

On the Number of Irreducible Characters and Their Anchor Groups**Manal H. Algreagri****Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia***Corresponding author: mhgreagri@uqu.edu.sa*

Abstract. We prove that for a finite group G , the number of height-zero characters with a fixed anchor subgroup A within a p -block B coincides with the number of height zero characters with anchor A in the Brauer correspondent block of $N_G(P)$. The proof proceeds via Brauer's First Main Theorem. We describe the anchor group of an irreducible character that is contained in the principal p -blocks containing at most five height zero irreducible ordinary characters. We also compute suitable examples.

1. INTRODUCTION

Consider p to be a fixed prime number and G to be a finite group. In this work, let (k, \mathcal{R}, F) be a p -modular system [10, 12, 32]. This system is made up of the following: \mathcal{R} is a complete discrete valuation ring with a characteristic zero and a unique maximal ideal $J(\mathcal{R})$; k is a field of fractions of \mathcal{R} of characteristic zero that contains a primitive $|G|^{\text{th}}$ root of unity for any finite group G we consider; and $F = \mathcal{R}/J(\mathcal{R})$ is the residue field, which is an algebraically closed field with characteristic p . Let $\text{Irr}(G)$ be the set of all ordinary irreducible characters of G .

To achieve more understanding of the relationships between modular and complex representations of a finite group G , Brauer divided the set of all p -modular and ordinary irreducible characters of G into naturally defined subsets known as p -blocks of G . The largest power of p that divides the natural number $\frac{|G|}{\chi(1)}$ is called a p -defect of the ordinary irreducible character χ . The greatest p -defect of ordinary irreducible characters within B represents the defect number of a p -block B . The height of the ordinary irreducible character χ in a p -block B is a non-negative integer produced by subtracting the defect number of χ from the defect number of B and is denoted by $ht(\chi)$. Height-zero ordinary irreducible characters within a p -block B are significant because they directly

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relate to several major issues in the field, such as the Alperin-McKay conjecture [1] and Brauer's height zero conjecture [7]. The p -block that contains the principal character is called the principal p -block, and is denoted by B_0 . In a principal p -block B_0 , height-zero characters are those whose degree is prime to p .

Our work is focused on the algebraic concept of the anchor group of an ordinary irreducible character φ . An anchor group of φ is the defect group of the primitive G -interior \mathcal{R} -algebra $(\mathcal{R}G) \cdot e_\varphi$, where e_φ is the unique central primitive idempotent in kG such that $\varphi(e_\varphi) \neq 0$. The notion of an anchor group of an irreducible character is a relatively recent concept introduced to bridge between ordinary irreducible characters and p -subgroups by R. Kessar, B. Külshammer, and M. Linckelmann in [19]. See [2, 3] for more details about the anchor group of an ordinary irreducible character. The issue of determining the structure of anchor groups for irreducible characters for a given number of ordinary irreducible characters with height-zero in the principal p -block B_0 can be viewed as a modular version of the problem of determining the structure of Sylow p -subgroups in finite groups for those irreducible characters in a principal p -block. As well as the issue of determining the structure of Sylow p -subgroups in finite groups with a specified number of irreducible characters in a principal p -block can be viewed as the classical problem of classifying finite groups with a specific number of conjugacy classes [31].

It is well known that the p -block contains a singleton ordinary irreducible character if and only if the defect group is trivial. In addition, in 1982, J. Brandt [5] demonstrated that the p -block has two ordinary irreducible characters if and only if the defect group has order two. In 1990, V.A. Belonogov [4] proved that if the principal p -block contains three ordinary irreducible characters, then the defect group has order three. In 2021, S. Koshitani and T. Sakurai [20] determined a Sylow p -subgroup of G when the principal p -block B_0 of G owns four ordinary irreducible characters. The same year, N. Rizo, A. A. Schaeffer Fry, and C. Vallejo [28] identified Sylow p -subgroups of G when the principal p -blocks contain exactly five irreducible characters. Thus, the contributions in [20] and [28] have identified the structure of Sylow p -subgroups of G for principal p -blocks that contain up to five irreducible characters. In 2023, the results mentioned in previous papers have been generalized in the case of height-zero characters by N. N. Hung, A. S. Fry, and C. Vallejo [17]. The most significant sources about Sylow theory, block theory, and defect theory are [8, 9, 11, 14, 15, 22–24, 30].

The purpose of this paper is to generalize the Alperin-McKay conjecture and Brauer's First Main Theorem to a modular version from the perspective of the anchor groups of irreducible characters with height zero. As well, we describe the structure of an anchor group of irreducible characters that is included in the principal p -blocks containing at most five height-zero irreducible ordinary characters.

We write $Irr_0(B)$ for the set of irreducible ordinary characters of G in a p -block of height zero and $Irr_0(B, A)$ for the set of irreducible ordinary characters of G in B of height zero whose anchor is A . We put $Irr_0(G, A) := \{\chi \in Irr(G) : ht(\chi) = 0 \text{ and } anchorgroup A\}$. Our first major aim is to show

that

$$|\text{Irr}_0(G, A)| = |\text{Irr}_0(N_G(A), A)|.$$

Our second major aim is to reformulate the theorem of N. N. Hung, A. S. Fry, and C. Vallejo [17, Theorem 1.2] to the case of anchor groups for irreducible characters.

Theorem 1.1. *Consider G to be a finite group and p a prime. Let B_0 be the principal p -block of G . Let A be an anchor group of an irreducible character χ of B_0 . We have*

- (1) For $r \in \{2, 3\}$, $|\text{Irr}_0(B_0)| = r$ if and only if A is cyclic of order r .
- (2) $|\text{Irr}_0(B_0)| = 4$, if and only if one of the following applies:
 - i.: $|A : A| = 4$.
 - ii.: $|A| = 5$ and $|N_G(A) : C_G(A)| = 2$.
- (3) $|\text{Irr}_0(B_0)| = 5$, if and only if one of the following applies:
 - i.: $|A| = 5$ and $|N_G(A) : C_G(A)| \in \{1, 4\}$.
 - ii.: $|A| = 7$ and $|N_G(A) : C_G(A)| \in \{2, 3\}$.

Theorem 1.1 describes the anchor group of an irreducible character that is contained in the principal p -blocks containing at most five height-zero irreducible ordinary characters. The paper is arranged as follows: In Section 2, we recast some well-known facts about the anchor group of irreducible characters, the block theory and the Alperin-McKay conjecture. In Section 3, we prove the main results along with some examples.

2. PRELIMINARIES

This section provides a summary of the results used in this work. First, let us summarize the most important features of the anchor group of irreducible characters used in this paper.

Theorem 2.1. *Suppose that B is a p -block of a finite group G with a defect group P . Let $\varphi \in \text{Irr}(B)$ have anchor group A .*

- (1) *The anchor group A is a subgroup of the defect group P (up to G -conjugacy).*
- (2) *If the defect group P is abelian, it is an anchor group of φ .*
- (3) *If φ has a height zero (full defect), then A is the defect group of B .*

Proof. Part (1) follows [19, Theorem 1.2 (a)] and (2), (3) follow [19, Theorem 1.3 (c)(d)]. □

Theorem 2.2. *If B_0 is a principal p -block of a finite group G , then the defect group of B_0 is the Sylow p -subgroup of G .*

Proof. This follows from [24, Chapter 5, Exercise 2.9], see also [23, Theorem 6.1.5]. □

Theorem 2.3 (Brauer's First Main Theorem). *Suppose that A is a p -subgroup of G . Then there is a bijection $BL(N_G(A) | A) \rightarrow BL(G | A)$. Here, $BL(G | A)$ represents the set of all p -blocks of G with defect group A .*

Proof. This follows from [6, Theorem 5] for the original statement; see also [14, Chapter 3, Theorem 9.7], [24, Chapter 5, Theorem 2.15], [23, Theorem 6.7.6 (i)] and [12, Theorem 6.2.6]. \square

Conjecture 2.1 (The Alperin-Mckay conjecture). *Suppose that B is a p -block of G , with defect group P , and b is the Brauer correspondent of B in $N_G(P)$. Then $|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$.*

Lemma 2.1. *Let p be a prime and $r \in \{2, 3\}$. If B is a p -block of positive defect of a finite group G , then:*

- (1) $r \mid |\text{Irr}_0(B)|$.
- (2) *If $p = 2$ and the defect γ of B is at least two, then $4 \mid |\text{Irr}_0(B)|$. Additionally, if B has no character of height 1, then $|\text{Irr}_0(B)| \equiv 2^\gamma \pmod{8}$.*

Proof. This follows from [21, Corollaries 1.3 and 1.6], (see also [26, Lemma 2.2] and [28, Theorems 1.6 and 1.7]). \square

Theorem 2.4. *Assume that G is p -solvable and that P is a Sylow p -subgroup of G . Then*

$$|\text{Irr}_0(B_0(G))| = |\text{Irr}_0(B_0(N_G(P)))|.$$

Proof. This is the main outcome in the block case in [13]; also see the main theorem in [27]. \square

Lemma 2.2. *If the Alperin-McKay conjecture holds for the principal p -block $B_0(G)$ of a finite group G , then $|\text{Irr}_0(B_0)| \geq 2\sqrt{p-1}$. In particular, if G is p -solvable or all the non-abelian composition factors of G have cyclic Sylow p -subgroups, then $|\text{Irr}_0(B_0)| \geq 2\sqrt{p-1}$.*

Proof. See [18, §2.1]. According to Theorem 2.4, the Alperin-McKay conjecture holds for p -solvable groups. \square

3. MAIN RESULTS

In this section, we introduce Brauer's First Main Theorem and the Alperin-McKay conjecture to a modular version from the perspective of the anchor groups of irreducible characters with height zero. We describe the anchor group of an irreducible character that is contained in the principal p -blocks containing at most five height-zero irreducible ordinary characters. We also provide appropriate examples for these theories.

Theorem 3.1. *Fix a prime p . Let G be a finite group. Let A be an anchor group of an irreducible character χ of G . We write*

$$\text{Irr}_0(G, A) := \{\chi \in \text{Irr}(G) : \text{ht}(\chi) = 0 \text{ and anchorgroup}A\}.$$

Then

$$|\text{Irr}_0(G, A)| = |\text{Irr}_0(N_G(A), A)|.$$

Proof. Partition $\text{Irr}_0(G)$ according to the p -blocks B of G that contain the irreducible characters of height zero. Similarly, partition $\text{Irr}_0(N_G(A))$ according to the Brauer correspondent of B in $N_G(A)$; say p -blocks b of $N_G(A)$ that contain the irreducible characters of height zero. From Theorem 2.1 (3), if χ has a height zero (full defect), then the anchor group A of χ is a defect group of the p -block

that contains χ . According to Brauer's First Main Theorem 2.3, there exists a bijection between the p -blocks of G that have defect group A and the p -blocks of $N_G(A)$ that also have defect group A . The Alperin-McKay conjecture 2.1 states that $|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$. Hence, $|\text{Irr}_0(B, A)| = |\text{Irr}_0(b, A)|$. By summing over all the p -blocks B of G that contain irreducible characters of height zero, as well as the Brauer correspondent of B in $N_G(A)$, we obtain the global equality. \square

The information for the following examples about the number of irreducible characters, the height of the irreducible characters, the structure of the Sylow p -subgroup of G , and its normalizer are from the GAP [29].

Example 3.1. Fix $p = 2$. Consider G to be a 2-constrained group with an order of 1920.

$$G := \text{WreathProduct}(\text{CyclicGroup}(2), \text{AlternatingGroup}(5)),$$

that has only the principal 2-block B_0 . We have $|\text{Irr}_0(G)| = 16$, and their anchor group is $A = C_2 \times (((C_2 \times C_2 \times C_2) : C_4) : C_2)$, which is isomorphic to a Sylow 2-subgroup of G . The normalizer of A in G is $C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) : C_2) : C_2) : C_3$, a group of order 384 with 7 generators. We have $|\text{Irr}_0(N_G(A))| = 16$ that all belong to the principal 2-block of $N_G(A)$, and their anchor group is isomorphic to A .

In the following, we demonstrate the proof of Theorem 1.1. First, we are recording results for small primes.

Theorem 3.2. Let G be a finite group. Let B_0 be the principal 2-block of G . Then the following are equivalent:

- (1) $|\text{Irr}(B_0)| = 2$.
- (2) $|\text{Irr}_0(B_0)| = 2$.
- (3) The anchor group of any irreducible character $\chi \in B_0$ is cyclic of order 2.

Proof. Suppose that $|\text{Irr}(B_0)| = 2$. As shown in [5, Theorem A], this is equivalent to the order of a Sylow p -subgroup of G being two. That means the Sylow p -subgroup of G is cyclic. The Sylow p -subgroup of G is the defect group of the principal p -block by Theorem 2.2. As is well known, the anchor group of an irreducible character φ with height zero (full defect) is the defect group of the p -block that contains φ by Theorem 2.1(3). Thus, the anchor group of an irreducible character φ with height zero in the principal p -block is the Sylow p -subgroup of G . Hence, the anchor group of any irreducible character $\varphi \in B_0$ is cyclic of order 2. Suppose that $|\text{Irr}_0(B_0)| = 2$. If $p = 2$, then the order of the defect group of B is of order 2 by Lemma 2.1. Hence, the anchor group is cyclic of order 2. Also $p \neq 3$ from Lemma 2.1. If $p \geq 5$, then by [16, Theorem A] G is p -solvable. Hence, according to Lemma 2.2 $|\text{Irr}_0(B_0)| \geq 2\sqrt{p-1} \geq 4$, which leads to a contradiction. \square

Example 3.2. Consider $p = 2$, and G is the dihedral group of order 10. This group has three 2-blocks and $|\text{Irr}_0(B_0)| = 2$. The anchor group of these two irreducible characters is C_2 , a cyclic group of order two. We have $N_G(A) \cong A$ is a self-normalizer that has two irreducible characters of height zero, and their anchor group is C_2 itself.

Theorem 3.3. Let G be a finite group. Let B_0 be the principal 3-block of G . Then the following are equivalent:

- (1) $|Irr(B_0)| = 3$.
- (2) $|Irr_0(B_0)| = 3$.
- (3) The anchor group of any irreducible character $\chi \in B_0$ is cyclic of order 3.

Proof. If $|Irr(B_0)| = 3$, then by the main result of [4] the Sylow p -subgroup of G is of order three. Hence, the defect group of the principal 3-block of G is cyclic of order 3. According to Theorem 2.1 (1),(2), every irreducible character in B_0 has an anchor group that is cyclic of order 3. Now, if the anchor group of any irreducible character $\chi \in B_0$ is cyclic of order 3, then it is the defect group of the principal p -block B_0 . Hence, the anchor group is a Sylow p -subgroup of G . From [25, Theorem 11.1], we have $|Irr(B_0)| = |Irr_0(B_0)| = 3$. Suppose that $|Irr_0(B_0)| = 3$. By [26, Theorem C], the order of G is divisible by 3 and not divisible by 9. Hence, the Sylow 3-subgroup of G is cyclic of order 3. From [19, Theorem 1.3], it is evident that the anchor group for every irreducible character $\chi \in B_0$ is cyclic of order 3. \square

Example 3.3. Fix $p = 3$. Let G be the symmetric group of degree five S_5 and order 120. This group has three 3-blocks. We have $|Irr_0(G)| = 6$, and three of them belong to the principal p -block of G . The anchor group for the irreducible characters of height zero is C_3 , the Sylow 3-subgroup of G . The normalizer of C_3 in S_5 is the dihedral group of degree 12. We have $|Irr_0(N_G(A))| = 6$, which belong in two 3-blocks. Also, the anchor group for the irreducible characters of height zero is C_3 , the Sylow 3-subgroup of $N_G(A)$.

Theorem 3.4. Consider G to be a finite group and p a prime. Let A be an anchor group of an irreducible character χ of G . We have $|Irr_0(B_0)| = 4$, if and only if one of the following applies:

- i.: $|A : \acute{A}| = 4$. Here, \acute{A} denotes the commutator subgroup of A .
- ii.: $|A| = 5$ and $|N_G(A) : C_G(A)| = 2$.

Proof. From Theorem 2.2, the defect group of the principal p -block is the Sylow p -subgroup of G . As well as, the anchor group of an irreducible character χ with height zero is the defect group of the p -block that contains χ by Theorem 2.1 (3). Hence, the anchor group of an irreducible character χ with height zero in the principal p -block is the Sylow p -subgroup of G . Therefore, the result can be demonstrated according to [17, Theorem 4.6]. \square

Example 3.4. Consider $p = 2$ and $G = SL(2,3)$ to be the special linear group of order 24. We see $|Irr_0(B_0)| = 4$. The anchor group for the irreducible characters of height zero is isomorphic to Q_8 , the quaternion group. The commutator subgroup of Q_8 is of order two and satisfies $|A : \acute{A}| = 4$.

Theorem 3.5. Consider G to be a finite group and p a prime. Let A be an anchor group of an irreducible character χ of G . We have $|Irr_0(B_0)| = 5$, if and only if one of the following applies:

- i.: $|A| = 5$ and $|N_G(A) : C_G(A)| \in \{1, 4\}$.
- ii.: $|A| = 7$ and $|N_G(A) : C_G(A)| \in \{2, 3\}$.

Proof. As we see in the proof of Theorem 3.4, the anchor group of an irreducible character χ with height zero in the principal p -block is the Sylow p -subgroup of G . Thus, the result can be demonstrated according to [17, Theorem 4.7]. \square

Example 3.5. Fix $p = 5$. Let G be the symmetric group of degree five S_5 and order 120. This group has three 5-blocks. We have $|\text{Irr}_0(B_0)| = 5$. The anchor group for the irreducible characters of height zero is isomorphic to C_5 , the Sylow 5-subgroup of G . The normalizer of C_5 in S_5 is the Frobenius group of order 20 and is isomorphic to the semidirect product $C_5 \rtimes C_4$. We have $|\text{Irr}_0(N_G(A))| = 5$, which belong in the principal 5-block. The anchor group of these irreducible characters is the Sylow 5-subgroup of $N_G(A)$. The centralizer of C_5 in S_5 is the cyclic group of order five. Hence, $|N_G(A) : C_G(A)| = 4$.

Proof of Theorem 1.1: According to Theorems 3.2 and 3.3, Theorem 1.1 (1) is proved. Theorems 3.4 and 3.5 establish parts (2) and (3) of Theorem 1.1, respectively. \square

4. DISCUSSION AND CONCLUSIONS

In [1], J. L. Alperin conjectured that "If B is a p -block of G , with defect group P , and b is the Brauer correspondent of B in $N_G(P)$, then $|\text{Irr}_0(B)| = |\text{Irr}_0(b)|$. In [20] and [28], the structure of Sylow p -subgroups of G for the principal p -blocks that contain up to five irreducible characters has been recognized. In [17], the structure of Sylow p -subgroups of G for the principal p -blocks that contain up to five irreducible characters of height zero. In this work, we generalize and reformulate the Alperin-McKay conjecture [1] to the anchor group of irreducible characters of G , proceeding via Brauer's First Main Theorem [6, Theorem 5]. Furthermore, we determine the structure of the anchor groups of irreducible characters when the number of irreducible characters of height zero in the principal p -block is known. In future work, we will generalize the theory of Sylow p -subgroups and the defect group of p -blocks to the case of the anchor group of irreducible characters.

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