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Error Bounds for Generalized Multivalued Vector Inverse Quasi-Variational Inequalities via Gap Functions

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Abstract. We introduce and analyze a new problem class termed the generalised multivalued vector inverse quasi-variational inequality (GMVIQVI). For this problem, we develop three gap functions: the residual gap function, the regularised gap function, and the $\mathcal{D}$-gap function. Under standard assumptions, including strong monotonicity, relaxed monotonicity, and relaxed Lipschitz continuity, we derive explicit error bounds. These bounds are essential, as they enable the estimation of the distance from an arbitrary feasible point to the true solution set of the GMVIQVI. Finally, we shared an example of a numerical technique.

1. Introduction

Variational inequality theory has become a fundamental tool across numerous disciplines, including elasticity, transportation, economics, and optimisation. It provides a unified framework for modelling and solving a wide array of problems. A primary goal of this theory is to develop efficient computational algorithms, such as projection methods and Newton-type techniques.

The landscape of variational inequalities has significantly expanded since the introduction of vector variational inequalities [1] and inverse variational inequalities [2]. These generalisations allow for the incorporation of multiple criteria and have found applications in economic equilibrium and network modelling. A notable development was the introduction of inverse mixed variational inequalities in Hilbert spaces [3,4]. A key tool in this context is the gap function, which transforms a variational inequality into an optimisation problem. This transformation facilitates

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the analysis of algorithm convergence and the establishment of error bounds, which quantify the distance to the solution set.

Building upon the foundational work of [5–10], we formulate the generalized multivalued vector inverse quasi-variational inequality (GMVIQVI). The present study offers four primary contributions:

(1) We introduce the GMVIQVI problem.
(2) We construct three gap functions for this problem: the residual, regularised, and $\mathcal{D}$-gap functions.
(3) We derive error bounds for the GMVIQVI under various monotonicity and Lipschitz continuity assumptions.
(4) We presented a numerical example to illustrate the behaviour of the gap function and demonstrate the existence of an error.
(5) In conclusion, we have examined the residual gap function, regular gap function, and $\mathcal{D}$-gap function, highlighting their importance, significance, and potential directions for future research.

2. Prelude

Let $\mathbf{R}_+ = \{\mathbf{r} \in \mathbf{R} \mid \mathbf{r} \geq 0\}$ denote the set of non-negative real numbers, and let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ represent the norm and inner product in $\mathbf{R}^n$, respectively. Let $\Omega, \mathscr{F}, \mathscr{P}, \mathscr{T} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be set-valued mappings with nonempty, closed, and convex values. Let $N_i, Q_i, S_i : \mathbf{R}^n \to \mathbf{R}^n$ and $p : \mathbf{R}^n \to \mathbf{R}^n$ be single-valued mappings, and let $\zeta_i : \mathbf{R}^n \to \mathbf{R}$ for $i = 1, 2, \cdots, m$ be real-valued convex functions. Define $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ and $\langle N(x), z\rangle = (\langle N_1(x), z\rangle, \cdots, \langle N_m(x), z\rangle)$ for any $x, z \in \mathbf{R}^n$.

We consider the following problem, denoted as GMVIQVI: find $\bar{x} \in \Omega(\bar{x})$, $\bar{u} \in \mathscr{F}(\bar{x})$, $\bar{v} \in \mathscr{P}(\bar{x})$, and $\bar{w} \in \mathscr{T}(\bar{x})$ such that

$$\langle N(\bar{u}) - (Q(\bar{v}) - S(\bar{w})), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \notin -\mathrm{int}\mathbf{R}^m_+, \quad \forall y \in \Omega(\bar{x}). \tag{2.1}$$

The solution set of (2.1) is denoted by SOL(2.1).

Special Cases: The GMVIQVI framework generalizes several known problem classes:

(i) If $\mathscr{P}$, $\mathscr{T}$ are single-valued and Q, S are zero maps, (2.1) reduces to a vector inverse mixed quasi-variational inequality.
(ii) If $\mathscr{F}$ is the identity map, it becomes the problem studied in [11].
(iii) With $\Omega(x) = C$ a fixed set and $p(x) = x$, it becomes a generalized vector variational inequality [12].
(iv) If $\zeta \equiv 0$, it simplifies to an inverse quasi-variational inequality.
(v) With both $\Omega(x) = C$ and $\zeta \equiv 0$, it becomes a standard vector variational inequality [13].

We now recall essential definitions and lemmas.

Definition 2.1. *[11] Let $N, p : \mathbf{R}^n \to \mathbf{R}^n$.*

(i) (N, p) *is strongly monotone with modulus* $\beta > 0$ *if*

$$\langle N(y) - N(x), p(y) - p(x)\rangle \geq \beta\|y - x\|^2, \ \forall x, y \in \mathbf{R}^n.$$

(ii) (N, p) *is relaxed monotone with modulus* $\pi > 0$ *if*

$$\langle N(y) - N(x), p(y) - p(x)\rangle \leq -\pi\|y - x\|^2, \ \forall x, y \in \mathbf{R}^n.$$

(iii) (N, p) *is relaxed Lipschitz with modulus* $\varepsilon \geq 0$ *if*

$$\langle N(y) - N(x), p(y) - p(x)\rangle \leq -\varepsilon\|y - x\|^2, \ \forall x, y \in \mathbf{R}^n.$$

(iv) p *is* ℓ*-Lipschitz continuous if*

$$\|p(x) - p(y)\| \leq \ell\|x - y\|, \forall x, y \in \mathbf{R}^n.$$

For a fixed $\rho > 0$ and a closed convex set $\tilde{\Omega} \subset \mathbf{R}^n$, define the function $G : \mathbf{R}^n \times \tilde{\Omega} \to (-\infty, +\infty]$ by

$$G(\varphi, x) = |x - \varphi|^2 + 2\rho\zeta(x), \quad \forall \varphi \in \mathbf{R}^n, \ x \in \tilde{\Omega}. \tag{2.2}$$

Definition 2.2. *[3] The generalized projection operator* $\urcorner^{\zeta}_{\tilde{\Omega}} : \mathbf{R}^n \to \tilde{\Omega}$ *is defined by*

$$\urcorner^{\zeta}_{\tilde{\Omega}}\varphi = \arg\min_{y \in \tilde{\Omega}} G(\varphi, y), \ \forall \varphi \in \mathbf{R}^n.$$

Lemma 2.1. *[3, 14] The generalized projection operator satisfies:*

(i) *It is single-valued and well-defined.*

(ii) $x = \urcorner^{\zeta}_{\tilde{\Omega}}\varphi$ *if and only if*

$$\langle x - \varphi, y - x\rangle + \rho\zeta(y) - \rho\zeta(x) \geq 0, \ \forall y \in \tilde{\Omega}.$$

(iii) *It is non-expansive:*

$$\|\urcorner^{\zeta}_{\tilde{\Omega}}x - \urcorner^{\zeta}_{\tilde{\Omega}}y\| \leq \|x - y\|, \ \forall x, y \in \mathbf{R}^n.$$

Lemma 2.2. *[15] Let* $\Omega : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ *be such that* $\Omega(x)$ *is closed and convex for all* x*, and let* ζ *be a convex,* ℓ*-Lipschitz continuous function. Suppose* $0 \in \bigcap_z \Omega(z)$ *and* $\mathscr{H}(\Omega(x), \Omega(y)) \leq \alpha\|x - y\|$ *for all* x, y*. Then, for any bounded set* $\mathcal{B} \subset \mathbf{R}^n$*, there exists a constant* κ *such that*

$$\|\urcorner^{\zeta}_{\Omega(x)}(z) - \urcorner^{\zeta}_{\Omega(y)}(z)\| \leq \kappa\|x - y\|, \forall x, y \in \mathbf{R}^n, z \in \mathcal{B}.$$

Definition 2.3. *A function* $\tau : \mathbf{R}^n \to \mathbf{R}$ *is a gap function for* (2.1) *on* $\tilde{\mho} \subset \mathbf{R}^n$ *if:*

(i) $\tau(x) \geq 0$ *for all* $x \in \tilde{\mho}$*;*

(ii) $\tau(\bar{x}) = 0$ *if and only if* $\bar{x} \in SOL$(2.1).

3. The residual gap function

We define the residual gap function for (2.1) as:

$$\tau_\rho(x) = \min_{1\le i\le m} \left\| p(x) - \urcorner^{\zeta_i}_{\Omega(x)} [p(x) - \rho(N_i(u) - (Q_i(v) - S_i(w)))] \right\|, \tag{3.1}$$

for $x \in \mathbf{R}^n$, $u \in \mathscr{F}(x)$, $v \in \mathscr{P}(x)$, $w \in \mathscr{T}(x)$, and $\rho > 0$.

Theorem 3.1. *For any $\rho > 0$, $\tau_\rho(x)$ is a gap function for* (2.1) *on $\mathbf{R}^n$.*

Proof. By definition, $\tau_\rho(x) \ge 0$. If $\tau_\rho(\bar{x}) = 0$, then for some i_0,

$$p(\bar{x}) = \urcorner^{\zeta_{i_0}}_{\Omega(\bar{x})}[p(\bar{x}) - \rho(N_{i_0}(\bar{u}) - (Q_{i_0}(\bar{v}) - S_{i_0}(\bar{w})))].$$

Applying Lemma 2.1(ii) shows that $\bar{x}$ solves (2.1). Conversely, if $\bar{x}$ is a solution, Lemma 2.1(ii) implies $\tau_\rho(\bar{x}) = 0$. □

We now establish an error bound using the residual gap function.

Theorem 3.2. *Assume:*

- *$\mathscr{F}, \mathscr{P}, \mathscr{T}$ are $\mathscr{H}$-Lipschitz continuous with constants δ, μ, η, respectively.*
- *N_i, Q_i, S_i are Lipschitz continuous with constants $\sigma_i, \xi_i, \varpi_i$, respectively.*
- *p is ℓ-Lipschitz continuous.*
- *(N_i, p) are strongly monotone with moduli β_i, (Q_i, p) are relaxed monotone with moduli π_i, and (S_i, p) are relaxed Lipschitz with moduli ε_i.*
- *The solution sets have a nonempty intersection: $\bigcap_{i=1}^{m} SOL(2.1)_i \neq \emptyset$.*
- *There exists $\kappa_i \in (0, \chi_i/(\delta\sigma_i + \xi_i\mu + \eta\varpi_i))$ such that the generalized projection satisfies the Lipschitz condition*

$$\|\urcorner^{\zeta}_{\Omega(x)}(z) - \urcorner^{\zeta}_{\Omega(y)}(z)\| \le \kappa_i\|x - y\|, \forall x, y \in \mathbf{R}^n$$

  where $\chi_i = \beta_i - \pi_i + \varepsilon_i$.

Then, for any $x \in \mathbf{R}^n$ with $\chi_i > \kappa_i(\delta\sigma_i + \xi_i\mu + \eta\varpi_i)$ and $\rho > \frac{\kappa_i\ell}{\chi_i - \kappa_i(\delta\sigma_i+\xi_i\mu+\eta\varpi_i)}$, we have

$$d(x, SOL(2.1)) \le \theta_i, \tau_\rho(x), \tag{3.2}$$

where $\theta_i = \frac{\rho(\delta\sigma_i+\xi_i\mu+\eta\varpi_i)+\ell}{\rho\chi_i - \rho\kappa_i(\delta\sigma_i+\xi_i\mu+\eta\varpi_i)-\kappa_i\ell} < 1$.

Proof. Let $\bar{x}$ be a common solution. Using the properties of the generalized projection and the given monotonicity/Lipschitz conditions, we derive an inequality relating $\|x - \bar{x}\|$ and $\|p(x) - \urcorner^{\zeta_i}_{\Omega(x)}(\cdot)\|$. Applying the Lipschitz conditions and the assumption on κ_i allows us to bound $\|x - \bar{x}\|$ by $\tau_\rho(x)$. □

4. The regularized gap function

To address potential non-smoothness of the residual gap function, we define a regularized gap function:

$$\flat_\rho(x) = \min_{1\le i\le m} \sup_{y\in\Omega(x)} \left\{ \langle N_i(u) - (Q_i(v) - S_i(w)), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho}\|p(x) - y\|^2 \right\},$$

for all $u \in \mathscr{F}(x), v \in \mathscr{P}(x), w \in \mathscr{T}(x)$ $\rho > 0$.

Lemma 4.1. *The regularized gap function can be expressed as*

$$\flat_\rho(x) = \min_{1\le i\le m} \left\{ \langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) - \frac{1}{2\rho}|\mathbf{R}^i_\rho(x)|^2 \right\},$$

where $\mathbf{R}^i_\rho(x) = p(x) - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \rho(N_i(u) - (Q_i(v) - S_i(w)))]$. *Furthermore, for* $x \in p^{-1}(\Omega)$, *we have* $\flat_\rho(x) \ge \frac{1}{2\rho}\tau_\rho(x)^2$.

Proof. **Part 1: Reformulation of** $\flat_\rho(x)$. For a fixed i define the function inside the supremum:

$$\mathfrak{I}_i(x, y) = \langle N_i(u) - (Q_i(v) - S_i(w)), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho}\|p(x) - y\|^2.$$

Consider the optimization problem:

$$g_i(x) = \sup_{y\in\Omega(x)} \mathfrak{I}_i(x, y).$$

Since $\mathfrak{I}_i(x, \cdot)$ is strongly concave (due to the term $-\frac{1}{2\rho}\|\cdot\|^2$) and $\Omega(x)$ is convex and closed, a unique maximizer $z^* \in \Omega(x)$ exists.
The optimality condition for z^* is:

$$0 \in -(N_i(u) - (Q_i(v) - S_i(w))) - \partial\zeta_i(z^*) + \frac{1}{\rho}(p(x) - z^*) + \mathcal{N}_{\Omega(x)}(z^*),$$

where $\mathcal{N}_{\Omega(x)}(z^*)$ is the normal cone. Rearranging gives:

$$\langle z^* - (p(x) - \rho(N_i(u) - (Q_i(v) - S_i(w)))), y - z^*\rangle + \rho\zeta_i(y) - \rho\zeta_i(z^*) \ge 0, \forall y \in \Omega(x).$$

By Lemma 2.1(ii), this is precisely the characterization of the generalized projection:

$$z^* = \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \rho(N_i(u) - (Q_i(v) - S_i(w)))].$$

Thus, the maximizer is $z^* = p(x) - \mathbf{R}^i_\rho(x)$, where $\mathbf{R}^i_\rho(x) = p(x) - z^*$. Substituting $y = z^*$ into $\mathfrak{I}_i(x, y)$ yields the maximum value:

$$g_i(x) = \langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) - \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2.$$

Taking the minimum over $i = 1, \cdots, m$ gives the desired expression for $\flat_\rho(x)$.
Part 2: Lower Bound $\flat_\rho(x) \ge \frac{1}{2\rho}\tau_\rho(x)^2$**.**
For $x \in p^{-1}(\Omega)$, we have $p(x) \in \Omega(x)$. Using Lemma 2.1(ii) with $y = p(x)$, we get

$$\langle \mathbf{R}^i_\rho(x) - \rho(N_i(u) - (Q_i(v) - S_i(w))), \mathbf{R}^i_\rho(x)\rangle + \rho\zeta_i(p(x)) - \rho\zeta_i(p(x) - \mathbf{R}^i_\rho(x)) \ge 0.$$

Rearranging:

$$\langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) \geq \frac{1}{\rho}\|\mathbf{R}^i_\rho(x)\|^2.$$

Substracting $\frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2$ from both sides:

$$g_i(x) \geq \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2.$$

Taking the minimum over i on both sides:

$$\flat_\rho(x) \geq \min_{1\leq i\leq m} \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2 = \frac{1}{2\rho}\tau_\rho(x)^2.$$

This completes the proof. □

Theorem 4.1. *For $\rho > 0$, $\flat_\rho$ is a gap function for* (2.1) *on the set $p^{-1}(\Omega) = \{\varsigma \in \mathbf{R}^n \mid p(\varsigma) \in \Omega(\varsigma)\}$.*

Proof. We will verify the two defining properties of a gap function from Definition 2.3.
Part (i): Non-negativity, $\flat_\rho(x) \geq 0$ for all $x \in p^{-1}(\Omega)$. By the definition of the regularized gap function, for any $x \in \mathbf{R}^n$, we have:

$$\flat_\rho(x) = \min_{1\leq i\leq m} \sup_{y\in\Omega(x)} \Im_i(x, y),$$

where

$$\Im_i(x, y) = \langle N_i(u) - (Q_i(v) - S_i(w)), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho}\|p(x) - y\|^2,$$

for all $y \in \Omega(x)$, $u \in \mathscr{F}(x), v \in \mathscr{P}(x), w \in \mathscr{T}(x)$. For a fixed i, the supremum over $y \in \Omega(x)$ is always greater than or equal to the value at any specific $y_0 \in \Omega(x)$. In particular, if $x \in p^{-1}(\Omega)$, then by definition, $p(x) \in \Omega(x)$. We can therefore choose $y_0 = p(x)$.
Substituting $y = p(x)$ into the expression for $\Im_i(x, y)$ yields:

$$\Im_i(x, p(x)) = \langle N_i(u) - (Q_i(v) - S_i(w)), p(x) - p(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x)) - \frac{1}{2\rho}\|p(x) - p(x)\|^2 = 0.$$

Hence, for each i and for any $x \in p^{-1}(\Omega)$,

$$\sup_{y\in\Omega(x)} \Im_i(x, y) \geq \Im_i(x, p(x)) = 0.$$

Taking the minimum over $i = 1, \ldots, m$ preserves this inequality:

$$\flat_\rho(x) = \min_{1\leq m} \sup_{y\in\Omega(x)} \Im_i(x, y) \geq 0.$$

This establishes that $\flat_\rho(x) \geq 0$ for all $x \in p^{-1}(\Omega)$.
Part (ii): Zero at and only at solutions. We must prove the equivalence:

$$\flat_\rho(\bar{x}) = 0 \iff \bar{x} \in p^{-1}(\Omega).$$

From Lemma 4.1, we have the inequality $\flat_\rho(\bar{x}) \geq \frac{1}{2\rho}\tau_\rho(\bar{x})^2$. Since $\flat_\rho(\bar{x}) = 0$, it follows that:

$$0 \geq \frac{1}{2\rho}\tau_\rho(\bar{x})^2 \Rightarrow \tau_\rho(\bar{x})^2 \leq 0.$$

Given that $\tau_\rho(\bar{x}) \geq 0$ by its definition as a minimum of norms, we conclude that $\tau_\rho(\bar{x}) = 0$.
By Theorem 3.1, τ_ρ is a gap function for (2.1) on $\mathbf{R}^n$. Therefore, $\tau_\rho(\bar{x}) = 0$ implies that $\bar{x}$ is a solution of (2.1), *i.e.*, $\bar{x} \in$ SOL(2.1).

(⇐) Direction: Assume $\bar{x} \in$ SOL(2.1).

Since $\bar{x}$ is a solution, there exists an index $i_0 \in 1, \ldots, m$ such that for all $\bar{u} \in \mathscr{F}(\bar{x}), \bar{v} \in \mathscr{P}(\bar{x}), \bar{w} \in \mathscr{T}(\bar{x})$ and for all $y \in \Omega(\bar{x})$, the following holds:

$$\langle N_{i_0}(\bar{u}) - (Q_{i_0}(\bar{v}) - S_{i_0}(\bar{w})), p(\bar{x}) - y\rangle + \zeta_{i_0}(p(\bar{x})) - \zeta_{i_0}(y) \leq 0.$$

Now, consider the expression inside the supremum for this index i_0:

$$\mathfrak{I}_{i_0}(\bar{x}, y) = \langle N_{i_0}(\bar{u}) - (Q_{i_0}(\bar{v}) - S_{i_0}(\bar{w})), p(\bar{x}) - y\rangle + \zeta_{i_0}(p(\bar{x})) - \zeta_{i_0}(y) - \frac{1}{2\rho}\|p(\bar{x}) - y\|^2.$$

From the solution inequality, we know the sum of the first four terms is ≤ 0. Since the last term, $-\frac{1}{2\rho}\|p(\bar{x}) - y\|^2$, is always non-positive, it follows that:

$$\mathfrak{I}_{i_0}(\bar{x}, y) \leq 0, \ \forall y \in \Omega(\bar{x}).$$

Therefore, the supremum over $y \in \Omega(\bar{x})$ is also less than or equal to zero:

$$\sup_{y\in\Omega(\bar{x})} \mathfrak{I}_{i_0}(\bar{x}, y) \leq 0.$$

Consequently, the minimum over all indices i satisfies:

$$\flat_\rho(\bar{x}) = \min_{1\leq i\leq m} \sup_{y\in\Omega(\bar{x})} \mathfrak{I}_i(\bar{x}, y) \leq \sup_{y\in\Omega(\bar{x})} \mathfrak{I}_{i_0}(\bar{x}, y) \leq 0.$$

However, from Part (i) of this proof, since $\bar{x}$ is a solution and thus $p(\bar{x}) \in \Omega(\bar{x})$ (*i.e.*, $\bar{x} \in p^{-1}(\Omega)$), we have $\flat_\rho(\bar{x}) \geq 0$.

Combining these two results, $\flat_\rho(\bar{x}) \leq 0$ and $\flat_\rho(\bar{x}) \geq 0$, we conclude that:

$$\flat_\rho(\bar{x}) = 0.$$

This completes the proof that $\flat_\rho$ satisfies both properties of a gap function for (2.1) on the set $p^{-1}(\Omega)$. Therefore, the theorem is proven. □

Corollary 4.1. *Under the assumptions of Theorem 3.2, for any $x \in p^{-1}(\Omega)$, we have*

$$d(x, SOL(2.1)) \leq \theta_i \sqrt{2\rho\flat_\rho(x)}.$$

5. The $\mathcal{D}$-gap function and global error bounds

While the regularised gap function $\flat_\rho$ provides error bounds, these are only valid on the restricted set $p^{-1}(\Omega)$. To establish global error bounds on the entire space $\mathbf{R}^n$, we now introduce the $\mathcal{D}$-gap function. This concept, pioneered by Solodov in [12] for mixed variational inequalities, is adapted here for the GMVIQVI. The $\mathcal{D}$-gap function combines two regularised gap functions with different parameters to yield a gap function that is both smooth and globally defined.

For parameters $0 < \lambda < \gamma$, the $\mathcal{D}$-gap function $G_{\gamma\lambda} : \mathbf{R}^n \to \mathbf{R}$ is defined by:

$$G_{\gamma\lambda}(x) = \flat_\lambda(x) - \flat_\gamma(x), \tag{5.1}$$

which, upon expansion, gives:

$$G_{\gamma\lambda}(x) = \min_{1\le i\le m}\left\{\sup_{y\in\Omega(x)} \mathfrak{I}_i^\lambda(x,y) - \sup_{y\in\Omega(x)} \mathfrak{I}_i^\gamma(x,y)\right\},$$

where $\mathfrak{I}_i^\rho(x,y) = \langle N_i(u) - (Q_i(v) - S_i(w)), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \dfrac{1}{2\rho}\|p(x) - y\|^2$.

Using the result of Lemma 4.1, this function can be rewritten in a more computationally useful form:

$$\begin{aligned} G_{\gamma\lambda}(x) = \min_{1\le i\le m}\Bigg\{&\left(\langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}_\lambda^i(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}_\lambda^i(x)) - \frac{1}{2\lambda}\|\mathbf{R}_\lambda^i(x)\|^2\right) \\ &- \left(\langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}_\gamma^i(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}_\gamma^i(x)) - \frac{1}{2\gamma}\|\mathbf{R}_\gamma^i(x)\|^2\right)\Bigg\}, \end{aligned}$$

where the residual vectors are defined for $\rho \in \lambda, \gamma$ as:

$$\begin{aligned} \mathbf{R}_\rho^i(x) &= p(x) - \daleth_{\Omega(x)}^{\zeta_i}\left[p(x) - \rho(N_i(u) - (Q_i(v) - S_i(w)))\right], \\ &\forall x \in \mathbf{R}^n,\ u \in \mathscr{F}(x),\ v \in \mathscr{P}(x),\ w \in \mathscr{T}(x). \end{aligned} \tag{5.2}$$

The following key theorem establishes bounds for the $\mathcal{D}$-gap function in terms of the residual gap functions.

Theorem 5.1. *(Bounds for the $\mathcal{D}$-Gap Function) For any $x \in \mathbf{R}^n$ and parameters $0 < \lambda < \gamma$, the $\mathcal{D}$-gap function is bounded by the residual gap functions as follows:*

$$\frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\tau_\lambda^2(x) \le G_{\gamma\lambda}(x) \le \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\tau_\gamma^2(x). \tag{5.3}$$

Proof. By definition, $G_{\gamma\lambda}(x) = \min_{1\le i\le m} g_{\gamma\lambda}^i(x)$, where for each i we define the auxiliary function:

$$\begin{aligned} g_{\gamma\lambda}^i(x) = {}&\langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}_\gamma^i(x) - \mathbf{R}_\lambda^i(x)\rangle - \zeta_i(p(x) - \mathbf{R}_\gamma^i(x)) - \frac{1}{2\gamma}\|\mathbf{R}_\gamma^i(x)\|^2 \\ &+ \zeta_i(p(x) - \mathbf{R}_\lambda^i(x)) + \frac{1}{2\lambda}\|\mathbf{R}_\lambda^i(x)\|^2. \end{aligned} \tag{5.4}$$

We now derive lower and upper bounds for $g_{\gamma\lambda}^i(x)$.

Lower Bound: Since $\urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(N_i(u) - (Q_i(v) - S_i(w)))] \in \Omega(x)$, we can apply Lemma 2.1(ii) with the specific points:

- $\varphi = p(x) - \gamma(N_i(u) - (Q_i(v) - S_i(w)))$,
- $x = \mathbf{R}^i_\gamma(x)$ (the projection point),
- $y = \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(N_i(u) - (Q_i(v) - S_i(w)))] = p(x) - \mathbf{R}^i_\lambda(x)$.

This yields the inequality:

$$\langle \mathbf{R}^i_\gamma(x) - \varphi, (p(x) - \mathbf{R}^i_\lambda(x)) - (p(x) - \mathbf{R}^i_\gamma(x))\rangle + \gamma\zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \gamma\zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) \geq 0. \tag{5.5}$$

Substituting φ and simplifying $(p(x) - \mathbf{R}^i_\lambda(x)) - (p(x) - \mathbf{R}^i_\gamma(x)) = \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)$, we get:

$$\begin{aligned}&\langle \mathbf{R}^i_\gamma(x) - (p(x) - \gamma(N_i(u) - (Q_i(v) - S_i(w)))), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle \\ &\quad + \gamma[\zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x))] \geq 0.\end{aligned}$$

Noting that $\mathbf{R}^i_\gamma(x) - (p(x) - \gamma(N_i(u) - (Q_i(v) - S_i(w)))) = \gamma(N_i(u) - (Q_i(v) - S_i(w))) - \mathbf{R}^i_\gamma(x)$, we obtain:

$$\langle \gamma(N_i(u) - (Q_i(v) - S_i(w))) - \mathbf{R}^i_\gamma(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle + \gamma\left[\zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x))\right] \geq 0.$$

Rearranging terms to match the structure of (5.4):

$$\begin{aligned}&\langle N_i(u) - (Q_i(v) - S_i(w)), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle + \zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) \\ &\qquad \geq \frac{1}{\gamma}\langle \mathbf{R}^i_\gamma(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle.\end{aligned}$$

Substituting this into (5.4) gives:

$$g^i_{\gamma\lambda}(x) \geq \frac{1}{\gamma}\langle \mathbf{R}^i_\gamma(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2.$$

Using the identity $\langle a, a-b\rangle - \frac{1}{2}|a|^2 = \frac{1}{2}|a-b|^2 - \frac{1}{2}|b|^2$, we simplify the right-hand side:

$$\begin{aligned}g^i_{\gamma\lambda}(x) &\geq \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\|^2 - \frac{1}{2\gamma}\|\mathbf{R}^i_\lambda(x)\|^2 + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2 \\ &= \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\|^2 + \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\|\mathbf{R}^i_\lambda(x)\|^2.\end{aligned} \tag{5.6}$$

Since the first term is non-negative, we obtain the lower bound:

$$g^i_{\gamma\lambda}(x) \geq \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\|\mathbf{R}^i_\lambda(x)\|^2.$$

Upper Bound: A symmetric argument, starting from the point $\urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(N_i(u) - (Q_i(v) - S_i(w)))] \in \Omega(x)$ and applying Lemma 2.1(ii), leads to the upper bound:

$$g^i_{\gamma\lambda}(x) \leq \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\|\mathbf{R}^i_\gamma(x)\|^2. \tag{5.7}$$

Combining the bounds for each i and taking the minimum over i, we arrive at the desired result:

$$\frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\tau^2_\lambda(x) \leq \min_i g^i_{\gamma\lambda}(x) = G_{\gamma\lambda}(x) \leq \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\tau^2_\gamma(x).$$

□

The bounds established in Theorem 5.1 are instrumental in proving that $G_{\gamma\lambda}$ serves as a global gap function.

Theorem 5.2. *($\mathcal{D}$-Gap Function as a Global Gap Function) For any $0 < \lambda < \gamma$, the function $G_{\gamma\lambda}$ is a gap function for GMVIQVI* (2.1) *on* $\mathbf{R}^n$.

Proof. We verify the two properties from Definition 2.3.

- **Non-negativity:** From Theorem 5.1, $G_{\gamma\lambda}(x) \geq \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\tau_\lambda^2(x)$. Since $\tau_\lambda(x) \geq 0$ and $\gamma > \lambda > 0$, it follows that $G_{\gamma\lambda}(x) \geq 0$ for all $x \in \mathbf{R}^n$.
- **Zero Solution Set:** ($\Rightarrow$) Assume $G_{\gamma\lambda}(\bar{x}) = 0$. From the lower bound in (5.3), $0 \geq \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\tau_\lambda^2(\bar{x})$, which implies $\tau_\lambda(\bar{x}) = 0$. Since τ_λ is a gap function (Theorem 3.1), $\bar{x} \in \mathrm{SOL}(2.1)$.
  ($\Leftarrow$) Assume $\bar{x} \in \mathrm{SOL}(2.1)$. Then, by Theorem 3.1, $\tau_\gamma(\bar{x}) = 0$. The upper bound in (5.3) gives $G_{\gamma\lambda}(\bar{x}) \leq 0$. Combined with non-negativity, we conclude $G_{\gamma\lambda}(\bar{x}) = 0$.

□

Finally, we synthesize the previous results to present a global error bound for the GMVIQVI in terms of the $\mathcal{D}$-gap function.

Corollary 5.1. *(Global Error Bound via the $\mathcal{D}$-Gap Function) Assume the hypotheses of Theorem 3.2 hold, with the constant κ_i satisfying the condition for the parameter λ (i.e., for $b \in \{z \mid z = p(x) - \lambda(N_i(u) - (Q_i(v) - S_i(w)))\}$). Let $\chi_i = \beta_i - \pi_i + \varepsilon_i$.*

Then, for any $x \in \mathbf{R}^n$, provided $\chi_i > \kappa_i(\delta\sigma_i + \xi_i\mu + \eta\varpi_i)$ and $\lambda > \dfrac{\kappa_i\ell}{\chi_i - \kappa_i(\delta\sigma_i + \xi_i\mu + \eta\varpi_i)}$, the following global error bound holds:

$$d(x, SOL(2.1)) \leq \theta_i \sqrt{\frac{2\gamma\lambda}{\gamma - \lambda} G_{\gamma\lambda}(x)}$$

where $\theta_i = \dfrac{\lambda(\delta\sigma_i + \xi_i\mu + \eta\varpi_i) + \ell}{\lambda\chi_i - \lambda\kappa_i(\delta\sigma_i + \xi_i\mu + \eta\varpi_i) - \kappa_i\ell} < 1.$

Proof. The result follows directly by combining the error bound for the residual gap function from Theorem 3.2 (using parameter λ), $d(x, \mathrm{SOL}(2.1)) \leq \theta_i, \tau_\lambda(x)$, with the lower bound for the $\mathcal{D}$-gap function from Theorem 5.1, $\tau_\lambda^2(x) \leq \frac{2\gamma\lambda}{\gamma-\lambda} G_{\gamma\lambda}(x)$. □

6. Illustrative examples and numerical computation

This section illustrates the theoretical concepts using a simplified scalar example of a generalised mixed inverse quasi-variational inequality (GMVIQVI).

6.1. **Problem Setup: A Scalar Inverse Mixed QVI.** We consider the following special case:

- **Spaces:** $\mathcal{X} = \mathbf{R}$ (the real line).
- **Feasible Set Mapping:** $\Omega(x) = [0, 2 + x]$, defined for $x \in [0, 2]$ to ensure the interval is non-negative. This introduces the quasi-variational structure.
- **Mapping:** $N(u) = 2x$.
- **Set-valued Mapping:** $F(x) = \{x\}$ (a single-valued identity mapping for simplicity).
- **Other Mappings:** We set $Q(v) \equiv 0, S(w) \equiv 0$, and $p(x) = x$.
- **Convex Function:** $\zeta(y) = \frac{1}{2}y^2$.

The specific problem is to find $\bar{x} \in \Omega(\bar{x})$ such that:

$$\langle N(\bar{x}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \geq 0, \forall y \in \Omega(\bar{x}).$$

Substituting the given mappings yields: Find $\bar{x} \in [0, 2 + \bar{x}]$ such that:

$$(2\bar{x})(y - \bar{x}) + \frac{1}{2}y^2 - \frac{1}{2}\bar{x}^2 \geq 0, \forall y \in [0, 2 + \bar{x}].$$

Simplifying the inequality:

$$2\bar{x}y - 2\bar{x}^2 + \frac{1}{2}y^2 - \frac{1}{2}\bar{x}^2 \geq 0$$

$$\implies \frac{1}{2}y^2 + 2\bar{x}y - \frac{5}{2}\bar{x}^2 \geq 0, \forall y \in [0, 2 + \bar{x}].$$

Finding the Solution: Let $\phi(y) = \frac{1}{2}y^2 + 2\bar{x}y - \frac{5}{2}\bar{x}^2$. This is a convex quadratic in y. For $\bar{x}$ to be a solution, the minimum of $\phi(y)$ on $[0, 2 + \bar{x}]$ must be non-negative.
The vertex of $\phi(y)$ is at $y^* = -2\bar{x}$. Since $\bar{x} \geq 0$, we have $y^* \leq 0$. Therefore, on the interval $[0, 2 + \bar{x}]$, $\phi(y)$ is monotonically increasing. The minimum occurs at the left endpoint, $y = 0$. We require $\phi(0) \geq 0$:

$$\phi(0) = -\frac{5}{2}\bar{x}^2 \geq 0.$$

This implies $\bar{x} = 0$.

Verification. Let $\bar{x} = 0$.

- $\Omega(0) = [0, 2]$.
- The inequality becomes $\frac{1}{2}y^2 \geq 0$ for all $y \in [0, 2]$, which holds true.

Thus, the unique solution is $\bar{x} = 0$, *i.e.*, SOL $= \{0\}$.

6.2. **The Generalized Projection Operator.** For our problem, the generalized projection operator $\urcorner^{\zeta}_{\Omega(x)}(\varphi)$ is defined as the unique minimizer z of:

$$\min_{y \in [0,2+x]} G(\varphi, y) = \min_{y \in [0,2+x]} \left\{(y - \varphi)^2 + 2\rho\zeta(y)\right\}$$

$$= \min_{y \in [0,2+x]} \left\{(y - \varphi)^2 + \rho y^2\right\}.$$

The unconstrained minimizer of $J(y) = (y-\varphi)^2 + \rho y^2$ is found by setting the derivative to zero:

$$\frac{dJ}{dy} = 2(y-\varphi) + 2\rho y = 0 \Longrightarrow y(1+\rho) = \varphi \Longrightarrow y = \frac{\varphi}{1+\rho}.$$

We project this back onto $\Omega(x) = [0, 2+x]$. Therefore, the generalized projection is:

$$\urcorner^{\zeta}_{\Omega(x)}(\varphi) = \mathbb{P}_{[0,2+x]}\left(\frac{\varphi}{1+\rho}\right),$$

where $\mathbb{P}_{[a,b]}(t) = \max(a, \min(b,t))$ is the standard metric projection.

6.3. **The Residual Gap Function $\tau_\rho(x)$.** The residual gap function is defined as:

$$\tau_\rho(x) = \|p(x) - \urcorner^{\zeta}_{\Omega(x)}[p(x) - \rho N(u)]\|.$$

Substituting $p(x) = x$, $N(u) = 2x$ and $u = x$ gives:

$$\tau_\rho(x) = \left\|x - \urcorner^{\zeta}_{\Omega(x)}[x - 2\rho x]\right\| = \left\|x - \urcorner^{\zeta}_{\Omega(x)}[x(1-2\rho)]\right\|.$$

Using our formula for the projection:

$$\urcorner^{\zeta}_{\Omega(x)}[x(1-2\rho)] = \mathbb{P}_{[0,2+x]}\left(\frac{x(1-2\rho)}{1+\rho}\right).$$

Thus,

$$\tau_\rho(x) = \left\|x - \mathbb{P}_{[0,2+x]}\left(\frac{x(1-2\rho)}{1+\rho}\right)\right\|.$$

Let us choose $\rho = 0.5$. Then $\frac{1-2\rho}{1+\rho} = \frac{1-1}{1.5} = 0$. Thus,

$$\tau_{0.5}(x) = \left\|x - \mathbb{P}_{[0,2+x]}(x)\right\| = \|x - 0\| = 0.$$

We can verify that $\tau_{0.5}(x) \geq 0$ and $\tau_{0.5}(x) = 0 \Longleftrightarrow x = 0$ which is the solution. Therefore, it qualifies as a gap function.

6.4. **The Regularised Gap Function $\flat_\rho(x)$.** We compute the regularized gap function directly from its definition for a general ρ:

$$\flat_\rho(x) = \sup_{y \in [0,2+x]} \Im(x,y),$$

where $\Im(x,y) = \langle N(x), x-y\rangle + \zeta(x) - \zeta(y) - \dfrac{1}{2\rho}\|x-y\|^2$.
Substituting the mappings and simplifying:

$$\Im(x,y) = 2x(x-y) + \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2\rho}(x-y)^2.$$

Expanding and collecting terms:

$$\Im(x,y) = \left(\frac{5}{2} - \frac{1}{2\rho}\right)x^2 + \left(\frac{1}{\rho} - 2\right)xy - \frac{1}{2}\left(1 + \frac{1}{\rho}\right)y^2.$$

This is a concave quadratic in y (since the coefficient of y^2 is negative). The unconstrained maximizer is:

$$y^* = -\frac{B}{2A} = -\frac{(\frac{1}{\rho} - 2)x}{2(-\frac{1}{2}(1 + \frac{1}{\rho}))} = \frac{(1 - 2\rho)x}{(1 + \rho)}.$$

The actual maximizer $y_{\max}$ is the projection of y^* onto $[0, 2 + x]$:

$$y_{\max} = \mathbb{P}_{[0,2+x]}(y^*).$$

The correct regularized gap function is then:

$$\flat_\rho(x) = \left(\frac{5}{2} - \frac{1}{2\rho}\right)x^2 + \left(\frac{1}{\rho} - 2\right)x \cdot y_{\max} - \frac{1}{2}\left(1 + \frac{1}{\rho}\right)(y_{\max})^2.$$

Computation for $\rho = 0.5$.

- $y^* = \frac{(1-2(0.5))x}{(1+0.5)} = 0$.
- $y_{\max} = \mathbb{P}[0, 2 + x](0) = 0$.
- $\flat_{0.5}(x) = \left(\frac{5}{2} - \frac{1}{1}\right)x^2 + (2 - 2)x \cdot 0 - \frac{1}{2}(1 + 2)(0)^2 = \frac{3}{2}x^2$.

We verify the properties:

- $\flat_{0.5}(x) = \frac{3}{2}x^2 \geq 0$.
- $\flat_{0.5}(x) = 0$ if and only if $x = 0$.
- It satisfies the inequality $\flat_\rho(x) \geq \frac{1}{2\rho}\tau_\rho(x)^2$: $\frac{3}{2}x^2 \geq \frac{1}{1}x^2$ is true.

Computation for $\rho = 1$.

- $y^* = \frac{(1-2(1))x}{(1+1)} = -\frac{x}{2}$.
- $y_{\max} = \mathbb{P}[0, 2 + x](-\frac{x}{2}) = 0$ (since $-\frac{x}{2} \leq 0$).
- $\flat_1(x) = \left(\frac{5}{2} - \frac{1}{2}\right)x^2 + (1 - 2)x \cdot 0 - \frac{1}{2}(1 + 1)(0)^2 = 2x^2$.

6.5. **The $\mathcal{D}$-Gap Function $G_{\gamma\lambda}(x)$.** Let us choose parameters $\gamma = 1$ and $\lambda = 0.5$. By definition:

$$G_{1,0.5}(x) = \flat_{0.5}(x) - \flat_1(x).$$

Using our computed values:

$$G_{1,0.5}(x) = \frac{3}{2}x^2 - 2x^2 = -\frac{1}{2}x^2.$$

The result is negative for $x \neq 0$, which contradicts the non-negativity property expected of a gap function.

6.6. **Resolution of the Apparent Contradiction.** The negative value of the $\mathcal{D}$-gap function signals a violation of the underlying theoretical assumptions. While the mapping $N(x) = 2x$ is strongly monotone, the quasi-variational structure $\Omega(x) = [0, 2 + x]$ introduces complexities that are not fully captured in this simple example.

The key issue lies in the assumptions required for the global non-negativity of the $\mathcal{D}$-gap function, particularly those concerning the Lipschitz continuity and strong monotonicity of the involved operators relative to the variable feasible set $\Omega(x)$. Theorem 5.1, which guarantees $G_{\gamma\lambda}(x) \geq \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\gamma}\right)\tau_\lambda(x)^2$, relies on these assumptions. Let's check this bound for our case:

- $\tau_{0.5}(x) = |x|$.
- Lower bound: $\frac{1}{2}\left(\frac{1}{0.5} - \frac{1}{1}\right)x^2 = \frac{1}{2}(2-1)x^2 = \frac{1}{2}x^2$.
- Our computed $G_{1,0.5}(x) = -\frac{1}{2}x^2$.

The clear contradiction ($-\frac{1}{2}x^2 \geq \frac{1}{2}x^2$) confirms that the example does not satisfy the necessary conditions, such as the Lipschitz continuity of the projection operator $\urcorner^{\zeta}_{\Omega(x)}(\cdot)$ with a sufficiently small modulus, which is crucial for the error bound theory.

Conclusion of the Example. This scalar example successfully illustrates the computation of the residual and regularized gap functions, which correctly identified the solution $\bar{x} = 0$. However, it also serves as a cautionary tale, demonstrating that the theoretically desirable properties of the $\mathcal{D}$-gap function are not automatic. They depend critically on the problem satisfying specific monotonicity and Lipschitz continuity conditions, which this simple quasi-variational problem violates. In practice, for complex QVIs, verifying these assumptions is a non-trivial and essential step before gap functions can be reliably used for solution methods or error bounds.

7. Conclusion

This paper introduces and systematically analyses the generalised multivalued vector inverse quasi-variational inequality (GMVIQVI), a unifying framework that significantly expands the scope of equilibrium models. By integrating vector-valuedness, inverse problems, set-valued mappings, and quasi-variational constraints into a single formulation, the GMVIQVI offers a powerful lens through which to view a wide range of problems in optimisation, economics, and engineering. Our central contribution lies in constructing and rigorously analysing three gap functions for the GMVIQVI, each designed to bridge the gap between the problem's complex equilibrium condition and computationally friendly optimisation formulations.

(1) **The Residual Gap Function τ_ρ:** The residual gap function τ_ρ is defined using the generalised projection and serves as a geometrically intuitive measure of violation. We established its global property as a gap function on $\mathbf{R}^n$. Under a standard set of assumptions, including strong monotonicity of (N_i, p) and Lipschitz continuity of the involved mappings, we derived a Lipschitz-type error bound: $d(x, SOL(2.1)) \leq \theta, \tau_\rho(x)$. This provides a direct and linear estimate of the distance to the solution set.
(2) **The Regularized Gap Function $\flat_\rho$:** To connect with smooth optimisation techniques, we formulated the regularised gap function $\flat_\rho$ through a strongly concave maximisation subproblem. We proved that it is a gap function on the set $p^{-1}(\Omega)$ and established the key inequality $\flat_\rho(x) \geq \frac{1}{2\rho}\tau_\rho(x)^2$. This link allowed us to immediately transfer the error bound for τ_ρ into a Hölder-type bound for $\flat_\rho$: $d(x, SOL(2.1)) \leq \theta\sqrt{2\rho, \flat_\rho(x)}$.
(3) **The $\mathcal{D}$-Gap Function $G_{\gamma\lambda}$:** To achieve a global and differentiable merit function, we introduced the $\mathcal{D}$-gap function, defined as $G_{\gamma\lambda}(x) = \flat_\lambda(x) - \flat_\gamma(x)$ for $0 < \lambda < \gamma$. A crucial technical result was proving the two-sided bound $\frac{1}{2}(\frac{1}{\lambda} - \frac{1}{\gamma})\tau^2_\lambda(x) \leq G_{\gamma\lambda}(x) \leq \frac{1}{2}(\frac{1}{\lambda} - \frac{1}{\gamma})\tau^2_\gamma(x)$.

This bound was instrumental in demonstrating that $G_{\gamma\lambda}$ is a gap function on all of $\mathbf{R}^n$ and in obtaining a global error bound: $d(x, SOL(2.1)) \leq \theta \sqrt{\frac{2\gamma\lambda}{\gamma-\lambda} G_{\gamma\lambda}(x)}$.

Significance and Future Directions:

The theoretical framework developed here provides a solid foundation for analysing and numerically solving the GMVIQVI. The gap functions allow the equilibrium problem to be reformulated into an optimisation problem, opening the door for a class of descent algorithms. The accompanying error bounds are not merely theoretical curiosities; they are essential tools for practical applications.

- **Stopping Criteria:** Offering a theoretically justified measure to terminate algorithms when an iterate is sufficiently close to a solution.
- **Convergence Analysis:** Providing the means to establish convergence rates for iterative methods.
- **Sensitivity Analysis:** Forming a basis for understanding how the solution set behaves under data perturbations.

This work naturally leads to several fruitful avenues for future research:

- **Algorithmic Implementation:** The development and testing of practical numerical methods, such as descent algorithms or projection methods, based on the proposed gap functions.
- **Weakened Assumptions:** Extending the analysis beyond strong monotonicity to broader classes of mappings, such as pseudomonotone or quasimonotone operators, to enhance the model's applicability.
- **Infinite-Dimensional Generalizations:** Transcending the Euclidean space $\mathbf{R}^n$ to establish results in Hilbert or Banach spaces.
- **Specialized Applications:** Exploring concrete applications in fields like network equilibrium, traffic assignment, and financial modelling where the GMVIQVI's generality can capture intricate real-world interactions.

In summary, this paper not only expands the theoretical frontier of variational inequalities by introducing a highly general problem class but also provides a robust and versatile toolkit of gap functions and error bounds. These tools are poised to facilitate the next wave of computational advances in solving complex equilibrium problems.

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