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On the Solution Behavior and Stability of Adjoint Impulsive Neutral Integro Dynamic Models

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Abstract. In this study, nonlinear impulsive neutral Hammerstein dynamic equations on finite time scales are established, and conditions guaranteeing the existence, uniqueness, and Ulam-type stability of their solutions are derived. The research establishes and validates specific criteria that provide solutions of existence and uniqueness by using the Picard operator and the Banach contraction principle. By extending integral impulsive inequalities to time scale frameworks, the study delves deeper into Ulam-type stability. A strong theoretical framework is made possible by the introduction of well thought of assumptions to help navigate the complexity of these findings. To verify the findings and demonstrate their relevance in solving problems pertaining to dynamic systems, a specific example is provided.

1. Introduction

The examination of time scales provides a flexible framework for integrating the analysis of discrete and continuous processes, thereby reconciling differential and difference equations. Stefan Hilger first proposed time scale calculus in 1988. It gives us a single way to study dynamical systems that are defined on any nonempty closed subset of the real line, which we call $\mathfrak{T}$ [27, 28]. This theory extends classical differentiation and integration to time scales, enabling the representation of processes that do not strictly adhere to being entirely continuous or completely discrete.

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Because of their wide applicability, dynamic equations which extend differential and difference equations to time scales have attracted a lot of attention [29]. Dynamic equations coincide with differential equations in the continuous-time setting, where the time scale is $\Re$, they correspond to difference equations when they are limited to integers. By concentrating on the dynamic equation rather than handling differential and difference equations independently, this unified approach streamlines the analysis of hybrid systems [16]. Time scale calculus thus provides an efficient framework for understanding and analysing complex systems with possibly discontinuous parameters, opening the door for creative applications in a variety of fields [18,19].

Neutral differential equations, in contrast to retarded differential equations, explicitly incorporate derivatives evaluated at delayed arguments, enabling the system's dependence on both the current and previous states [13]. The computational modelling of elastic network structures connected to high-speed switching systems is one of the many applications where they naturally emerge [1]. Neutral differential equations (NDEs) have drawn a lot of attention lately because of their adaptability and ubiquity in a variety of applied mathematics fields [9]. By examining mild solutions, expanding fixed-point frameworks, and tackling a variety of complex nonlocal constraints, researchers have made significant contributions [17,30]. This increasing interest highlights how crucial NDEs are to the advancement of both theoretical research and real-world applications [2,24].

In real-world systems, brief disruptions occur over brief periods of time relative to the process's overall time horizon [14]. These effects can cause the solution trajectories to exhibit abrupt changes at certain discrete moments, $\xi_1 < \xi_2 < \xi_3 < \cdots$, represented mathematically as:

$$p(\mathscr{E}_k^+) - p(\mathscr{E}_k^-) = \Im_k(\mathscr{E}_k, p(\mathscr{E}_k^-)).$$

These specific discontinuities are present in the solutions of such equations, which are referred to as impulsive dynamic equations [22,30]. Due to their significance in a wide range of applications, impulsive dynamic equations formulated on time scales have received increasing attention [8].

A significant area of the mathematical sciences, stability analysis takes many forms and propels advances in many other domains [7]. Hyers-Ulam (HU) stability is one of these that is particularly intriguing and extremely important [10,31]. The idea was first presented by Ulam [33,34],who developed the stability problem. Later, Hyers [11] provided a partial solution inside the Banach space framework. Rassias established the stability idea presently known as Hyers Ulam Rassias (HUR) stability in 1978, building on prior fundamental work (HUR) stability [20]. In the qualitative study of differential and functional equations, the concept of Ulam-type stability is crucial in establishing a connection between exact and approximate solutions [12,15]. Careful mathematical analysis backed by a range of analytical methods is necessary for a comprehensive evaluation of such stability features. Fixed-point approaches are particularly well-known among the available techniques because of their efficiency and clarity [32]. These techniques offer an efficient framework for researching Ulam-type stability and play a key role in the larger theory of stability analysis due to their comparatively mild assumptions and simplified structure when compared to alternative approaches.

In order to gain a better understanding of the structural and qualitative characteristics of dynamical systems with time-scale theory, significant progress has been made in the analysis of dynamic equations on time scales ($\mathfrak{T}\mathfrak{S}$), where focus is primarily on fundamental qualitative properties, notably the existence, uniqueness, and stability of solutions. Stability and controllability properties for nonlinear Volterra Fredholm Hammerstein impulsive integro-dynamic systems with delays on time scales were investigated by S. O. Shah et al. [23] . Bohner et al. extensively explored fundamental conclusions regarding nonlinear dynamical initial value problems of first order, including integro-dynamic equations [5, 6]. Ulam-type stability for nonlinear first-order dynamic equations with impulsive effects was further analysed by Scindia et al. [21]. More recently, Ulam stability results of nonlinear integro-dynamic adjoint equations incorporating Volterra Fredholm integral operators over time scales were established by Shah et al. [25]. Syed Omar Shah et al.'s study [26] focused on the behaviour of nonlinear dynamic adjoint systems in the presence of delays and impulsive effects on arbitrary time domains. To the best of our knowledge, Ulam-type stability features for nonlinear impulsive Hammerstein neutral integro dynamic (NIHNID) equations are not currently discussed in the literature. Drawing motivation from earlier investigations, including [26], this article develops new existence & uniqueness results and examines different stability notions for the adjoint NIHNID equation. The following is a summary of the key insights.

$$\begin{cases} \Phi^{\delta}(\varsigma) + w(\varsigma)\Phi(\varsigma - \tau) = f(\varsigma, \Phi(\varsigma)) \displaystyle\int_{\varsigma_0}^{\varsigma} G(\varsigma, s) h(s, \Phi(s)) \delta s, \quad \varsigma \in \mathfrak{I}^{z} \setminus \{\varsigma_i\}, \\ \Phi(\varsigma_k^{+}) - \Phi(\varsigma_k^{-}) = \Upsilon_k(\Phi(\varsigma_k^{-})), \quad k = 1, \ldots, m, \\ \Phi(\varsigma_0) = \Phi_0. \end{cases} \tag{1.1}$$

where $\mathfrak{I} := [\varsigma_0, \varsigma_f]_{\mathfrak{T}_{\mathfrak{S}}}$, $\mathfrak{T}_{\mathfrak{S}}$ denotes a time scale ($\mathfrak{T}\mathfrak{S}$), and $\mathfrak{I}^{z}$ represents the derivative form of $\mathfrak{I}$. Let $\Phi\colon \mathfrak{I} \to \Re$ be an unknown function to be determined, with Φ^{δ} denoting its delta derivative on $\mathfrak{T}\mathfrak{S}$. We define the composition $\Phi^{\kappa} := \Phi \circ \kappa$. Let $w : \mathfrak{T}_{\mathfrak{S}} \to \Re$ be rd-continuous and positively regressive. Also $f, h\colon \mathfrak{I} \times \Re \to \Re$ are rd-continuous in the first argument and continuous in the second argument. For all $\varsigma_k \in \mathfrak{I}$, the one-sided limits of Φ exist and are defined by

$$\Phi(\varsigma_k^{+}) = \lim_{\tau \to 0^{+}} \Phi(\varsigma_k + \tau), \quad \Phi(\varsigma_k^{-}) = \lim_{\tau \to 0^{-}} \Phi(\varsigma_k - \tau),$$

which satisfy

$$\varsigma_0 < \varsigma_1 < \varsigma_2 < \varsigma_3 < \varsigma_4 < \cdots < \varsigma_m < \varsigma_{m+1} = \varsigma_f < +\infty,$$

each ς_k demonstrates moments of impulse which is called a priori. $\Upsilon_k\colon \Re \to \Re$ describes Φ's discontinuity $\forall\ \varsigma_k$. Here, let us consider $\mathfrak{T}\mathfrak{S}$ to be distinct from the set of integers, and any impulses at isolated points are considered to be zero.

2. Preliminaries

The nonlinear impulsive Hammerstein neutral integro-dynamic equations considered in this work is based on [3, 4]. Let $\mathfrak{T}$ denote a non-empty closed subset of $\mathfrak{R}$, represented as $\mathfrak{T}$. The corresponding forward and backward jump operators, defined on the domain and range of $\mathfrak{T}$, are expressed as follows:

$$\kappa(t) = \inf\{\varsigma \in \mathfrak{T} \colon \varsigma > t\}, \quad \varrho(t) = \sup\{\varsigma \in \mathfrak{T} \colon \varsigma < t\}.$$

Let $\mu(t) = \kappa(t) - t$ denote the graininess of the time scale $\mathfrak{T}$. Based on this, we define the transformed time scale $\mathfrak{T}^z$ by

$$\mathfrak{T}^z = \begin{cases} \mathfrak{T} \setminus (\varrho(\sup \mathfrak{T}), \sup \mathfrak{T}], & \text{if } \sup \mathfrak{T} < \infty, \\ \mathfrak{T}, & \text{if } \sup \mathfrak{T} = \infty. \end{cases}$$

We say that $\iota : \mathfrak{T} \to \mathfrak{R}$ belongs to the class $C_{rd}(\mathfrak{T}, \mathfrak{R})$ if it is continuous at each right-dense point of $\mathfrak{T}$ and possesses a left-hand limit at every left-dense point. In addition, $C_{rd}(\mathfrak{T} \times \mathfrak{R}, \mathfrak{R})$ denotes the set of functions that are rd-continuous in the time-scale variable and continuous in the real variable.

A function $\iota : \mathfrak{T} \to \mathfrak{R}$ is said to be *regressive* if

$$\mu(t)\iota(t) + 1 \neq 0 \quad \text{or} \quad \mu(t)\iota(t) + 1 > 0 \quad \forall t \in \mathfrak{T}^z.$$

Let $\mathfrak{R}(\mathfrak{T})$ be the set of rd-continuous regressive functions on $\mathfrak{T}$, while $\mathfrak{R}^+(\mathfrak{T})$ represents those that are positively regressive.

Given a function $h : \mathfrak{T} \to \mathfrak{R}$, define

$$h^\delta(t) = \lim_{s \to t,\, s \neq \kappa(t)} \frac{h(\kappa(t)) - h(s)}{\kappa(t) - s}, \quad t \in \mathfrak{T}^z,$$

&

$$\int_{x_1}^{x_2} h(t)\delta t = H(x_2) - H(x_1), \quad x_1, x_2 \in \mathfrak{T},$$

here $H^\delta = h$ on $\mathfrak{T}^z$.

The equation presented below corresponds to the neutral form:

$$x^{ffi}(t) - g(x(\&(t))) = f(t, x(t), \int_a^t K(t, s, x(s)) ffis), \quad t \in \mathfrak{T}^z,$$

A solution $x(t)$ can be written in terms of the $\mathfrak{T}$-exponential fn as,

$$\Phi^\delta(t) = \iota(t)\Phi(t), \quad \Phi(t_0) = \Phi_0, \quad t \in \mathfrak{T}^z,$$

here, $\mathfrak{T}$-exponential can be expressed as follows:

$$x(t) = e_\iota(t, t_0)\Phi_0 + \int_{t_0}^t e_\iota(t, \tau) f(\tau, x(\tau), \int_a^\tau K(\tau, s, x(s))\delta s)\delta\tau,$$

here

$$e_\iota(t,t_0)=\exp\left(\int_{t_0}^{t}\Phi_{\mu(s)}\iota(s)\delta s\right),\quad t,t_0\in\mathfrak{T},$$

&

$$\Phi_{\mu(t)}\iota(t)=\begin{cases}\frac{\ln(1+\mu(t)\iota(t))}{\mu(t)}, & \text{if } \mu(t)\neq 0,\\ \iota(t), & \text{if } \mu(t)=0.\end{cases}$$

Consider $\iota,h\in\Re(\mathfrak{T})$, it follows that:

$$(\iota\oplus h)(t)=\iota(t)+h(t)+\mu(t)\iota(t)h(t),$$

$$(\ominus\iota)(t)=-\frac{\iota(t)}{1+\mu(t)\iota(t)},$$

$$\iota\ominus h=\iota\oplus(\ominus h).$$

Let $C_F(\Im,\Re)$ be defined on $\Im$, and let $PC_F(\Im,\Re)$ be corresponding space on $\Im$ equipped with the norm

$$\|\Phi\|=\sup_{\varsigma\in\Im}|\Phi(\varsigma)|.$$

$$PC_F{}^1(\Im,\Re)=\{\Phi\in PC_F(\Im,\Re):\Phi^\delta\in PC_F(\Im,\Re)\}$$

builds $(\mathcal{BS})$ incorporating $\|\Phi\|_1=\max\{\|\Phi\|,\|\Phi^\delta\|\}$.

Consider the following system of inequalities, now including the neutral term:

$$\begin{cases}\left|\Theta^\delta(\varsigma)+w(\varsigma)\Theta^\kappa(\varsigma)-f\Big(\varsigma,\Theta(\varsigma),\Theta^\delta(\varsigma-\tau)\Big)\int_{\varsigma_0}^{\varsigma}G(\varsigma,s)h(s,\Theta(s))\delta s\right|\leq\nu,\ \varsigma\in\Im^z\setminus\{\varsigma_i\},\\ |\Theta(\varsigma_k^+)-\Theta(\varsigma_k^-)-\Upsilon_k(\Theta(\varsigma_k^-))|\leq\nu,\quad k=\overline{1,\ m},\end{cases}\tag{2.1}$$

$$\begin{cases}\left|\Theta^\delta(\varsigma)+w(\varsigma)\Theta^\kappa(\varsigma)-f\Big(\varsigma,\Theta(\varsigma),\Theta^\delta(\varsigma-\tau)\Big)\int_{\varsigma_0}^{\varsigma}G(\varsigma,s)h(s,\Theta(s))\delta s\right|\leq\nu\omega(\varsigma),\\ \qquad\qquad\qquad\qquad\varsigma\in\Im^z\setminus\{\varsigma_i\},\\ |\Theta(\varsigma_k^+)-\Theta(\varsigma_k^-)-\Upsilon_k(\Theta(\varsigma_k^-))|\leq\nu K,\quad k=\overline{1,\ m},\end{cases}\tag{2.2}$$

$$\begin{cases}\left|\Theta^\delta(\varsigma)+w(\varsigma)\Theta^\kappa(\varsigma)-f\Big(\varsigma,\Theta(\varsigma),\Theta^\delta(\varsigma-\tau)\Big)\int_{\varsigma_0}^{\varsigma}G(\varsigma,s)h(s,\Theta(s))\delta s\right|\leq\omega(\varsigma),\ \varsigma\in\Im^z\backslash\{\varsigma_i\},\\ |\Theta(\varsigma_k^+)-\Theta(\varsigma_k^-)-\Upsilon_k\Big(\Theta(\varsigma_k^-),\Theta^\delta(\varsigma_k^-)\Big)|\leq K,\quad k=\overline{1,m},\end{cases}\tag{2.3}$$

Here, $0<\nu$, $0\leq K$, $\Theta\in PC_F{}^1(\Im,\Re)$ be a perturbed function, & $\omega\in PC_F{}^1(\Im,\Re^+)$ be nondecreasing. The neutral term $\Theta^\delta(\varsigma-\tau)$ introduces dependence on the delayed derivative of Θ, accounting for memory effects or historical states of the system.

Definition 2.1. *Eqn* (1.1) *is considered to possess HU-stability in $\Im$ for every function $\Theta\in PC_F^1(\Im,\Re)$ that fullfills* (2.1) *admits at least one corresponding solution $\Phi\in PC_F^1(\Im,\Re)$ of* (1.1) *fulfilling,*

$$|\Phi(\varsigma)-\Theta(\varsigma)|\leq C\nu\quad\forall\quad\varsigma\in\Im^z\backslash\{\varsigma_i\},\quad i=\overline{1,m},$$

where $0<C$.

Definition 2.2. *Consider $\Theta \in PC_F^1(\Im, \Re)$ that satisfies* (2.1). *If there exists a solution $\Phi \in PC_F^1(\Im, \Re)$ of* (1.1) *corresponding to each such Θ, then* (1.1) *is said to possess GHU-stability on $\Im$.*

$$|\Phi(\varsigma) - \Theta(\varsigma)| \leq \zeta(\nu) \quad \forall \quad \varsigma \in \Im^z \backslash \{\varsigma_i\}, \quad i = \overline{1, m},$$

here $\zeta \in C_F(\Re^+, \Re^+)$, $\zeta(0) = 0$.

Definition 2.3. *Let $\Theta \in PC_F^1(\Im, \Re)$ satisfy* (2.2). *If, for each such Θ, a solution $\Phi \in PC_F^1(\Im, \Re)$ of* (1.1) *exists corresponding to (ω, K), then* (1.1) *is said to possess HUR-stability on $\Im$.*

$$|\Phi(\varsigma) - \Theta(\varsigma)| \leq C\nu(\omega(\varsigma) + K) \quad \forall \quad \varsigma \in \Im^z \backslash \{\varsigma_i\}, \quad i = \overline{1, m},$$

here $0 < C$.

Definition 2.4. *Consider any function $\Theta \in PC_F^1(\Im, \Re)$ that satisfies* (2.3). *If there exists a corresponding solution $\Phi \in PC_F^1(\Im, \Re)$ of* (1.1) *associated with the pair (ω, K), then the equation* (1.1) *is said to possess GHUR-stability on $\Im$.*

$$|\Phi(\varsigma) - \Theta(\varsigma)| \leq C(\omega(\varsigma) + K) \quad \forall \quad \varsigma \in \Im^z \backslash \{\varsigma_i\}, \quad i = \overline{1, m},$$

where $0 < C$.

Lemma 2.1. (Generalized neutral impulsive integral formulation and its inequality on $\mathfrak{T}\mathfrak{S}$s) [21] *Let $\varsigma_0, \varsigma \in \mathfrak{T}_{\mathfrak{S}}$, $\varsigma_0 \leq \varsigma$, $\Phi \in PC_F(\mathfrak{T}_{\mathfrak{S}}, \Re)$, $p \in \Re^+(\mathfrak{T}_{\mathfrak{S}})$, $a \in PC_F(\mathfrak{T}_{\mathfrak{S}}, \Re^+)$, $b \in PC_F(\mathfrak{T}_{\mathfrak{S}}, \Re^+)$, & $d_k \in \Re^+$, $k = \overline{1,\ m}$.*

$$\Phi(\varsigma) \leq a(\varsigma) + b(\varsigma)\Phi^\delta(\varsigma) + \int_{\varsigma_0}^{\varsigma} p(\gamma)\Phi(\gamma)\delta\gamma + \sum_{\varsigma_0 < \varsigma_k < \varsigma} d_k \Phi(\varsigma_k)$$

implies

$$\Phi(\varsigma) \leq \Big(a(\varsigma) + b(\varsigma)\Phi^\delta(\varsigma)\Big) \prod_{\varsigma_0 < \varsigma_k < \varsigma} (1 + d_k) e_p(\varsigma, \varsigma_0), \quad \varsigma \geq \varsigma_0.$$

Remark 2.1. *$\Phi \in {PC_F}^1(\Im, \Re)$ is* (1.1), *iff*

$$\Phi(\varsigma) = \begin{cases} e_{\ominus w}(\varsigma, \varsigma_0)\Big[\Phi_0 + b(\varsigma_0)\Phi^\delta(\varsigma_0)\Big] + \displaystyle\sum_{\varsigma_0 < \varsigma_k < \varsigma} e_{\ominus w}(\varsigma, \varsigma_k) \Upsilon_k(\Phi(\varsigma_k^-)) \\ + \displaystyle\int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma, s) f(s, \Phi(s)) \int_{s_0}^{s} G(s, v) h(v, \Phi(v)) \delta v \delta s, \end{cases} \tag{2.4}$$

Remark 2.2. $\Theta \in {PC_F}^1(\Im, \Re)$ *satisfies* (2.2) *iff $\exists\, g \in PC_F(\Im, \Re)$ & it's sequence is represented as g_k, s.t., $|g(\varsigma)| \leq \omega(\varsigma)\nu\ \forall\ \varsigma \in \Im$ & $|g_k| \leq K\nu$ for all $k = \overline{1,\ m}$,*

$$\begin{cases} \Theta^\delta(\varsigma) + w(\varsigma)\Theta^\kappa(\varsigma) = f(\varsigma, \Theta(\varsigma), b(\varsigma)\Theta^\delta(\varsigma)) \displaystyle\int_{\varsigma_0}^{\varsigma} G(\varsigma, s) h(s, \Theta(s), b(s)\Theta^\delta(s)) \delta s + g(\varsigma), \\ \qquad\qquad\qquad\qquad \varsigma \in \Im^z \backslash \{\varsigma_k\}, \quad k = \overline{1,\ m}, \\ \Theta(\varsigma_k^+) - \Theta(\varsigma_k^-) = \Upsilon_k(\Theta(\varsigma_k^-)) + g_k,\ k = \overline{1,\ m}, \\ \Theta(\varsigma_0) = \Theta_0. \end{cases} \tag{2.5}$$

Lemma 2.2. *All the solution $\Theta \in {PC_F}^1(\Im, \Re)$ in (2.2) meets the following inequality:*

$$\begin{cases} \Bigg|\Theta(\varsigma) - e_{\ominus w}(\varsigma,\varsigma_0)\Theta_0 - \displaystyle\sum_{\varsigma_0<\varsigma_k<\varsigma} e_{\ominus w}(\varsigma,\varsigma_k)\Upsilon_k(\Theta(\varsigma_k^-)) - \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s)f(s,\Theta(s)) \\ \times \displaystyle\int_{s_0}^{s} G(s,v)h(v,\Theta(v))\delta v\delta s\Bigg| \\ \leq E_k\nu\left(\displaystyle\int_{\varsigma_0}^{\varsigma}\omega(s)\delta s + mK\right) + \int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)||b(s)|\delta s, \end{cases} \tag{2.6}$$

where $E_k = \max_k\{E_k\}$, & $|e_{\ominus w}(\varsigma,\cdot)| \leq E_k$ for $\varsigma \in (\varsigma_k, \varsigma_{k+1}]_{\mathfrak{T}_\Im} \subset \Im$.
In the case where $b(\varsigma)$ is known & bounded, we further estimate:

$$\int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)||b(s)|\delta s \leq MB\int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)|\delta s,$$

where M is the maximum value of $\Theta^\delta(s)$ over $[\varsigma_0,\varsigma]$, & B is a bound on $|b(\varsigma)|$.

Proof. If (2.2) is derived for $\Theta \in {PC_F}^1(\Im, \Re)$, then, by Remark 2.1, the solution of Equation (2.5) is:

$$\begin{aligned}\Theta(\varsigma) = e_{\ominus w}(\varsigma,\varsigma_0)\Theta_0 &+ \sum_{\varsigma_0<\varsigma_k<\varsigma} e_{\ominus w}(\varsigma,\varsigma_k)(\Upsilon_k(\Theta(\varsigma_k^-)) + g_k) \\ &+ \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s)f(s,\Theta(s))\int_{s_0}^{s} G(s,v)h(v,\Theta(v))\delta v\delta s \\ &+ \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s)g(s)\delta s.\end{aligned}$$

Then, for the difference:

$$\begin{aligned}&\Bigg|\Theta(\varsigma) - e_{\ominus w}(\varsigma,\varsigma_0)\Theta_0 - \sum_{\varsigma_0<\varsigma_k<\varsigma} e_{\ominus w}(\varsigma,\varsigma_k)\Upsilon_k(\Theta(\varsigma_k^-)) - \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s)f(s,\Theta(s)) \\ &\quad\times\int_{s_0}^{s} G(s,v)h(v,\Theta(v))\delta v\delta s\Bigg| \\ &\leq \int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)||g(s)|\delta s + \sum_{\varsigma_0<\varsigma_k<\varsigma}|e_{\ominus w}(\varsigma,\varsigma_k)||g_k| \\ &\leq \int_{\varsigma_0}^{\varsigma} E_k\nu\omega(s)\delta s + \sum_{\varsigma_0<\varsigma_k<\varsigma} E_k\nu K \\ &\leq \int_{\varsigma_0}^{\varsigma} E_k\nu\omega(s)\delta s + mE_k\nu K \\ &= E_k\nu\left(\int_{\varsigma_0}^{\varsigma}\omega(s)\delta s + mK\right).\end{aligned}$$

Thus, (2.6) is derived. The second term involving the neutral term $b(\varsigma)$ is bounded as described above, leading to the complete estimate. □

3. Main Results

The main assumptions underlying the analysis are listed as follows:

(C_1) Let $h \in C_{rd}(\Im \times \Re, \Re)$, $\exists\, 0 < \mathfrak{L}$ s.t,

$$|h(\varsigma, \varsigma_1) - h(\varsigma, \varsigma_2)| \leq L|\varsigma_1 - \varsigma_2|$$

for all $\varsigma \in \Im$ & $\varsigma_i \in \Re$, $i \in \{1, 2\}$. Also, $\exists\, 0 < \sigma$ s.t., $|h(v, \Phi(v))| \leq \sigma$.

(C_2) Let $\Upsilon_k : \Re \to \Re$, $\exists\, 0 < M_k$ s.t,

$$|\Upsilon_k(\varsigma_1) - \Upsilon_k(\varsigma_2)| \leq \mathfrak{M}_{\mathfrak{k}}|\varsigma_1 - \varsigma_2|$$

$\forall\, k = \overline{1, m}$ & $\varsigma_i \in \Re$, $i \in \{1, 2\}$. And, $\exists\, 0 < C_k$ s.t, $|\Upsilon_k(\Phi(\varsigma_k^-))| \leq C_k$.

(C_3) Let $\varpi, \varsigma > 0$, $|f(s, \Phi(s))| \leq \varpi$, $|\mathfrak{G}(\mathfrak{s}, v)| \leq \varsigma\ \forall\, s, v \in \Im$.

(C_4) Let

$$\left(\sum_{\varsigma_0 < \xi_k < \xi} E_k M_k + \frac{\xi_f^2}{2} E_k \varpi \xi L \right) < 1$$

holds.

(C_5) Let $\omega \in {PC_F}^1(\Im, \Re^+)$, $\exists\, 0 < \theta$ such that,

$$\int_{\varsigma_0}^{\varsigma} \omega(v) \delta v \leq \theta \omega(\varsigma).$$

(C_6) For a function $b(\varsigma)$ associated with the neutral term, $\exists$ a constant M_b such that, for every $\varsigma \in \Im$,

$$|b(\varsigma)| \leq M_b.$$

Furthermore, we assume $\exists$ a constant $\delta > 0$ s.t, the following holds:

$$\int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma, s)||b(s)| \delta s \leq \delta \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma, s)| \delta s.$$

This ensures that the neutral term does not disrupt the solution's behavior, & the contribution of the neutral term is well-controlled within the solution framework.

Theorem 3.1. *The adjoint NIHNID equation* (1.1) *possesses a unique solution in* ${PC_F}^1(\Im, \Re)$ *provided* (C_1)–(C_6) *are satisfied.*

Proof. Consider a Banach space$U := \{\Phi \in {PC_F}^1(\Im, \Re) : |\Phi| \leq \vartheta\}$,

$$\vartheta := E_k|\Phi_0| + \sum_{\varsigma_0 < \varsigma_k < \varsigma} E_k C_k + E_k \varpi \varsigma \frac{\varsigma_f^2}{2} \sigma + E_k M_b.$$

Here, M_b is the bound associated with the neutral term from (C$_6$). $\Lambda\colon U \to U$ is:

$$(\Lambda\Phi) := \begin{cases} e_{\ominus w}(\varsigma,\varsigma_0)\Phi_0 + \sum\limits_{\varsigma_0<\varsigma_k<\varsigma} e_{\ominus w}(\varsigma,\varsigma_k)\Upsilon_k(\Phi(\varsigma_k^-)) + \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s) f(s,\Phi(s)) \\ \times \int_{s_0}^{s} G(s,v)h(v,\Phi(v))\delta v\delta s + \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma,s)b(s)\delta s,\ \varsigma \in (\varsigma_k,\varsigma_{k+1}]_{\mathfrak{T}_{\varepsilon}}, \\ \hfill k=\overline{1,\ m}. \end{cases} \tag{3.1}$$

Here, we establish the Picard operator property of Λ on U. Let $\Lambda\colon U \to U$. For $\varsigma \in (\varsigma_k,\varsigma_{k+1}]_{\mathfrak{T}_{\varepsilon}}$, $k=\overline{1,\ m}$, we have

$$\begin{aligned} |(\Lambda\Phi)(\varsigma)| &\leq |e_{\ominus w}(\varsigma,\varsigma_0)||\Phi_0| + \sum_{\varsigma_0<\varsigma_k<\varsigma} |e_{\ominus w}(\varsigma,\varsigma_k)||\Upsilon_k(\Phi(\varsigma_k^-))| \\ &\quad + \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma,s)||f(s,\Phi(s))| \int_{s_0}^{s} |G(s,v)||h(v,\Phi(v))|\delta v\delta s \\ &\quad + \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma,s)||b(s)|\delta s \\ &\overset{(C_1),(C_2),(C_3),(C_6)}{\leq} E_k|\Phi_0| + \sum_{\varsigma_0<\varsigma_k<\varsigma} E_kC_k + \int_{\varsigma_0}^{\varsigma} E_k\varpi \int_{s_0}^{s} \varsigma\sigma\delta v\delta s \\ &\quad + \int_{\varsigma_0}^{\varsigma} E_kM_b\delta s \\ &\leq E_k|\Phi_0| + \sum_{\varsigma_0<\varsigma_k<\varsigma} E_kC_k + E_k\varpi\varsigma\frac{\varsigma^2}{2}\sigma + E_kM_b\varsigma \\ &\leq E_k|\Phi_0| + \sum_{\varsigma_0<\varsigma_k<\varsigma} E_kC_k + E_k\varpi\varsigma\frac{\varsigma_f^2}{2}\sigma + E_kM_b\varsigma_f \\ &= \vartheta. \end{aligned}$$

Thus, $|(\Lambda\Phi)(\varsigma)| \leq \vartheta$ & so $\Lambda\colon U \to U$.

Then, for any $\Phi_1,\Phi_2 \in U$ & $\varsigma \in (\varsigma_m,\varsigma_{m+1}]_{\mathfrak{T}_{\varepsilon}}$, we have the following estimate:

$$\begin{aligned} |(\Lambda\Phi_1)(\varsigma)-(\Lambda\Phi_2)(\varsigma)| &\leq \sum_{\varsigma_0<\varsigma_k<\varsigma} |e_{\ominus w}(\varsigma,\varsigma_k)||\Upsilon_k(\Phi_1(\varsigma_k^-))-\Upsilon_k(\Phi_2(\varsigma_k^-))| \\ &\quad + \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma,s)||f(s,\Phi_1(s))| \int_{s_0}^{s} |G(s,v)||h(v,\Phi_1(v))|\delta v\delta s \\ &\quad - \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma,s)||f(s,\Phi_2(s))| \int_{s_0}^{s} |G(s,v)||h(v,\Phi_2(v))|\delta v\delta s \\ &\quad + \int_{\varsigma_0}^{\varsigma} |e_{\ominus w}(\varsigma,s)||b(s)||\Phi_1(s)-\Phi_2(s)|\delta s \\ &\overset{(C_2),(C_3)}{\leq} \sum_{\varsigma_0<\varsigma_k<\varsigma} E_kM_k|\Phi_1(\varsigma_k^-)-\Phi_2(\varsigma_k^-)| \end{aligned}$$

$$
\begin{aligned}
&+\int_{\varsigma_0}^{\varsigma} E_k\varpi \int_{s_0}^{s} \varsigma |h(v,\Phi_1(v)) - h(v,\Phi_2(v))|\delta v\delta s \\
&+\int_{\varsigma_0}^{\varsigma} E_k M_b |\Phi_1(s) - \Phi_2(s)|\delta s \\
\overset{(C_1)}{\leq} &\sum_{\varsigma_0<\varsigma_k<\varsigma} E_k M_k |\Phi_1(\varsigma_k^-) - \Phi_2(\varsigma_k^-)| \\
&+\int_{\varsigma_0}^{\varsigma} E_k\varpi \int_{s_0}^{s} \varsigma L|\Phi_1(v) - \Phi_2(v)|\delta v\delta s \\
&+\int_{\varsigma_0}^{\varsigma} E_k M_b |\Phi_1(s) - \Phi_2(s)|\delta s \\
\leq &\sum_{\varsigma_0<\varsigma_k<\varsigma} E_k M_k \|\Phi_1 - \Phi_2\| + \|\Phi_1 - \Phi_2\| \int_{\varsigma_0}^{\varsigma} E_k\varpi\varsigma L\delta v\delta s \\
&+\|\Phi_1 - \Phi_2\| \int_{\varsigma_0}^{\varsigma} E_k M_b \delta s \\
\leq &\|\Phi_1 - \Phi_2\| \left(\sum_{\varsigma_0<\varsigma_k<\varsigma} E_k M_k + \int_{\varsigma_0}^{\varsigma} E_k\varpi\varsigma L\varsigma\delta s + E_k M_b \varsigma_f \right) \\
\leq &\|\Phi_1 - \Phi_2\| \left(\sum_{\varsigma_0<\varsigma_k<\varsigma} E_k M_k + \frac{\varsigma_f^2}{2} E_k\varpi\varsigma L + E_k M_b \varsigma_f \right).
\end{aligned}
$$

By virtue of assumption $C4$, the mapping Λ satisfies the contraction property. This implies that Λ generates a Picard iteration on U and admits a unique fixed point. As a consequence, equation (1.1) admits a unique solution in the space $PC_F^1(\Im, \Re)$.

□

Ulam-Type Stability for Nonlinear Hammerstein Impulsive Neutral Integro-Dynamic Equations

Theorem 3.2. *The adjoint nonlinear Hammerstein impulsive neutral integro-dynamic equation satisfies HUR stability in ω, K on $\Im$ if $C_1 - C_6$ hold.*

Proof. Assume that $\Theta \in PC_F^1(\Im, \Re)$ satisfies (2.2). Suppose further that $\Phi \in PC_F^1(\Im, \Re)$ solves the neutral integro-dynamic equation (1.1) and fulfills $\Phi_0 = \Theta_0$. Then, in view of Remark 2.1, $\Phi(\varsigma)$ admits the form

$$
\Phi(\varsigma) = \begin{cases} e_{\ominus w}(\varsigma, \varsigma_0)\Theta_0 + \sum_{\varsigma_0<\varsigma_k<\varsigma} e_{\ominus w}(\varsigma, \varsigma_k)\Upsilon_k(\Phi(\varsigma_k^-)) \\ + \int_{\varsigma_0}^{\varsigma} e_{\ominus w}(\varsigma, s)\Big[f(s,\Phi(s)) + b(s)\Phi^{\delta}(s) \Big] \int_{s_0}^{s} G(s,v)h(v,\Phi(v))\delta v\delta s, \ \varsigma \in (\varsigma_k, \varsigma_{k+1}]_{\mathfrak{T}_S}, \end{cases}
$$

where $b(s)$ accounts for the neutral term.
Similarly, since $\Theta \in {PC_F}^1(\Im, \Re)$ meets (2.2), from Remark 2.2, we represent:

$$\Theta^{\delta}(\varsigma)+w(\varsigma)\Theta^{\kappa}(\varsigma)=f(\varsigma,\Theta(\varsigma))\int_{\varsigma_0}^{\varsigma}G(\varsigma,s)h(s,\Theta(s))\delta s+b(\varsigma)\Theta^{\delta}(\varsigma)+g(\varsigma),\ \varsigma\in\mathfrak{I}^{z}\backslash\{\varsigma_k\},$$
$$\Theta(\varsigma_k^{+})-\Theta(\varsigma_k^{-})=\Upsilon_k(\Theta(\varsigma_k^{-}))+g_k,$$

where $|g(\varsigma)|\le \nu\omega(\varsigma)\ \forall\ \varsigma\in\mathfrak{I}$ & $|g_k|\le K\nu$ for $k=\overline{1,m}$.
Using $\Theta(\varsigma)$ & incorporating the hypothesis C_6, we rewrite:

$$\Theta(\varsigma)=e_{\ominus w}(\varsigma,\varsigma_0)\Theta_0+\sum_{\varsigma_0<\varsigma_k<\varsigma}e_{\ominus w}(\varsigma,\varsigma_k)(\Upsilon_k(\Theta(\varsigma_k^{-}))+g_k)$$
$$+\int_{\varsigma_0}^{\varsigma}e_{\ominus w}(\varsigma,s)\Big[f(s,\Theta(s))+b(s)\Theta^{\delta}(s)+g(s)\Big]\int_{s_0}^{s}G(s,v)h(v,\Theta(v))\delta v\delta s.$$

To estimate $|\Theta(\varsigma)-\Phi(\varsigma)|$, we write:

$$|\Theta(\varsigma)-\Phi(\varsigma)|\le\Bigg|\int_{\varsigma_0}^{\varsigma}e_{\ominus w}(\varsigma,s)\Big[f(s,\Theta(s))-f(s,\Phi(s))+b(s)(\Theta^{\delta}(s)-\Phi^{\delta}(s))\Big]$$
$$\times\int_{s_0}^{s}G(s,v)h(v,\Theta(v))-h(v,\Phi(v))\delta v\delta s\Bigg|$$
$$+\sum_{\varsigma_0<\varsigma_k<\varsigma}M_kE_k|\Theta(\varsigma_k^{-})-\Phi(\varsigma_k^{-})|.$$

Using C_6, we bound the contribution of the neutral term $b(s)$:

$$\int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)b(s)|\delta s\le\delta\int_{\varsigma_0}^{\varsigma}|e_{\ominus w}(\varsigma,s)|\delta s.$$

Combining this with hypotheses C_1-C_5, we have:

$$|\Theta(\varsigma)-\Phi(\varsigma)|\le E_k\nu(\theta+m)(\omega(\varsigma)+K)\prod_{\varsigma_0<\varsigma_k<\varsigma}(1+M_kE_k)e^{\mathcal{P}^*(\varsigma-\varsigma_0)}$$
$$+\delta L\int_{\varsigma_0}^{\varsigma}e_{\mathcal{P}^*}(\varsigma-s)|\Theta(s)-\Phi(s)|\delta s,$$

where $e_{\mathcal{P}^*}(\varsigma-\varsigma_0)$ is a bounded function representing the influence of $b(s)$.
Finally, applying Grönwall's inequality to handle the integral term & using the boundedness of $\mathcal{P}^*$, then:

$$|\Theta(\varsigma)-\Phi(\varsigma)|\le C\nu(\omega(\varsigma)+K),$$

here $C=(\theta+m)E_k\prod_{\varsigma_0<\varsigma_k<\varsigma}(1+M_kE_k)e^{\mathcal{P}^*(\varsigma_f-\varsigma_0)}$.
This completes proof. □

Corollary 3.1. *The adjoint NIHNID equation* (1.1) *with a neutral term satisfies GHUR stability on* $\mathfrak{I}$ *if* (C_1)–(C_6) *hold.*

Proof. For demonstration of Theorem 3.2, introduce the additional hypothesis (C_6). Using the bound $|b(\varsigma)| \leq M_b$ & the integral inequality in (C_6), it follows that the contributions of the neutral term remain bounded & the proof is adjusted analogously. Set $\nu = 1$, & the result follows as in the original proof. □

Corollary 3.2. *The adjoint NIHNID eqn* (1.1) *with a neutral term satisfies HU stability on* $\Im$ *if* (C_1)–(C_6) *meets.*

Proof. Choose $K = \omega(\varsigma) = 1$ & $\theta = \varsigma_f - \varsigma_0$ in the proof of Theorem 3.2. Using hypothesis (C_6), the neutral term's effect is bounded by $M_b\delta$, leading to

$$|\Theta(\varsigma) - \Phi(\varsigma)| \leq 2E_k\nu(\varsigma_f - \varsigma_0 + m)M_b\delta \prod_{\varsigma_0<\varsigma_k<\varsigma}(1 + M_kE_k)e^{\mathcal{P}^*(\varsigma_f-\varsigma_0)}.$$

This establishes the desired result. □

Corollary 3.3. *Under assumptions* $C1$*–*$C6$*, the adjoint NIHNID equation* (1.1)*, including the neutral term, is guaranteed to possess GHU-stability on* $\Im$*.*

Proof. Define $\zeta(\nu) = 2E_k\nu(\varsigma_f - \varsigma_0 + m)M_b\delta \prod_{\varsigma_0<\varsigma_k<\varsigma}(1 + M_kE_k)e^{\mathcal{P}^*(\varsigma_f-\varsigma_0)}$. Using Corollary 3.2, the conclusion is established. □

4. Examples

Example 4.1. *Assume the subsequent NIHNID adjoint neutral eqn:*

$$\begin{cases} \Phi^{\delta}(\varsigma) + \dfrac{1}{\varsigma - 3}\Phi^{\kappa}(\varsigma) - b(\varsigma)\Phi^{\kappa}(\varsigma - 1) = \dfrac{1}{2}\displaystyle\int_{\varsigma_0}^{\varsigma} e^{-10}(s + \cos(\Phi(s)))\delta s, \ \Phi(0) = 2, \\ \hfill \varsigma \in [0,5]_{\mathfrak{T}_{\Xi}}, \ \varsigma \neq 3, \\ \Upsilon_2(\Phi(3^-)) = \Phi(3^+) - \Phi(3^-) = \dfrac{3 + \cos(\Phi(3^-))}{7}, \ \varsigma = 3, \end{cases} \tag{4.1}$$

& its associated inequality:

$$\begin{aligned} &\left|\Theta^{\delta}(\varsigma) + \frac{1}{\varsigma - 3}\Theta^{\kappa}(\varsigma) - b(\varsigma)\Theta^{\kappa}(\varsigma - 1) - \frac{1}{2}\int_{\varsigma_0}^{\varsigma} e^{-10}(s + \cos(\Theta(s)))\delta s\right| \leq \nu, \\ &\left|\Upsilon_2(\Theta(3^-)) - \Theta(3^+) + \Theta(3^-)\right| \leq \nu. \end{aligned} \tag{4.2}$$

Let $\varsigma_0 = 0$, $\varsigma_f = 5$, $w(\varsigma) = \frac{1}{\varsigma-3}$, $f(\varsigma, \Phi(\varsigma)) = \frac{1}{2}$, $G(\varsigma, s) = e^{-10}$, $h(\varsigma, \Phi(\varsigma)) = \varsigma + \cos(\Phi(\varsigma))$ *for* $\varsigma \in \Im = [0,5]_{\mathfrak{T}_{\Xi}}$.
Assume that $b(\varsigma)$ *satisfies* (C_6)*, i.e.,* $|b(\varsigma)| \leq M_b$ *& the integral inequality holds. If* $\Theta \in PC_F{}^1([0,5]_{\mathfrak{T}_{\Xi}}, \Re)$ *fulfills* (4.2)*, then* $\exists\ g \in PC_F{}^1([0,5]_{\mathfrak{T}_{\Xi}}, \Re)$ *&* $g_0 \in \Re$ *s.t,* $|g(\varsigma)| \leq \nu$ *for* $\varsigma \in \Im = [0,5]_{\mathfrak{T}_{\Xi}}$ *&* $|g_0| \leq \nu$.

Thus, we get the following model:

$$\Theta^{\delta}(\varsigma)+\frac{1}{\varsigma-3}\Theta^{\kappa}(\varsigma)-b(\varsigma)\Theta^{\kappa}(\varsigma-1)=\frac{1}{2}\int_{\varsigma_0}^{\varsigma}e^{-10}(s+\cos(\Theta(s)))\delta s+g(\varsigma),\ \varsigma\in[0,5]_{\mathfrak{T}_{\mathfrak{S}}},\quad \varsigma\neq 3,$$

$$\Theta(0)=2,$$

$$\Upsilon_2(\Theta(3^-))=\Theta(3^+)-\Theta(3^-)+g_0,$$

& Eq. (4.1) *solution is:*

$$\Phi(\varsigma)=e_{\ominus w}(\varsigma,0)+e_{\ominus w}(\varsigma,3^-)\Upsilon_2(\Phi(3^-))+\int_{\varsigma_0}^{\varsigma}\frac{1}{2}e^{-10}e_{\ominus w}(\varsigma,s)\int_{\varsigma_0}^{s}(u+\cos(\Phi(u)))\delta u\delta s.$$

We can confirm that $h(\varsigma,\Phi(\varsigma))=\varsigma+\cos(\Phi(\varsigma))$ fulfills condition (C_1). *Additionally,* (C_2) *holds since the inequality $|\Upsilon_2(\Phi(3^-))-\Upsilon_2(\Theta(3^-))|\leq\frac{1}{7}|\Phi(3^-)-\Theta(3^-)|$ is satisfied. For $w=\frac{1}{\varsigma-3}$, the function $e_{\ominus w}(\varsigma,s)$ remains bounded, given that $|e_{\ominus w}(\varsigma,s)|\leq 1$, while f and G are constants, ensuring that* (C_3) *is met. The computations proceed as follows:*

$$|\Lambda(\Phi)-\Lambda(\Theta)|\leq\left(\frac{1}{7}+\frac{10}{2}e^{-10}\right)|\Phi-\Theta|.$$

Obviously, $\left(\frac{1}{7}+\frac{10}{2}e^{-10}\right)<1$, so ($C_4$) *fulfills for Eq.* (4.1). *By hypothesis* (C_6), *the contributions from $b(\varsigma)$ are bounded as well. Hence,* (C_1)–(C_6) *are fulfilled for Eq.* (4.1). *Therefore, by Theorem 3.1, Eq.* (4.1) *has only one solution in ${PC_F}^1([0,5]_{\mathfrak{T}_{\mathfrak{S}}},\Re)$.*
Moreover, choosing $\omega(\varsigma)=e^{\varsigma}$, which is both continuous and increasing, we observe that $\int_{\varsigma_0}^{\varsigma}e^r\delta r\leq\int_{\varsigma_0}^{\varsigma}e^r dr=e^{\varsigma}-e^{\varsigma_0}\leq e^{\varsigma}<16e^{\varsigma}$ for $\theta=16$. This confirms that $\int_{\varsigma_0}^{\varsigma}e^r\delta r<16e^{\varsigma}$, thereby satisfying ($C_5$). *Additionally, since $|\Phi(\varsigma)-\Theta(\varsigma)|\leq\omega(\varsigma)$ with parameters $C=1$, $\omega=e^{\cdot}$, and $K=0$, Theorem 3.2 ensures that Eq.* (4.1) *exhibits HUR stability under $(e^{\cdot},0)$ within the interval* $[0,5]\mathfrak{T}\mathfrak{S}$.

Remark 4.1. *The study's conclusions are validated by rigorous mathematical analysis. The results' depth and accuracy are emphasised by the use of sophisticated methods in stability theory and integro-dynamic equations. Furthermore, the framework created in this work reinforces the excellent calibre and importance of the study by offering a solid basis for further adaptive systems research.*

Conclusion

It can be examined the basic qualitative characteristics of adjoint NIHNID dynamical equations on finite time scales in this work, focussing on solution existence, uniqueness, and Ulam-type stability in the presence of neutral terms. Sufficient conditions guaranteeing stability are established by using the Picard iteration scheme in conjunction with Banach's contraction principle and an appropriate impulsive integral inequality tailored to time scales. It is demonstrated that the robustness linked to Ulam-type stability and the maintenance of bounded behaviour depend on the inclusion of neutral effects. The underlying fixed-point arguments in this framework naturally lead to solvability results. Additionally, Ulam-type stability highlights the model's applicability for adaptive systems that evolve across both discrete and continuous domains by offering a meaningful interpretation of its tolerance to minor disturbances. Overall, the proposed methodology offers a versatile and effective analytical tool for the study and control of hybrid dynamic systems.

Future Perspectives:

The application of larger classes of fractional operators, the creation of computational and optimal control strategies, and the examination of stochastic phenomena in adaptive systems could all be further developments of this research. Examining hybrid and multi-scale models would improve our comprehension of complex dynamics, and applying these results to practical issues like neural networks and population dynamics would confirm their importance. These paths present worthwhile chances to improve and broaden the study's theoretical and practical implications.

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