q^* - x_{n_3} + x_{n_3} - z^*\| + \|z^* - x_{n_3}\| \\
&\quad + \|x_{n_3} - q^*\| \\
&\leq L[\|q^* - x_{n_3}\| + \|x_{n_3} - z^*\|] + \|z^* - x_{n_3}\| \\
&\quad + \|x_{n_3} - q^*\| \\
&\leq (L+1)\|q^* - x_{n_3}\| + (L+1)\|z^* - x_{n_3}\| \\
&< (L+1).\frac{\varepsilon_1}{2(L+1)} + (L+1).\frac{\varepsilon_1}{2(L+1)} \\
&< \varepsilon_1. \qquad (3.14)
\end{aligned}$$

This implies that $q^* \in F(T)$. Similarly, we can show that $q^* \in F(S)$. Since S is asymptotically quasi-nonexpansive mapping, so it is uniformly quasi-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\} \geq 1$ (by remark 1.1). So, as above we can show that $q^* \in F(S)$. Thus $q^* \in F = F(S) \cap F(T)$, that is, q^* is a hybrid fixed point of the mappings S and T. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $S\colon K \to K$ be a asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \in [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $T\colon K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$ and D_n be taken as in Theorem 3.1. Suppose that the mappings S and T satisfy the following conditions:*

(C_1) $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$,

(C_2) *there exists a constant $A > 0$ such that*

$$\Big\{a_1\|x_n - Sx_n\| + a_2\|x_n - Tx_n\|\Big\} \geq Ad(x_n, F)$$

for all $n \geq 1$, where a_1 and a_2 are two nonnegative real numbers such that $a_1 + a_2 = 1$.

Then $\{x_n\}$ converges strongly to a hybrid fixed point of the mappings S and T.

Proof. From conditions (C_1) and (C_2), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a hybrid fixed point of the mappings S and T. This completes the proof. □

Theorem 3.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $S\colon K \to K$ be a asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \in [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $T\colon K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$ and D_n be taken as in Theorem 3.1. Suppose $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. If at least one of the mappings S and T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a hybrid fixed point of the mappings S and T.*

Proof. Without loss of generality, we may assume that T is semi-compact. By Theorem 3.1, $\{x_n\}$ is bounded and by assumption of the theorem $\lim_{n\to\infty}$ $\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. This means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ as $n_k \to \infty$. Now again by the hypothesis of the theorem, we find

$$\|x^* - Tx^*\| = \lim_{n_k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$$

and

$$\|x^* - Sx^*\| = \lim_{n_k\to\infty}\|x_{n_k} - Sx_{n_k}\| = 0.$$

This shows that $x^* \in F$. According to Theorem 3.1, the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n_k\to\infty}\|x_{n_k} - x^*\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$, that is, the sequence $\{x_n\}$ converges strongly to a hybrid fixed point of the mappings S and T. This completes the proof. □

Theorem 3.4. *Let E be a real Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $S\colon K \to K$ be a asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \in [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $T\colon K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$ and D_n be taken as in Theorem 3.1. Suppose that S and T have a hybrid fixed point, $I - S$ and $I - T$ are demiclosed at zero and $\{x_n\}$ is an approximating hybrid fixed point sequence for S and T, that is, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges weakly to a hybrid fixed point of the mappings S and T.*

Proof. Let p be a hybrid fixed point of S and T. Then $\lim_{n\to\infty}\|x_n - p\|$ exists as proved in Theorem 3.1. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F = F(S) \cap F(T)$. For, let u and v be weak limits of the subsequences

$\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By hypothesis of the theorem, we know that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed at zero, therefore we obtain $Su = u$. Similarly, $Tu = u$. Thus $u \in F(S) \cap F(T)$. Again in the same fashion, we can prove that $v \in F(S) \cap F(T)$. Next, we prove the uniqueness. To this end, if u and v are distinct then by Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - v\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - u\| \\
&= \lim_{n\to\infty} \|x_n - u\|.
\end{aligned}$$

This is a contradiction. Hence $u = v \in F$. Thus $\{x_n\}$ converges weakly to a hybrid fixed point of the mappings S and T. This completes the proof. □

Remark 3.5. Our results extend, improve and generalize many known results from the existing literature (see, e.g., [1], [3], [4], [6]-[8], [10]-[14] and many others).

4. **Conclusion**

The results proved in this paper of the iteration scheme (1.3) for a pair of asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in the intermediate sense which is wider than that the class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings. Thus our results are good improvement and extension of some corresponding previous work from the existing literature.

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