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SECOND HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. Let S denote the class of analytic and univalent functions in the open unit disk $D = \{z : |z| < 1\}$ with the normalization conditions. In the present article an upper bound for the second Hankel determinant $\left|a_2a_4 - a_3^2\right|$ is obtained for the analytic functions defined by Ruscheweyh derivative.

1. INTRODUCTION

Let D be the unit disk $\{z : |z| < 1\}$, $\mathcal{A}$ be the class of functions analytic in D, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1.1}$$

Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

because of the conditions (1.1). Let S denote class of analytic and univalent functions in D with the normalization conditions (1.1).

The q^{th} determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas [13] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & & a_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

This determinant has also been considered by several authors. For example, Noor in [14] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. Ehrenborg in [2] stadied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman's article [9]. It is well known that [1] that for $f \in S$ and given by (1.2) the sharp inequality $\left|a_3 - a_2^2\right| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $k = 1$. After that, Fekete-Szegö further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ with real μ and $f \in S$. For a given class of functions in $\mathcal{A}$, the sharp bound for the nonlinear functional $\left|a_2a_4 - a_3^2\right|$ is known as the second Hankel determinant. This corresponds to the Hankel determinant

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with $q = 2$ and $k = 2$. In particular, sharp bounds on $H_2(2)$ were obtained by several authors of articles [7], [17], [5], [6], [18] and [12] for different subclasses of univalent functions.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. The Hadamard product (convolution) of f and g, denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{D}. \tag{1.4}$$

Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The Ruscheweyh derivative [15] of the n^{th} order of f, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \tag{1.5}$$

The Ruscheweyh derivative gave an impulse for various generalization of well known classes of functions. By using the Ruscheweyh Derivative, we can generalize the class of the starlike and convex functions functions, denoted by S^* and C,which are defined as

$$S^* = \left\{ f(z) \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{D} \right\} \tag{1.6}$$

and

$$C = \left\{ f(z) \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in \mathbb{D} \right\}. \tag{1.7}$$

The class R_n was studied by Singh and Singh [16], which is given by the following definition

$$\operatorname{Re} \frac{z\left(D^n f(z)\right)'}{D^n f(z)} > 0, \ z \in \mathbb{D}. \tag{1.8}$$

We denote that $R_0 = S^*$ and $R_1 = C$. In the present paper, we obtain an upper bound for functional $\left|a_2 a_4 - a_3^2\right|$ in the class R_n.

2. Preliminary Results

The following lemmas are required to prove our main results. Let P be the family of all functions p analytic in $\mathbb{D}$ for which $\operatorname{Re}(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z + \cdots. \tag{2.1}$$

Lemma 1. (*Duren,* [1]) *If* $p \in P$, *then* $|c_k| \leq 2$ *for each* $k \in \mathbb{N}$.

Lemma 2. (*Grenander&Szegö,* [4]) *The power series for* $p(z)$ *given by* (2.1) *converges in* $\mathbb{D}$ *to a function in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \ n = 1, 2, \cdots. \tag{2.2}$$

and $c_{-k} = \overline{c_k}$, *are all nonnegative. They are strictly positive except for* $p(z) = \sum_{k=1}^{m} \rho_k p_0\left(e^{it_k z}\right)$, $\rho_k > 0$, t_k *real and* $t_k \neq t_j$ *for* $k \neq j$; *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \geq m$.

We may assume that without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = 8 + 2\operatorname{Re}\left\{c_1^2 c_2\right\} - 2\left|c_2\right|^2 - 4c_1^2 \geq 0, \tag{2.3}$$

which is equivalent to

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right) \tag{2.4}$$

for some x, $|x| \leq 1$. If we consider the determinant

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0, \tag{2.5}$$

we get the following inequality

$$\left|\left(4c_3 - 4c_1c_2 + c_1^3\right)\left(4 - c_1^2\right) + c_1\left(2c_2 - c_1^2\right)^2\right| \leq 2\left(4 - c_1^2\right)^2 - 2\left|\left(2c_2 - c_1^2\right)\right|^2. \tag{2.6}$$

From (2.4) and (2.6), it is obtained that

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2c_1\left(4 - c_1^2\right)\left(1 - |x|^2\right)z \tag{2.7}$$

for some z, $|z| \leq 1$.

3. Main Results

We prove the following theorem by using thecniques of Libera and Zlotkiewicz [10], [11].

Theorem 1. *Let the function* f *given by* (1.2) *be in the class in* R_n. *Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} 1, & n = 0 \\ \frac{1}{8}, & n = 1 \\ \frac{12(n-1)}{(n+1)^2(n+2)^2(n+3)}, & n > 1 \end{cases} \tag{3.1}$$

Proof. Since $f \in R_n$, there exists an analytic function $p \in P$ in the unit disk D with $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ such that

$$\frac{z\left(D^n f(z)\right)'}{D^n f(z)} = p(z) \tag{3.2}$$

Let

$$F(z) = D^n f(z) = z + \sum_{k=2}^{\infty} A_k z^k, \tag{3.3}$$

where

$$A_k = \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!}a_k, \tag{3.4}$$

then we have

$$\frac{zF'(z)}{F(z)} = p(z). \tag{3.5}$$

By using the series expansion of $F(z)$ and $p(z)$ as in (3.3) and (2.1), equating coefficients in (3.5) yields

$$\begin{aligned} a_2 &= \frac{1}{n+1} c_1 \\ a_3 &= \frac{1}{(n+1)(n+2)} \left\{c_2 + c_1^2\right\} \\ a_4 &= \frac{1}{(n+1)(n+2)(n+3)} \left\{2c_3 + 3c_1c_2 + c_1^3\right\}. \end{aligned} \tag{3.6}$$

Hence, we get from (3.6)

$$a_2a_4 - a_3^2 = A(n)\left\{2c_1c_3 + 3c_1^2c_2 + c_1^4 - B(n)\left(c_2 + c_1^2\right)^2\right\}, \tag{3.7}$$

where

$$A(n) = \frac{1}{(n+1)(n+2)(n+3)}, \tag{3.8}$$

and

$$B(n) = \left(\frac{n+3}{n+2}\right), \; n = 0, 1, 2, \cdots. \tag{3.9}$$

Using (2.4) and (2.7) in (3.7) ,we get

$$\left|a_2a_4 - a_3^2\right| = A(n)\left|2c_1c_3 + 3c_1^2c_2 + c_1^4 - B(n)\left(c_2^2 + 2c_1c_2 + c_1^4\right)\right|$$

and

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = A(n)\Bigg| & 3\left(1 - \frac{3}{4}B(n)\right)c_1^4 + \frac{3}{2}\left(1 - B(n)\right)c_1^2 x\left(4 - c_1^2\right) \\ & -\frac{c_1^2}{2}\left(4 - c_1^2\right)x^2 + c_1\left(4 - c_1^2\right)\left(1 - |x|^2\right)z - B(n)\frac{x^2\left(4 - c_1^2\right)^2}{4}\Bigg| \end{aligned} \tag{3.10}$$

Since the function $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ is also in the class P, we assume that without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$, $c \in [0, 2]$. Applying the triangle inequality with the assumptions $c_1 = c \in [0, 2]$, $|x| = \rho$ and $|z| \le 1$, it is obtained that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \le A(n)\Bigg\{ & 3\left|1 - \frac{3}{4}B(n)\right|c^4 + \frac{3}{2}\left(B(n) - 1\right)c^2\rho\left(4 - c^2\right) \\ & +\rho^2\frac{\left(4 - c^2\right)c\left(c - 2\right)}{2} + c\left(4 - c^2\right) + B(n)\rho^2\frac{\left(4 - c^2\right)^2}{4}\Bigg\} \\ = G(c, \rho). & \end{aligned} \tag{3.11}$$

We now maximize the function $G(c, \rho)$ on the closed square $[0, 2] \times [0, 1]$. Since

$$\frac{\partial G(c,\rho)}{\partial \rho} = \frac{3}{2}\left(B(n) - 1\right)c^2\left(4 - c^2\right) - \rho\left(4 - c^2\right)\left(2 - c\right)\left\{c - \frac{B(n)}{2}\left(2 + c\right)\right\} \tag{3.12}$$

and $B(n) \in \left[1, \frac{3}{2}\right]$, we get the following inequality

$$\frac{\partial G(c,\rho)}{\partial \rho} \geq \frac{\rho\left(4-c^2\right)(2-c)(6-c)}{4} > 0. \tag{3.13}$$

Hence, $G(c,\rho)$ can not have a maximum in the interior of the closed square $[0,2] \times [0,1]$. Hence, for fixed $c \in [0,2]$

$$\max_{0\leq\rho\leq1} G(c,\rho) = G(c,1) = F(c). \tag{3.14}$$

One can obtain that

$$\left|a_2a_4 - a_3^2\right| \leq A(n)F(c), \tag{3.15}$$

where
(3.16)

$$F(c) = 3\left|1-\frac{3}{4}B(n)\right|c^4 + \frac{3}{2}\left(B(n)-1\right)c^2\left(4-c^2\right) + \frac{c\left(4-c^2\right)}{2} + B(n)\frac{\left(4-c^2\right)^2}{4}.$$

Since
(3.17)

$$F'(c) = \begin{cases} \frac{25}{3}c^3 + c\left(4-c^2\right) + \frac{3}{2}c^3, \ n=0 \\ \frac{8}{3}c\left(1-c^2\right), \ n=1 \\ \left(12-9B(n)\right)c^3 + \left(B(n)-1\right)c\left(4-c^2\right) - 3\left(B(n)-2\right)c^3, \ n>1 \end{cases},$$

we have to consider following three cases:

Case 1. *For $n=0$, $F'(c)>0$. Hence $F(c) \leq F(2)$. We get the following result*

$$\left|a_2a_4 - a_3^2\right| \leq A(0)\left\{48\left|1-\frac{3}{4}B(0)\right|\right\} = 1. \tag{3.18}$$

This one coincides with the result in the article [8].

Case 2. *After necessarly calculations, it is obtained that*

$$F'(0) = 0 \text{ and } F'(1) = 0. \tag{3.19}$$

Since

$$F''(0) > 0 \text{ and } F''(1) < 0,$$

$F(c)$ has a maximum at $c=1$. Hence, we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{8}, \tag{3.20}$$

which is also stated in [8].

Case 3. *Let $n>1$. Then, $F'(c)$ can be rewrite as*

$$F'(c) = c\left\{\left(20-14B(n)\right)c^2 + 8\left(B(n)-1\right)\right\}. \tag{3.21}$$

Since $20-14B(n)>0$ and $B(n)-1>0$, we get $F'(0)=0$, $F''(0)>0$ and $F'(c)>0$ in the interval $(0,2]$. Therefore, it is obvious that

$$\left|a_2a_4 - a_3^2\right| \leq A(n)\left\{48\left|1-\frac{3}{4}B(n)\right|\right\} = \frac{12\,(n-1)}{(n+1)^2\,(n+2)^2\,(n+3)}. \tag{3.22}$$

This completes the proof of theorem.

□

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