International Journal of Analysis and Applications

Fixed Point Results for Two Mappings on Closed Balls in Continuous Controlled Metric Spaces with Applications to Urysohn Integral Equations

Mahmood Akhtar Khan[1,*], Abdullah Shoaib[1], Thabet Abdeljawad[2,3,4], Aiman Mukheimer[2]

[1]*Department of Mathematics and Statistics, Riphah International University, Islamabad, 44000, Pakistan*
[2]*Department of Mathematics and Sciences, Prince Sultan University, Riyadh-11586, Saudi Arabia*
[3]*Department of Fundamental Sciences, Faculty of Engineering and Architecture, Istanbul Gelisim University, Avcılar-İstanbul, 34310, Türkiye*
[4]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Garankuwa, Medunsa 0204, South Africa*

Corresponding author: mahmoodmath212@gmail.com

Abstract. This manuscript investigates complete continuous controlled metric spaces (CCMSs). New fixed point theorems are established under Geraghty and Geraghty-Ćirić type contractive conditions, formulated on a closed ball of a CCMS rather than on the entire space. This localized approach extends the existing fixed point theory in controlled metric spaces, where such results on closed balls have not been previously reported. Concrete examples are provided to illustrate the effectiveness of the obtained theorems. Furthermore, the applicability of the theoretical results is demonstrated by proving the existence and uniqueness of solutions for a system of Urysohn-type integral equations.

1. Introduction

A fundamental concept in functional analysis is that of a fixed point. Fixed point theory has numerous applications in solving problems arising in engineering and mathematics. The fundamental idea was formally introduced and established by Banach through the celebrated Banach contraction principle.

Currently, fixed point theory plays a vital role in many disciplines, including economics, computer science, psychology, ecology, game theory and decision-making theory [6, 8, 9, 22, 34–36].

The notion of b-metric space generalizes the classical metric space by relaxing the triangle inequality. Various results have been derived by considering different types of contractions for

Received: Mar. 13, 2026.

2020 *Mathematics Subject Classification.* 54H25, 47H10.

Key words and phrases. fixed point; Geraghty-Ćirić contraction; Geraghty contraction; continuous controlled metric space; closed ball; Urysohn integral equation.

https://doi.org/10.28924/2291-8639-24-2026-279
ISSN: 2291-8639

single and multi-valued mappings, with applications in several areas [18,19,21,22,30,37]. Ran and Reurings [26] applied fixed point techniques in ordered metric spaces to derive solutions of matrix equations.

The study of fixed point results restricted to closed balls in metric-type spaces constitutes an important topic in nonlinear analysis. The iterative approximation method is enhanced, ensuring that all successive iterates are bounded within fixed radius. This technique has proven effective in situations where classical global approaches fail. By introducing modified contractions Arshad et al. [32] established fixed points results restricted to closed balls of a prescribed radius [36,38,39].

Mlaiki et al. [20] were the first who introduced controlled metric space. By introducing a control function, the traditional idea of contraction mappings was generalized and later on it was extended to double controlled with various applications [1].

The Geraghty contraction, was introduced in 1973, extends the Banach contraction principle by replacing the fixed contraction constant with a variable function. The Geraghty-Ćirić contraction was proposed by including extra distance terms. Many results are derived by the researchers by extending the fixed point theorems using Geraghty Contraction and Geraghty-Ćirić contraction in b-metric spaces and other extended metric spaces [17,24,25].

In this study, we analyze the fixed point results for a pair of maps on a closed ball in a complete CCMS for Geraghty and Geraghty-Ćirić contraction. This restricted study is motivated by [27,28,39] to find the desired results.

2. Preliminaries

Definition 2.1. *Let $\xi \neq \emptyset$ be a set and $\beta : \xi \times \xi \to [1,\infty)$. A mapping $v : \xi \times \xi \to [0,\infty)$ is called a controlled metric if, for all $s,h,p \in \xi$, the following conditions hold:*

(1) $v(s,h) = 0 \iff s = h$,
(2) $v(s,h) = v(h,s)$,
(3) $v(s,h) \leq \beta(s,p)v(s,p) + \beta(p,h)v(p,h)$.

The pair (ξ, v) is called a controlled metric space.

Remark 2.1. *In Definition 2.1, if there exists a parameter $t \geq 1$ such that $\beta(s,h) = t$ for all $s,h \in \xi$, then (ξ, v) is called a b-metric space. Therefore, every b-metric space is considered as a specific type of controlled metric space.*

Definition 2.2. *Let (ξ, v) be a controlled metric space. We say $v : \xi \times \xi \to [0,\infty)$ is a continuous function if $s_n \to s$ and $h_n \to h$ implies $v(s_n,h_n) \to v(s,h)$.*

Definition 2.3. *Let (ξ, v) be a continuous controlled metric space (CCMS).*

(i) *A sequence $\{s_n\}$ converges to $s \in \xi$ if $\lim_{n\to\infty} v(s_n,s) = 0$, or equivalently, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $v(s_n,s) < \varepsilon$.*
(ii) *A sequence $\{s_n\}$ is Cauchy if for $l,n \in \mathbb{N}$ with $l > n$, $\lim_{n,l\to\infty} v(s_l,s_n) = 0$, or equivalently, for any $\varepsilon > 0$, there exists $l_0 \in \mathbb{N}$ such that for all $n,l \in \mathbb{N}$ with $l > n \geq l_0$, $v(s_l,s_n) \leq \varepsilon$.*

(iii) *A CCMS (ξ, v) is complete if every Cauchy sequence converges to some $s \in \xi$.*

Definition 2.4. *Let (ξ, v) be a complete CCMS. The closed ball with center s_0 and radius r is defined as*

$$B_v(s_0, r) = \{s \in \xi : v(s_0, s) \leq r\}.$$

Proposition 2.1. *Let (ξ, v) be a complete CCMS. Then, every closed ball is a closed set.*

Proof. Let $b_n \in B_v(s_0, r)$ and $\lim_{n\to\infty} b_n = b$. By using sequential continuity, for all $n \in \mathbb{N}$, $v(b_n, s_0) \leq r$ implies $v(b, s_0) \leq r$. Hence, $b \in B_v(s_0, r)$, which implies the set is closed. □

Definition 2.5. *Consider two self-maps f and g on a metric space (ξ, v). Suppose $\sigma \in \Psi$ such that the inequality*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h)$$

holds, where $E_{f,g}(s,h) = v(s,h) + |v(s,fs) - v(h,gh)|$. Then, we say that the mappings f and g satisfy the Geraghty contraction of type $E_{f,g}$. Here, Ψ denotes the family of functions $\sigma : [0,\infty) \to [0, 1/\mu)$ such that

$$\lim_{n\to\infty} \sigma(t_n) = \frac{1}{\mu} \Rightarrow \lim_{n\to\infty} t_n = 0.$$

Definition 2.6. *Consider two self-maps f and g on a metric space (ξ, v). Suppose $\sigma \in \Psi$ such that the inequality*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h)$$

holds, where $E_{f,g}(s,h) = \max\{v(s,h), v(s,fs), v(h,gh)\}$. Then, we say that the mappings f and g satisfy the Geraghty-Ćirić contraction of type $E_{f,g}$.

3. Main Results

Now, we present the main result on closed balls for Geraghty contractions and Geraghty-Ćirić contractions within the framework of a complete CCMS.

Theorem 3.1. *Let (ξ, v) be a complete CCMS. Let $f, g : \xi \to \xi$ be two maps satisfying a Geraghty-type contraction such that*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h), \tag{3.1}$$

and

$$\mu v(s_0, gs_0) \leq \frac{(1-\mu\theta)(1+\tau)}{(1+3\tau)} r, \tag{3.2}$$

for all $s, h \in B_v(s_0, r)$, where $\theta = \frac{2\tau}{1+\tau}$, $\tau \in (0,1)$, and $\sigma \in \Psi_\alpha$ is a family of control bounded functions with $\sigma(t) \leq \theta < \frac{1}{\mu}$.

Moreover, for $s_0 \in \xi$, take $s_{2n+2} = fs_{2n+1}$ and $s_{2n+1} = gs_{2n}$, for all $n \in \mathbb{N}$. Assume the controlled function meets

$$\beta(s_i, s_j) \leq \gamma \quad \text{for all } i, j \geq 0 \text{ and } \gamma \in [1, 1/\tau), \tag{3.3}$$

$$\sup \lim_{m\to\infty} \frac{\beta(s_{2i+1}, s_{2i+2})\beta(s_{2i+1}, s_{2m})}{\beta(s_{2i}, s_{2i+1})} < \frac{1}{\theta}. \tag{3.4}$$

There exists $s^* \in B_v(s_0, r)$ such that $s_n \to s^*$. In addition,

$$\lim_{n\to\infty} \beta(s_{2n}, s^*) \text{ and } \lim_{n\to\infty} \beta(s^*, s_{2n}) \text{ exist and are finite.} \tag{3.5}$$

Then, the mappings f and g have a unique common fixed point s^* in $B_v(s_0, r)$.

Proof. Consider the sequence starting from $s_0 \in \xi$ such that $s_1 = gs_0, s_2 = fs_1, s_3 = gs_2, s_4 = fs_3, \ldots,$ and in general $s_{2n+1} = gs_{2n}$, $s_{2n+2} = fs_{2n+1}$ for all $n \in \mathbb{N}$.

From (3.1), we have:

$$\begin{aligned} v(s_{2n}, s_{2n+1}) &\leq \sigma(E_{f,g}(s_{2n-1}, s_{2n}))E_{f,g}(s_{2n-1}, s_{2n}) \\ &\leq \tau E_{f,g}(s_{2n-1}, s_{2n}) \\ &\leq \tau[v(s_{2n-1}, s_{2n}) + |v(s_{2n-1}, s_{2n}) - v(s_{2n}, s_{2n+1})|] \\ &\leq \frac{2\tau}{1+\tau} v(s_{2n-1}, s_{2n}). \end{aligned}$$

Let $\theta = \frac{2\tau}{1+\tau}$. Since $\tau \in (0,1)$, then $\theta \in (0,1)$, and we have

$$v(s_{2n}, s_{2n+1}) \leq \theta v(s_{2n-1}, s_{2n}) \leq \theta^2 v(s_{2n-2}, s_{2n-1}) \leq \cdots \leq \theta^{2n} v(s_0, s_1). \tag{3.6}$$

From (3.2):

$$v(s_0, gs_0) \leq \frac{(1-\gamma\theta)(1+\tau)}{\gamma(1+3\tau)} r.$$

Since $\frac{1+\tau}{1+3\tau} < 1$ and $1 - \gamma\tau < 1$, we have $v(s_0, s_1) \leq r$, which implies $s_1 \in B_v(s_0, r)$.

Using the triangle inequality, (3.1) and (3.2):

$$\begin{aligned} v(s_0, s_2) &\leq \beta(s_0, s_1)v(s_0, s_1) + \beta(s_1, s_2)v(s_1, s_2) \\ &\leq \mu v(s_0, s_1) + \mu \frac{2\tau}{1+\tau} v(s_0, s_1) \\ &\leq \left(1 + \frac{2\tau}{1+\tau}\right) \mu v(s_0, s_1) \\ &\leq \mu \frac{1+3\tau}{1+\tau} v(s_0, s_1) \\ &\leq \mu \frac{1+3\tau}{1+\tau} \cdot \frac{(1-\mu\theta)(1+\tau)}{\mu(1+3\tau)} r \\ &\leq (1-\mu\theta)r \leq r. \end{aligned}$$

Thus, $s_2 \in B_v(s_0, r)$.

Now, suppose that $s_3, s_4, \ldots, s_{2n} \in B_v(s_0, r)$ for some $n \in \mathbb{N}$. Using (3.1), (3.2) and (3.3):

$$\begin{aligned} v(s_0, s_{2n+1}) &\leq \beta(s_0, s_1)v(s_0, s_1) + \beta(s_1, s_{2n+1})v(s_1, s_{2n+1}) \\ &\leq \beta(s_0, s_1)v(s_0, s_1) + \beta(s_1, s_{2n+1})\beta(s_1, s_2)v(s_1, s_2) \\ &\quad + \beta(s_1, s_{2n+1})\beta(s_2, s_{2n+1})v(s_2, s_{2n+1}) \\ &\leq \beta(s_0, s_1)v(s_0, s_1) + \sum_{i=1}^{2n-1} \prod_{j=1}^{i} \beta(s_j, s_{2n+1})\beta(s_i, s_{i+1})v(s_i, s_{i+1}) \end{aligned}$$

$$
\begin{aligned}
&+\prod_{j=1}^{2n}\beta(s_j,s_{2n+1})v(s_{2n},s_{2n+1})\\
&\le \gamma v(s_0,s_1)+\sum_{i=1}^{2n-1}\gamma^{i+1}\theta^i v(s_0,s_1)+(\gamma\theta)^{2n}v(s_0,s_1)\\
&\le v(s_0,s_1)\left(\gamma+\gamma\sum_{i=1}^{2n-1}(\gamma\theta)^i\right)+(\gamma\theta)^{2n}v(s_0,s_1)\\
&\le \gamma v(s_0,s_1)\left(1+\sum_{i=1}^{2n-1}(\gamma\theta)^i\right)+(\gamma\theta)^{2n}v(s_0,s_1)\\
&\le \gamma v(s_0,s_1)\sum_{i=0}^{2n-1}(\gamma\theta)^i+(\gamma\theta)^{2n}v(s_0,s_1).
\end{aligned}
$$

The series $\sum_{i=0}^{\infty}(\gamma\theta)^i$ converges as $\gamma\theta<1$, with $\sum_{i=0}^{2n-1}(\gamma\theta)^i=\frac{1-(\gamma\theta)^{2n}}{1-\gamma\theta}$. Thus,

$$v(s_0,s_{2n+1})\le \gamma v(s_0,s_1)\frac{1-(\gamma\theta)^{2n}}{1-\gamma\theta}+(\gamma\theta)^{2n}v(s_0,s_1).$$

Simplifying:

$$v(s_0,s_{2n+1})\le \frac{1-(\gamma\theta)^{2n+1}}{1-\gamma\theta}\gamma v(s_0,s_1)\le \frac{1-(\gamma\theta)^{2n+1}}{1-\gamma\theta}\gamma\cdot\frac{(1-\gamma\theta)(1+\tau)}{\gamma(1+3\tau)}r.$$

Since $\theta\in(0,1)$, $\gamma\theta<1$, then $1-(\gamma\theta)^{2n+1}<1$ and $\frac{1+\tau}{1+3\tau}<1$, we have

$$v(s_0,s_{2n+1})\le r.$$

Thus, $s_{2n+1}\in B_v(s_0,r)$. Hence $s_{2n}\in B_v(s_0,r)$ for all $n\in\mathbb{N}$.

By using techniques analogous to those in [20], we can show that $\{s_{2n}\}$ is Cauchy in $B_v(s_0,r)$. Thus,

$$\lim_{n,m\to\infty}v(s_{2n},s_{2m})=0. \tag{3.7}$$

Since every closed ball is a closed set in a CCMS (Proposition 2.1), $(B_v(s_0,r),v)$ is a complete metric space, and every Cauchy sequence converges in $B_v(s_0,r)$. Thus, there exists $s^*\in B_v(s_0,r)$ such that

$$\lim_{n\to\infty}v(s_{2n},s^*)=0.$$

For the fixed point property:

$$
\begin{aligned}
v(s^*,gs^*)&\le \beta(s^*,s_{2n})v(s^*,s_{2n})+\beta(s_{2n},gs^*)v(s_{2n},gs^*)\\
&\le \beta(s^*,s_{2n})v(s^*,s_{2n})+\theta\beta(s_{2n},gs^*)v(s_{2n-1},s^*).
\end{aligned}
$$

Taking $n\to\infty$ and using (3.5), we get $v(s^*,gs^*)=0$, which implies $s^*=gs^*$ is a fixed point. Similarly, we can derive a fixed point for f.

For uniqueness, consider another fixed point $h=gh=fh$. Then

$$v(s^*,h)=v(fs^*,gh)\le\theta v(s^*,h),$$

but $\theta < 1$ implies $v(s^*, h) = 0$. Thus, the two mappings f and g have a unique common fixed point $s^* \in B_v(s_0, r)$. □

Example 3.1. *Let $\xi = \{0, 1, 2\}$ with metric $v : \xi \times \xi \to \mathbb{R}$ defined as*

$$v(0,0) = v(1,1) = v(2,2) = 0,$$
$$v(0,1) = v(1,0) = 1.5,$$
$$v(0,2) = v(2,0) = 0.4,$$
$$v(1,2) = v(2,1) = 0.8,$$

and control function $\beta : \xi \times \xi \to [1, \infty)$ given by

$$\beta(0,0) = \beta(1,1) = \beta(2,2) = 1,$$
$$\beta(0,1) = \beta(1,0) = 1.3,$$
$$\beta(0,2) = \beta(2,0) = 1.2,$$
$$\beta(1,2) = \beta(2,1) = 1.25.$$

Let $g, f : \xi \to \xi$ be mappings defined by $f(s) = g(s) = 2$ for all $s \in \xi$. Taking $s_0 = 0$, $r = 3$, $\gamma = 1.1$, $\tau = 0.4$, and $\theta = 0.571$, we have $E_{f,g}(s,h) = v(s,h) + |v(s,2) - v(h,2)|$. Clearly, the conditions are satisfied since $v(fs, gh) = 0$ and

$$\gamma v(s_0, gs_0) \leq \frac{[1 - 1.1(0.571)](1 + 0.4)}{1 + 1.2} = 0.7101 \in B_v(s_0, r).$$

All assumptions (3.3)–(3.5) *are satisfied, and* 2 *is a unique common fixed point of f and g.*

Corollary 3.1. *Let (ξ, ω) be a complete metric space. Let $f, g : \xi \to \xi$ be two maps with Geraghty contraction such that*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h),$$

and

$$v(s_0, gs_0) \leq \frac{(1-\theta)(1+\tau)}{(1+3\tau)}r$$

for all $s, h \in B_v(s_0, r)$, where $\tau \in (0,1)$ and $\sigma \in \Psi_\alpha$ is a family of control bounded functions with $\sigma(t) \leq \tau < 1$. Then, s^* is a unique common fixed point of f and g in $B_v(s_0, r)$.

Proof. By choosing the constant function $\beta(s_i, h_j) = 1$ for all $i, j \in \mathbb{N}$, the given conditions in the corollary directly imply conditions (3.1)–(3.4) of Theorem 3.1. Thus, f and g have a unique common fixed point in $B_v(s_0, r)$. □

Corollary 3.2. *Assume (ξ, v) is a complete continuous b-metric space. Suppose $f, g : \xi \to \xi$ are two maps with Geraghty contraction such that*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h),$$

and

$$\mu v(s_0, gs_0) \leq \frac{(1-\mu\theta)(1+\tau)}{(1+3\tau)} r$$

for all $s, h \in B_v(s_0, r)$, where $\tau \in (0,1)$ and $\sigma \in \Psi_\alpha$ is a family of control bounded functions with $\sigma(t) \leq \tau < \frac{1}{\mu}$. Then, the mappings f and g have a unique common fixed point in $B_v(s_0, r)$.

Proof. By choosing the constant function $\beta(s_i, h_j) = \mu$ for all $i, j \in \mathbb{N}$, the given conditions in the corollary directly imply conditions (3.1)–(3.4) of Theorem 3.1. Consequently, f and g have a unique common fixed point in $B_v(s_0, r)$. □

Theorem 3.2. *Let (ξ, v) be a complete CCMS. Let $f, g : \xi \to \xi$ be two maps with Geraghty-Ćirić contraction such that*

$$v(fs, gh) \leq \sigma(E_{f,g}(s,h))E_{f,g}(s,h), \tag{3.8}$$

and

$$\gamma v(s_0, gs_0) \leq \frac{(1-\tau)(1+\tau)}{(1+3\tau)} r \tag{3.9}$$

for all $s, h \in B_v(s_0, r)$, where $\tau \in (0,1)$ and $\sigma \in \Psi_\alpha$ is a family of control bounded functions with $\sigma(t) \leq \tau < \frac{1}{\gamma}$.

Moreover, for $s_0 \in \xi$, take $s_{2n+2} = fs_{2n+1}$ and $s_{2n+1} = gs_{2n}$, for all $n \in \mathbb{N}$. Choose the controlled function satisfying

$$\beta(s_i, s_j) \leq \gamma \quad \text{for all } i, j \geq 0 \text{ and } \gamma \in [1, 1/\tau), \tag{3.10}$$

$$\sup \lim_{m\to\infty} \frac{\beta(s_{i+1}, s_{i+2})\beta(s_{i+1}, s_{2m})}{\beta(s_i, s_{i+1})} < \frac{1}{\tau}, \tag{3.11}$$

where $\theta \in (0,1)$. Then, there exists $s^* \in B_v(s_0, r)$ such that $s_n \to s^*$. Moreover,

$$\lim_{n\to\infty} \beta(s_n, s^*) \text{ and } \lim_{n\to\infty} \beta(s^*, s_n) \text{ exist and are finite.} \tag{3.12}$$

Then, the mappings f and g have a unique common fixed point s^* in $B_v(s_0, r)$.

Proof. Consider the sequence starting from $s_0 \in \xi$ such that $s_1 = gs_0, s_2 = fs_1, s_3 = gs_2, s_4 = fs_3, \ldots,$ and in general $s_{2n+1} = gs_{2n}$, $s_{2n+2} = fs_{2n+1}$ for all $n \in \mathbb{N}$.

From (3.8):

$$\begin{aligned} v(s_{2n}, s_{2n+1}) &\leq \sigma(E_{f,g}(s_{2n-1}, s_{2n}))E_{f,g}(s_{2n-1}, s_{2n}) \\ &\leq \tau E_{f,g}(s_{2n-1}, s_{2n}) \\ &\leq \tau \max\{v(s_{2n-1}, s_{2n}), v(s_{2n-1}, s_{2n}), v(s_{2n}, s_{2n+1})\} \\ &\leq \tau \max\{v(s_{2n-1}, s_{2n}), v(s_{2n}, s_{2n+1})\}. \end{aligned}$$

If $v(s_{2n}, s_{2n+1}) \leq \tau v(s_{2n}, s_{2n+1})$, then $v(s_{2n}, s_{2n+1}) \leq \tau v(s_{2n}, s_{2n+1})$, which is only possible if $v(s_{2n}, s_{2n+1}) = 0$. Therefore, we have $v(s_{2n}, s_{2n+1}) \leq \tau v(s_{2n-1}, s_{2n})$, which gives

$$v(s_{2n}, s_{2n+1}) \leq \tau v(s_{2n-1}, s_{2n}) \leq \tau^2 v(s_{2n-2}, s_{2n-1}) \leq \cdots \leq \tau^{2n} v(s_0, s_1).$$

From (3.9):

$$v(s_0, gs_0) \leq \frac{(1-\tau)(1+\tau)}{\gamma(1+3\tau)} r.$$

Since $\frac{1+\tau}{1+3\tau} < 1$, $\gamma \geq 1$, and $1 - \tau < 1$, we get $v(s_0, s_1) \leq r$, which implies $s_1 \in B_v(s_0, r)$.

Using the triangle inequality, (3.8) and (3.9):

$$\begin{aligned} v(s_0, s_2) &\leq \beta(s_0, s_1)v(s_0, s_1) + \beta(s_1, s_2)v(s_1, s_2) \\ &\leq \gamma v(s_0, s_1) + \gamma\tau v(s_0, s_1) \\ &= (\gamma + \gamma\tau)v(s_0, s_1) \\ &= \gamma(1+\tau)v(s_0, s_1) \\ &\leq \gamma(1+\tau) \cdot \frac{(1-\tau)(1+\tau)}{\gamma(1+3\tau)} r \\ &= \frac{(1-\tau^2)(1+\tau)}{(1+3\tau)} r. \end{aligned}$$

Since $\frac{1+\tau}{1+3\tau} < 1$ and $1 - \tau^2 < 1$, we have $v(s_0, s_2) \leq r$, which implies $s_2 \in B_v(s_0, r)$.

By using techniques analogous to those in Theorem 3.1, it can be established that the sequence $\{s_{2n}\}$ is Cauchy in $B_v(s_0, r)$. Thus, we have $v(s_0, s_{2n+1}) \leq r$, and therefore $s_{2n+1} \in B_v(s_0, r)$. Hence $s_{2n} \in B_v(s_0, r)$ for all $n \in \mathbb{N}$.

By a similar method as in Theorem 3.1, the sequence $\{s_{2n}\}$ is Cauchy. Hence,

$$\lim_{n,m\to\infty} v(s_{2n}, s_{2m}) = 0. \tag{3.13}$$

Since every closed ball is closed in a CCMS (Proposition 2.1), $(B_v(s_0, r), v)$ is complete. Thus, every Cauchy sequence converges in $B_v(s_0, r)$. Therefore, there exists $s^* \in B_v(s_0, r)$ such that

$$\lim_{n\to\infty} v(s_{2n}, s^*) = 0. \tag{3.14}$$

For the fixed point property:

$$\begin{aligned} v(s^*, gs^*) &\leq \beta(s^*, s_{2n})v(s^*, s_{2n}) + \beta(s_{2n}, gs^*)v(s_{2n}, gs^*) \\ &\leq \beta(s^*, s_{2n})v(s^*, s_{2n}) + \tau\beta(s_{2n}, gs^*)v(s_{2n-1}, s^*). \end{aligned}$$

Taking $n \to \infty$ and using (3.14), we get $v(s^*, gs^*) = 0$, which implies $s^* = gs^*$ is a fixed point.

To prove s^* is a fixed point of f is similar to that for g. For uniqueness, consider another fixed point $h = fh = gh$. Then

$$v(s^*, h) = v(fs^*, gh) \leq \tau \max\{v(s^*, h), v(s^*, fs^*), v(h, gh)\} = \tau v(s^*, h).$$

Thus, $\tau < 1$ implies $s^* = h$, so s^* is a unique common fixed point of f and g in $B_v(s_0, r)$. □

Example 3.2. *Assume $\xi = \{0, 1, 2\}$ with metric $v : \xi \times \xi \to \mathbb{R}$ and control function as defined in Example 3.1. Let $g, f : \xi \to \xi$ be mappings defined by $f(s) = g(s) = 2$ for all $s \in \xi$. Taking $s_0 = 2$, $r = 3$, $\mu = 1.2$, and $\tau = 0.4$, clearly all conditions* (3.8)–(3.12) *hold, and* 2 *is a unique common fixed point of f and g.*

Corollary 3.3. *Suppose (ξ, v) is a complete CCMS in which $B_v(s_0, r)$ is compact. Let $f, g : (\xi, v) \to (\xi, v)$ be two mappings satisfying conditions* (3.8), (3.9), (3.10), *and* (3.12) *of Theorem 3.2. Then f and g have a unique common fixed point s^* in $B_v(s_0, r)$.*

Proof. Since $B_v(s_0, r)$ is compact, any sequence $\{s_n\} \in B_v(s_0, r)$ has a convergent subsequence. By (3.9), the sequence $\{s_n\}$ is Cauchy. Hence, all assumptions of Theorem 3.2 are fulfilled. Therefore, f and g have a unique common fixed point s^* in $B_v(s_0, r)$. □

Corollary 3.4. *Let (ξ, v) be a complete continuous b-metric space with parameter μ. Let $f, g : \xi \to \xi$ be two maps with Geraghty-Ćirić contraction such that*

$$v(fs, gh) \le \tau E_{f,g}(s, h),$$

and

$$\gamma v(s_0, gs_0) \le \frac{(1-\tau)(1+\tau)}{(1+3\tau)} r$$

for all $s, h \in B_v(s_0, r)$, where $\tau \in (0, 1)$ and $\sigma \in \Psi_\alpha$ is a family of control functions with $\sigma(t) = \tau < \frac{1}{\gamma}$. Then, the mappings f and g have a unique common fixed point s^* in $B_v(s_0, r)$.

Proof. By choosing the constant function $\beta(s_i, s_j) = \gamma$ for all $i, j \in \mathbb{N}$, the given assumptions imply conditions (3.1)–(3.4) of Theorem 3.2. Thus, by Theorem 3.2, f and g have a unique fixed point in $B_v(s_0, r)$. □

4. Applications

Let $\xi = (C[0, 1], \mathbb{R})$ be the set of all real continuous functions. We aim to find the unique solution for the Urysohn integral equation

$$s(\sigma) = \int_a^b K_i(t, q, s(q))\, dq + g(t), \quad i = 1, 2 \text{ for all } t \in [0, 1], \tag{4.1}$$

where the Lipschitz kernel K is continuous. Define the b-metric v by

$$v(s, h) = \sup_{t \in [0,1]} |s(t) - h(t)|^m$$

with $m \ge 1$ and parameter $\mu = 2^{m-1}$, and the closed ball

$$B_v(s_0, r) = \{s \in C([0, 1], \mathbb{R}) : v(s, s_0) \le r\}$$

with $r > 0$.

Theorem 4.1. *Consider the integral equation* (4.1) *and suppose that:*

(1) *$K : [0, 1] \times [0, 1] \times C[0, 1] \to \mathbb{R}$ is continuous for all $t \in [0, 1]$ such that for $\tau \in (0, 1)$,*

$$|K_1(t, q, s(q)) - K_2(t, q, h(q))| \le \tau^{1/m} |s(q) - h(q)|$$

for all $s, h \in C[0, 1]$ such that $s(t), h(t) \in [s_0(t) - r^{1/m}, s_0(t) + r^{1/m}]$ for all $t \in [0, 1]$ and $s_0 \in C[0, 1]$.

(2) *The mappings $f, g : \xi \to \xi$ are defined by*

$$gs(t) = \int_a^b K_1(t, q, s(q))\, dq + g(t)$$

and

$$fs(t) = \int_a^b K_2(t, q, s(q))\, dq + g(t).$$

(3) *There exists*

$$\left| s_0(q) - \int_0^1 K_1(\tau, t, s_0(q))\, dt \right|^m \leq \frac{(1-\mu\tau)(1+\tau)}{\mu(1+3\tau)} r,$$

where $\mu = 2^{m-1}$, $r > 0$, and $\mu\tau < 1$.

Then, there exists a unique $s^* \in C[0,1]$ such that s^* is the unique solution of (4.1) *and $s^*(t) \in [s_0(t) - r^{1/m}, s_0(t) + r^{1/m}]$.*

Proof. Let $s, h \in C[0,1]$ such that $s(t), h(t) \in [s_0(t) - r^{1/m}, s_0(t) + r^{1/m}]$. Then

$$\begin{aligned} |gs(t) - fh(t)|^m &= \left| \int_0^1 [K_1(t, q, s(q)) - K_2(t, q, h(q))]\, dq \right|^m \\ &\leq \int_0^1 |K_1(t, q, s(q)) - K_2(t, q, h(q))|^m\, dq. \end{aligned}$$

By condition (1),

$$|gs(t) - fh(t)|^m \leq \int_0^1 |\tau^{1/m}[s(q) - h(q)]|^m\, dq = \tau \int_0^1 |s(q) - h(q)|^m\, dq.$$

Taking supremum, we get

$$\sup_{t\in[0,1]} |gs(t) - fh(t)|^m \leq \tau \sup_{t\in[0,1]} |s(q) - h(q)|^m \int_0^1 dq = \tau \sup_{t\in[0,1]} |s(q) - h(q)|^m.$$

This implies $v(fs, gh) \leq \tau v(s, h)$. But $\tau v(s, h) \leq \tau E_{f,g}(s, h)$, so we have

$$v(fs, gh) \leq \sigma(E_{f,g}(s, h)) E_{f,g}(s, h).$$

From condition (3),

$$\left| s_0(q) - \int_0^1 K_1(t, q, s_0(q))\, dq \right|^m \leq \frac{(1-\tau)(1+\tau)}{\mu(1+3\tau)} r,$$

so

$$\sup_{t\in[0,1]} |s_0(q) - gs_0|^m \leq \frac{(1-\tau)(1+\tau)}{\mu(1+3\tau)} r,$$

which implies

$$\mu v(s_0, gs_0) \leq \frac{(1-\tau)(1+\tau)}{(1+3\tau)} r.$$

Hence, all the axioms of Corollary 3.4 hold, and s^* is the unique fixed point of f and g in $C[0,1]$. Thus, there exists a unique s^* such that

$$s^*(t) = fs^*(t) = gs^*(t) = \int_0^1 K_i(t, q, s^*(q))\, dq + g(t), \quad i = 1, 2 \text{ for all } t \in [0,1],$$

and $s^* \in C[0,1]$ is the unique solution of (4.1) with $s^*(t) \in [s_0(t) - r^{1/m}, s_0(t) + r^{1/m}]$. □

Acknowledgments: The authors T.Abdeljawad and A.Mukheimer gratefully acknowledge Prince Sultan University for financial support through the TAS research laboratory and for covering the APC.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double Controlled Metric Type Spaces and Some Fixed Point Results, Mathematics 6 (2018), 320. https://doi.org/10.3390/math6120320.

[2] S. Aggarwal, I. Uddin, J.J. Nieto, A Fixed-Point Theorem for Monotone Nearly Asymptotically Nonexpansive Mappings, J. Fixed Point Theory Appl. 21 (2019), 91. https://doi.org/10.1007/s11784-019-0728-5.

[3] C. Alaca, Fixed Point Results for Mappings Satisfying a General Contractive Condition of Operator Type in Dislocated Fuzzy Quasi-Metric Spaces, J. Comput. Anal. Appl. 12 (2010), 361–368.

[4] I. Altun, H. Simsek, Some Fixed Point Theorems on Ordered Metric Spaces and Application, Fixed Point Theory Appl. 2010 (2010), 621469. https://doi.org/10.1155/2010/621469.

[5] I. Altun, N. Al Arifi, M. Jleli, A. Lashin, B. Samet, A New Approach for the Approximations of Solutions to a Common Fixed Point Problem in Metric Fixed Point Theory, J. Funct. Spaces 2016 (2016), 1–5. https://doi.org/10.1155/2016/6759320.

[6] M. Arshad, A. Shoaib, I. Beg, Fixed Point of a Pair of Contractive Dominated Mappings on a Closed Ball in an Ordered Dislocated Metric Space, Fixed Point Theory Appl. 2013 (2013), 115. https://doi.org/10.1186/1687-1812-2013-115.

[7] J.P. Aubin, Mathematical Methods of Game and Economic Theory, North-Holland Publishing Company, 1979.

[8] H.F. Bohnenblust, S. Karlin, On a Theorem of Ville, in: Contributions to the Theory of Games (AM-24), Volume I, Princeton University Press, pp. 155–160, (1951). https://doi.org/10.1515/9781400881727-014.

[9] S. Carl, S. Heikkilä, Fixed Point Theory in Ordered Sets and Applications, Springer New York, 2011. https://doi.org/10.1007/978-1-4419-7585-0.

[10] K.B. Chander, T.V.P. Kumar, Common Fixed Points of Two Pairs of Selfmaps Satisfying a Geraghty-Berinde Type Contraction Condition in b-Metric Spaces, Electron. J. Math. Anal. Appl. 9 (2021), 12–36.

[11] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11.

[12] S. Furqan, H. Işık, N. Saleem, Fuzzy Triple Controlled Metric Spaces and Related Fixed Point Results, J. Funct. Spaces 2021 (2021), 1–8. https://doi.org/10.1155/2021/9936992.

[13] J. Harjani, K. Sadarangani, Generalized Contractions in Partially Ordered Metric Spaces and Applications to Ordinary Differential Equations, Nonlinear Anal. Theory Methods Appl. 72 (2010), 1188–1197. https://doi.org/10.1016/j.na.2009.08.003.

[14] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of b-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[15] A. Hussain, M. Arshad, M. Nazam, S.U. Khan, New Type of Results Involving Closed Ball with Graphic Contraction, J. Inequal. Spec. Funct. 7 (2016), 36–48.

[16] G. Jungck, Compatible Mappings and Common Fixed Points, Int. J. Math. Math. Sci. 9 (1986), 771–779. https://doi.org/10.1155/S0161171286000935.

[17] A.G. Kalo, K.K. Tola, H.E. Yesuf, Fixed Point Results for Geraghty–Ćirić-Type Contraction Mappings in b-Metric Space with Applications, Fixed Point Theory Appl. Sci. Eng. 2024 (2024), 8. https://doi.org/10.1186/s13663-024-00764-3.

[18] E. Karapinar, S. Czerwik, H. Aydi, (α,ψ)-Meir-Keeler Contraction Mappings in Generalized b-Metric Spaces, J. Funct. Spaces 2018 (2018), 1–4. https://doi.org/10.1155/2018/3264620.

[19] M.A. Khamsi, W.A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, Wiley, 2001. https://doi.org/10.1002/9781118033074.

[20] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled Metric Type Spaces and the Related Contraction Principle, Mathematics 6 (2018), 194. https://doi.org/10.3390/math6100194.

[21] M. Mudhesh, H.A. Hammad, E. Ameer, A. Ali, Fixed Point Results under New Contractive Conditions on Closed Balls, Appl. Math. Inf. Sci. 16 (2022), 555–564. https://doi.org/10.18576/amis/160408.

[22] M.A. Navascués, R.N. Mohapatra, Fixed Point Dynamics in a New Type of Contraction in b-Metric Spaces, Symmetry 16 (2024), 506. https://doi.org/10.3390/sym16040506.

[23] D. Rakić, A. Mukheimer, T. Došenović, Z.D. Mitrović, S. Radenović, On Some New Fixed Point Results in Fuzzy b-Metric Spaces, J. Inequal. Appl. 2020 (2020), 99. https://doi.org/10.1186/s13660-020-02371-3.

[24] R. Pant, R. Panicker, Geraghty and Ćirić Type Fixed Point Theorems in b-Metric Spaces, J. Nonlinear Sci. Appl. 9 (2016), 5741–5755. https://doi.org/10.22436/jnsa.009.11.03.

[25] R.P. Pant, Common Fixed Points of Noncommuting Mappings, J. Math. Anal. Appl. 188 (1994), 436–440. https://doi.org/10.1006/jmaa.1994.1437.

[26] A.C.M. Ran, M.C.B. Reurings, A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations, Proc. Amer. Math. Soc. 132 (2004), 1435–1443. https://doi.org/10.1090/S0002-9939-03-07220-4.

[27] T. Rasham, A. Shoaib, G. Marino, B.A.S. Alamri, M. Arshad, Sufficient Conditions to Solve Two Systems of Integral Equations via Fixed Point Results, J. Inequal. Appl. 2019 (2019), 182. https://doi.org/10.1186/s13660-019-2130-7.

[28] T. Rasham, A. Shoaib, Common Fixed Point Results for Two Families of Multivalued A–Dominated Contractive Mappings on Closed Ball with Applications, Open Math. 17 (2019), 1350–1360. https://doi.org/10.1515/math-2019-0114.

[29] S. Sessa, On a Weak Commutativity Condition of Mappings in Fixed Point Considerations, Publ. Inst. Math. (Beograd) 32(46) (1982), 149–153.

[30] A. Shoaib, α-η Dominated Mappings and Related Common Fixed Point Results in Closed Ball, J. Concr. Appl. Math. 13 (2015), 152–170.

[31] A. Shoaib, S. Mustafa, A. Shahzad, Common Fixed Point of Multivalued Mappings in Ordered Dislocated Quasi G-Metric Spaces, Punjab Univ. J. Math. 52 (2020), 1–23.

[32] A. Shoaib, A. Azam, M. Arshad, E. Ameer, Fixed Point Results for Multivalued Mappings on a Sequence in a Closed Ball with Applications, J. Math. Comput. Sci. 17 (2017), 308–316. https://doi.org/10.22436/jmcs.017.02.13.

[33] A. Shoaib, A. Azam, M. Arshad, A. Shahzad, Fixed Point Results for the Multivalued Mapping on Closed Ball in Dislocated Fuzzy Metric Space, J. Math. Anal. 8 (2017), 98–106.

[34] A. Shoaib, A. Azam, A. Shahzad, Common Fixed Point Results for the Family of Multivalued Mappings Satisfying Contractions on a Sequence in Hausdorff Fuzzy Metric Space, J. Comput. Anal. Appl. 24 (2018), 692–699.

[35] A. Shoaib, M. Arshad, M.A. Kutbi, Common Fixed Points of a Pair of Hardy–Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, J. Comput. Anal. Appl. 17 (2014), 255–264.

[36] A. Shoaib, A. Hussain, M. Arshad, A. Azam, Fixed Point Results for α_*-ψ-Ćirić Type Multivalued Mappings on an Intersection of a Closed Ball and a Sequence with Graph, J. Math. Anal. 7 (2016), 41–50.

[37] N. Souayah, N. Mlaiki, A Fixed Point Theorem in S_b-Metric Spaces, J. Math. Comput. Sci. 16 (2016), 131–139. https://doi.org/10.22436/jmcs.016.02.01.

[38] L. Subashi, Some Topological Properties of Extended b-Metric Space, in: Proceedings of the 5th International Virtual Conference on Advanced Scientific Results, 2017, pp. 164–167. https://doi.org/10.18638/scieconf.2017.5.1.451.

[39] T.V. An, L.Q. Tuyen, N.V. Dung, Stone-Type Theorem on b-Metric Spaces and Applications, Topol. Appl. 185-186 (2015), 50–64. https://doi.org/10.1016/j.topol.2015.02.005.