

On the Well-Posedness and Asymptotic Behavior of an Initial-Boundary Value Problem for a Heterogeneous Hot Standby Renewable Reliability System

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Abstract. In this research, we investigated the well-posedness and asymptotic behavior of an initial-boundary value problem (IBVP) associated with a heterogeneous hot-standby renewable reliability system. First, we established the well-posedness of the system model by leveraging C_0 -semigroup theory from functional analysis to ensure the existence and uniqueness of solutions. Second, by analyzing the spectral properties of the operator corresponding to the IBVP, we demonstrated that zero is an eigenvalue for the operator and its adjoint with geometric multiplicity one. Further, we showed that all non-zero points on the imaginary axis lie within the resolvent set of the operator. Based on these findings, we concluded that the time-dependent solution of the IBVP converges strongly to its steady-state solution as time approaches infinity.

1. INTRODUCTION

Reliability represents a significant challenge within the manufacturing industry. The inherent complexity of systems exacerbates the difficulties associated with their effective maintenance and operation^[1–5]. Owing to the high cost and unreliability of components, system analysts have encountered a variety of problems. Consequently, to enhance system reliability, it is crucial to develop effective modeling, monitoring, and control procedures.

However, in system reliability analysis and evaluation, several researchers (e.g., Hsu et al. [6], Ke et al. [7], Kumar and Jain [8], Wu et al. [9], Yang and Tsao [10]) have assumed that the failures of all system components are mutually independent. Under this assumption, the qualitative analysis and quantitative calculation of system reliability metrics often result in excessive errors and may even lead to incorrect conclusions. Therefore, it is necessary and significant to incorporate dependent failures into systems with redundant components.

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Redundancy techniques and maintenance through failed component repair can significantly enhance the performance of repairable systems. In reliability theory, three categories of standby components are widely employed: cold, warm, and hot standby. Cold standby is characterized by a zero failure rate when a component is in the standby state [11, 12]. Warm standby implies that an inactive component in the standby state can fail at a non-zero rate lower than that of an active component [13, 14]. Hot standby components have the same failure rate as those in operation [15–19]. In this study, we focus on a hot-standby system established by Rykov and Kozyrev [17]. In 2017, Rykov and Kozyrev [17] considered a heterogeneous hot double-redundant repairable reliability system. They assumed that the lifetimes of components are exponentially distributed random variables, the repair times of components have absolutely continuous distributions, and all lifetimes and repair times are independent. Under these assumptions, they established the system model using the supplementary variable method [20, 21] and subsequently analyzed the stationary and quasi-stationary probability distributions of the system under the condition that the time-dependent solution converges to its steady-state solution. However, the reliability model established by Rykov and Kozyrev using the supplementary variable method is described by partial differential equations (PDEs) with integral boundary conditions. As a result, obtaining exact solutions is difficult. Therefore, Rykov and Kozyrev [17] assumed that the time-dependent solution of the hot double-redundant system model converges to a steady-state solution; however, they did not report whether this assumption holds. Hence, we need to study the existence (well-posedness) and asymptotic behavior of the time-dependent solutions of the aforementioned reliability model. To achieve this, first, by selecting a suitable state space and introducing corresponding operators and their domains, we convert the heterogeneous hot double-redundant repairable reliability system model into an abstract Cauchy problem. Next, under certain conditions, we prove that the corresponding operator of the resulting Cauchy problem generates a positive contraction C_0 -semigroup. Further, we derive the well-posedness of the system described by the Cauchy problem. Finally, by studying the spectrum of the underlying operator on the imaginary axis, we conclude that the time-dependent solution of the Cauchy problem strongly converges to its steady-state solution.

Based on Rykov and Kozyrev's [17] work, the heterogeneous hot-standby renewable reliability system can be characterized through the following PDE system:

$$\frac{d\pi_0(t)}{dt} = -(\delta_1 + \delta_2)\pi_0(t) + \sum_{i=1}^3 \int_0^\infty \pi_i(t, y)\beta_i(y)dy, \quad (1.1)$$

$$\frac{\partial \pi_1(t, y)}{\partial t} + \frac{\partial \pi_1(t, y)}{\partial y} = -(\delta_2 + \beta_1(y))\pi_1(t, y), \quad (1.2)$$

$$\frac{\partial \pi_2(t, y)}{\partial t} + \frac{\partial \pi_2(t, y)}{\partial y} = -(\delta_1 + \beta_2(y))\pi_2(t, y), \quad (1.3)$$

$$\frac{\partial \pi_3(t, y)}{\partial t} + \frac{\partial \pi_3(t, y)}{\partial y} = -\beta_3(y)\pi_3(t, y), \quad (1.4)$$

$$\pi_1(t, 0) = \delta_1 \pi_0(t), \quad (1.5)$$

$$\pi_2(t, 0) = \delta_2 \pi_0(t), \quad (1.6)$$

$$\pi_3(t, 0) = \delta_1 \int_0^\infty \pi_2(t, y) dy + \delta_2 \int_0^\infty \pi_1(t, y) dy, \quad (1.7)$$

$$\pi_0(0) = 1, \pi_i(0, y) = 0, i = 1, 2, 3. \quad (1.8)$$

Here, $(t, y) \in [0, \infty) \times [0, \infty)$; $\pi_0(t)$ is the probability that both components in the system are working at time t , showing the chance of full system operation; $\pi_1(t, y)$ is the probability that the first component is under repair while the second is working at time t , with y reflecting the repair progress of the first component; $\pi_2(t, y)$ is the probability that the second component is being repaired and the first is working at time t , where y accounts for the repair status of the second component; and $\pi_3(t, y)$ is the probability that both components have failed, causing system breakdown at time t , with y related to repair - influenced failure factors. Further, $\beta_i(y) (i = 1, 2, 3)$ represents the repair rate of the system at state i , and $\delta_i (i = 1, 2)$ represents the failure rate of unit i .

2. REFORMULATION OF THE SYSTEM MODEL

In this section, we reformulate the heterogeneous double-redundant hot-standby renewable reliability system into an abstract Cauchy problem defined on an appropriately chosen Banach space. To accomplish this, we first define the state space Y as follows:

$$Y = \left\{ \pi \in \mathbb{R} \times \left(L^1[0, \infty) \right)^3 \left| \|\pi\| = |\pi_0| + \sum_{i=1}^3 \|\pi_i\|_{L^1[0, \infty)} < \infty \right. \right\},$$

which is clearly a Banach space. For convenience in our further analysis, we introduce

$$w = \begin{pmatrix} e^{-y} & 0 & 0 & 0 \\ \delta_1 e^{-y} & 0 & 0 & 0 \\ \delta_2 e^{-y} & 0 & 0 & 0 \\ 0 & \delta_2 & \delta_1 & 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix},$$

and define some operators as well as their domains as follows:

$$A_1 \pi = \begin{pmatrix} -(\delta_1 + \delta_2) & 0 & 0 & 0 \\ 0 & -\frac{d}{dy} - (\delta_2 + \beta_1(y)) & 0 & 0 \\ 0 & 0 & -\frac{d}{dy} - (\delta_1 + \beta_2(y)) & 0 \\ 0 & 0 & 0 & -\frac{d}{dy} - \beta_3(y) \end{pmatrix} \pi,$$

$$D(A_1) = \left\{ \pi \in Y \left| \begin{array}{l} \frac{d\pi_i}{dy} \in L^1[0, \infty), \pi_i(y) \text{ are absolutely continuous} \\ \text{functions and } \pi(0) = \int_0^\infty w\pi(y) dy, i = 1, 2, 3 \end{array} \right. \right\},$$

$$A_2\pi = \begin{pmatrix} \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \\ 0 \\ 0 \\ 0 \end{pmatrix}, D(A_2) = Y.$$

Subsequently, the system of Equations (1.1)-(1.8) can be formulated as an abstract Cauchy problem within the state space Y :

$$\begin{cases} \frac{d\pi(t)}{dt} = (A_1 + A_2) \pi(t), & \forall t \in [0, \infty), \\ \pi(0) = (1, 0, 0, 0). \end{cases} \quad (2.1)$$

3. WELL-POSEDNESS OF THE SYSTEM (2.1)

In this section, we first demonstrate that the operator $A_1 + A_2$ generates a positive contraction C_0 -semigroup $T(t)$ on the state space Y . Subsequently, we verify that the operator $A_1 + A_2$ is conservative, allowing us to conclude that the semigroup $T(t)$ is isometric. Finally, by combining these findings, we establish the well-posedness of System (2.1).

Theorem 3.1. *If $\overline{\beta_i} = \sup_{y \in [0, \infty)} \beta_i(y) < \infty$ ($i = 1, 2, 3$), then the operator $A_1 + A_2$ generates a positive contraction C_0 -semigroup $T(t)$.*

Proof. We prove the theorem via a four-step approach. First, we estimate the norm of $(\widetilde{\gamma}I - A_1)^{-1}$, Second, we demonstrate that the domain of the operator A_1 , denoted as $D(A_1)$, is dense in state space Y . This allows us to conclude, using the Hille-Yosida theorem, that the operator A_1 generates a C_0 -semigroup. Third, we check that A_2 is a bounded linear operator, and establish that $A_1 + A_2$ generates a C_0 -semigroup $T(t)$ by applying the perturbation theory of C_0 -semigroups. Finally, we show that $A_1 + A_2$ is dispersive and conclude that $T(t)$ is a positive contraction C_0 -semigroup by invoking the Phillips theorem.

For any given $v = (v_0, v_1(y), v_2(y), v_3(y)) \in Y$, consider the equation $(\widetilde{\gamma}I - A_1) \pi = v$, i.e.,

$$(\widetilde{\gamma} + \delta_1 + \delta_2) \pi_0 = v_0, \quad (3.1)$$

$$\frac{d\pi_1(y)}{dy} = -(\widetilde{\gamma} + \delta_2 + \beta_1(y)) \pi_1(y) + v_1(y), \quad (3.2)$$

$$\frac{d\pi_2(y)}{dy} = -(\widetilde{\gamma} + \delta_1 + \beta_2(y)) \pi_2(y) + v_2(y), \quad (3.3)$$

$$\frac{d\pi_3(y)}{dy} = -(\widetilde{\gamma} + \beta_3(y)) \pi_3(y) + v_3(y), \quad (3.4)$$

$$\pi_1(0) = \delta_1 \pi_0, \quad (3.5)$$

$$\pi_2(0) = \delta_2 \pi_0, \quad (3.6)$$

$$\pi_3(0) = \delta_1 \int_0^\infty \pi_2(y) dy + \delta_2 \int_0^\infty \pi_1(y) dy. \quad (3.7)$$

By solving (3.1)-(3.4), we deduce

$$\pi_0 = \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} v_0, \quad (3.8)$$

$$\pi_1(y) = \tau_1 e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} + e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \int_0^y v_1(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta, \quad (3.9)$$

$$\pi_2(y) = \tau_2 e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} + e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \int_0^y v_2(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta, \quad (3.10)$$

$$\pi_3(y) = e^{-\bar{\gamma}y - \int_0^y \beta_3(\xi) d\xi} \left(\tau_3 + \int_0^y v_3(\eta) e^{\bar{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} d\eta \right). \quad (3.11)$$

From (3.9)-(3.11), and (3.5)-(3.7), we calculate

$$\tau_1 = \pi_1(0) = \delta_1 \pi_0, \quad (3.12)$$

$$\tau_2 = \pi_2(0) = \delta_2 \pi_0, \quad (3.13)$$

$$\begin{aligned} \tau_3 &= \pi_3(0) = \delta_1 \int_0^\infty \pi_2(y) dy + \delta_2 \int_0^\infty \pi_1(y) dy \\ &= \delta_1 \int_0^\infty \left[\delta_2 \pi_0 e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \right. \\ &\quad \left. + e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \int_0^y v_2(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta \right] dy \\ &\quad + \delta_2 \int_0^\infty \left[\delta_1 \pi_0 e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \right. \\ &\quad \left. + e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \int_0^y v_1(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta \right] dy \\ &= \delta_1 \delta_2 \pi_0 \int_0^\infty e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} dy \\ &\quad + \delta_1 \int_0^\infty e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \int_0^y v_2(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta dy \\ &\quad + \delta_1 \delta_2 \pi_0 \int_0^\infty e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} dy \\ &\quad + \delta_2 \int_0^\infty e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \int_0^y v_1(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta dy. \end{aligned} \quad (3.14)$$

Based on (3.14), the Fubini theorem and the inequalities $e^{-\int_0^y \beta_i(\xi) d\xi} \leq 1$, $e^{-\int_\eta^y \beta_i(\xi) d\xi} \leq 1$ ($y \geq \eta \geq 0$, $i = 1, 2, 3$), we estimate

$$\begin{aligned} |\tau_3| &\leq \delta_1 \delta_2 |\pi_0| \int_0^\infty e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} dy \\ &\quad + \delta_1 \int_0^\infty e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \int_0^y |v_2(\eta)| e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta dy \\ &\quad + \delta_1 \delta_2 |\pi_0| \int_0^\infty e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} dy \end{aligned}$$

$$\begin{aligned}
& + \delta_2 \int_0^\infty e^{-(\tilde{\gamma}+\delta_2)y-\int_0^y \beta_1(\xi)d\xi} \int_0^y |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta+\int_0^\eta \beta_1(\xi)d\xi} d\eta dy \\
& \leq \delta_1 \delta_2 |\pi_0| \int_0^\infty e^{-(\tilde{\gamma}+\delta_1)y} dy \\
& + \delta_1 \int_0^\infty |v_2(\eta)| e^{(\tilde{\gamma}+\delta_1)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_1)y} \cdot e^{-\int_\eta^y \beta_2(\xi)d\xi} dy d\eta \\
& + \delta_1 \delta_2 |\pi_0| \int_0^\infty e^{-(\tilde{\gamma}+\delta_2)y} dy \\
& + \delta_2 \int_0^\infty |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_2)y} \cdot e^{-\int_\eta^y \beta_1(\xi)d\xi} dy d\eta \\
& \leq \frac{\delta_1 \delta_2}{\tilde{\gamma} + \delta_1} |\pi_0| + \delta_1 \int_0^\infty |v_2(\eta)| e^{(\tilde{\gamma}+\delta_1)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_1)y} dy d\eta \\
& + \frac{\delta_1 \delta_2}{\tilde{\gamma} + \delta_2} |\pi_0| + \delta_2 \int_0^\infty |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_2)y} dy d\eta \\
& = \frac{\delta_1 \delta_2}{\tilde{\gamma} + \delta_1} |\pi_0| + \frac{\delta_1}{\tilde{\gamma} + \delta_1} \|v_2\|_{L^1[0,\infty)} \\
& + \frac{\delta_1 \delta_2}{\tilde{\gamma} + \delta_2} |\pi_0| + \frac{\delta_2}{\tilde{\gamma} + \delta_2} \|v_1\|_{L^1[0,\infty)} \\
& = \frac{\delta_1 \delta_2 (2\tilde{\gamma} + \delta_1 + \delta_2)}{(\tilde{\gamma} + \delta_1)(\tilde{\gamma} + \delta_2)(\tilde{\gamma} + \delta_1 + \delta_2)} |v_0| \\
& + \frac{\delta_2}{\tilde{\gamma} + \delta_2} \|v_1\|_{L^1[0,\infty)} + \frac{\delta_1}{\tilde{\gamma} + \delta_1} \|v_2\|_{L^1[0,\infty)}. \tag{3.15}
\end{aligned}$$

Similarly, using the aforementioned expressions (3.9)-(3.15), and the inequalities $e^{-\int_0^y \beta_i(\xi)d\xi} \leq 1$, $e^{-\int_\eta^y \beta_i(\xi)d\xi} \leq 1$ ($y \geq \eta \geq 0$, $i = 1, 2, 3$), we further estimate $\|\pi_i\|_{L^1[0,\infty)}$ ($i = 1, 2, 3$) as follows

$$\begin{aligned}
\|\pi_1\|_{L^1[0,\infty)} & \leq \int_0^\infty |\tau_1| e^{-(\tilde{\gamma}+\delta_2)y-\int_0^y \beta_1(\xi)d\xi} dy \\
& + \int_0^\infty e^{-(\tilde{\gamma}+\delta_2)y-\int_0^y \beta_1(\xi)d\xi} \int_0^y |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta+\int_0^\eta \beta_1(\xi)d\xi} d\eta dy \\
& \leq |\tau_1| \int_0^\infty e^{-(\tilde{\gamma}+\delta_2)y} dy + \int_0^\infty |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta+\int_0^\eta \beta_1(\xi)d\xi} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_2)y-\int_0^y \beta_1(\xi)d\xi} dy d\eta \\
& = \frac{|\tau_1|}{\tilde{\gamma} + \delta_2} + \int_0^\infty |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_2)y} \cdot e^{-\int_\eta^y \beta_1(\xi)d\xi} dy d\eta \\
& \leq \frac{\delta_1}{\tilde{\gamma} + \delta_2} |\pi_0| + \int_0^\infty |v_1(\eta)| e^{(\tilde{\gamma}+\delta_2)\eta} \int_\eta^\infty e^{-(\tilde{\gamma}+\delta_2)y} dy d\eta \\
& = \frac{\delta_1}{\tilde{\gamma} + \delta_2} |\pi_0| + \frac{1}{\tilde{\gamma} + \delta_2} \int_0^\infty |v_1(\eta)| d\eta \\
& = \frac{\delta_1}{(\tilde{\gamma} + \delta_2)(\tilde{\gamma} + \delta_1 + \delta_2)} |v_0| + \frac{1}{\tilde{\gamma} + \delta_2} \|v_1\|_{L^1[0,\infty)}, \tag{3.16}
\end{aligned}$$

$$\|\pi_2\|_{L^1[0,\infty)} \leq \frac{\delta_2}{(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| + \frac{1}{\widetilde{\gamma} + \delta_1} \|v_2\|_{L^1[0,\infty)}, \quad (3.17)$$

$$\begin{aligned} \|\pi_3\|_{L^1[0,\infty)} &\leq |\tau_3| \int_0^\infty e^{-\widetilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi} dy + \int_0^\infty e^{-\widetilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi} \int_0^y |v_3(\eta)| e^{\widetilde{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} d\eta dy \\ &\leq |\tau_3| \int_0^\infty e^{-\widetilde{\gamma}y} dy + \int_0^\infty |v_3(\eta)| e^{\widetilde{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} \int_\eta^\infty e^{-\widetilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi} dy d\eta \\ &\leq \frac{1}{\widetilde{\gamma}} |\tau_3| + \frac{1}{\widetilde{\gamma}} \|v_3\|_{L^1[0,\infty)} \\ &\leq \frac{\delta_1 \delta_2 (2\widetilde{\gamma} + \delta_1 + \delta_2)}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| + \frac{\delta_2}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_2)} \|v_1\|_{L^1[0,\infty)} \\ &\quad + \frac{\delta_1}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)} \|v_2\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}} \|v_3\|_{L^1[0,\infty)}. \end{aligned} \quad (3.18)$$

From (3.8) and (3.16)-(3.18), we obtain

$$\begin{aligned} \|\pi\| &= |\pi_0| + \sum_{i=1}^3 \|\pi_i\|_{L^1[0,\infty)} \\ &\leq \frac{1}{\widetilde{\gamma} + \delta_1 + \delta_2} |v_0| + \frac{\delta_1}{(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| \\ &\quad + \frac{1}{\widetilde{\gamma} + \delta_2} \|v_1\|_{L^1[0,\infty)} + \frac{\delta_2}{(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| \\ &\quad + \frac{1}{\widetilde{\gamma} + \delta_1} \|v_2\|_{L^1[0,\infty)} + \frac{\delta_1 \delta_2 (2\widetilde{\gamma} + \delta_1 + \delta_2)}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| \\ &\quad + \frac{\delta_2}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_2)} \|v_1\|_{L^1[0,\infty)} + \frac{\delta_1}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)} \|v_2\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}} \|v_3\|_{L^1[0,\infty)} \\ &= \left[\frac{1}{\widetilde{\gamma} + \delta_1 + \delta_2} + \frac{\delta_2}{(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_1 + \delta_2)} + \frac{\delta_1}{(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} \right. \\ &\quad \left. + \frac{\delta_1 \delta_2 (2\widetilde{\gamma} + \delta_1 + \delta_2)}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} \right] |v_0| \\ &\quad + \left[\frac{1}{\widetilde{\gamma} + \delta_2} + \frac{\delta_2}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_2)} \right] \|v_1\|_{L^1[0,\infty)} \\ &\quad + \left[\frac{1}{\widetilde{\gamma} + \delta_1} + \frac{\delta_1}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)} \right] \|v_2\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}} \|v_3\|_{L^1[0,\infty)} \\ &= \frac{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_2) + \delta_2 \widetilde{\gamma}(\widetilde{\gamma} + \delta_2) + \delta_1 \widetilde{\gamma}(\widetilde{\gamma} + \delta_1) + \delta_1 \delta_2 (2\widetilde{\gamma} + \delta_1 + \delta_2)}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)(\widetilde{\gamma} + \delta_2)(\widetilde{\gamma} + \delta_1 + \delta_2)} |v_0| \\ &\quad + \frac{\widetilde{\gamma} + \delta_2}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_2)} \|v_1\|_{L^1[0,\infty)} \\ &\quad + \frac{\widetilde{\gamma} + \delta_1}{\widetilde{\gamma}(\widetilde{\gamma} + \delta_1)} \|v_2\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}} \|v_3\|_{L^1[0,\infty)} \\ &\leq \frac{1}{\widetilde{\gamma} - \delta_1 - \delta_2}. \end{aligned} \quad (3.19)$$

(3.19) shows, that the norm of $(\tilde{\gamma}I - A_1)^{-1}$ exists, when $\tilde{\gamma} > \delta_1 + \delta_2$ and

$$(\tilde{\gamma}I - A_1)^{-1} : Y \rightarrow D(A_1), \|(\tilde{\gamma}I - A_1)^{-1}\| \leq \frac{1}{\tilde{\gamma} - (\delta_1 + \delta_2)}.$$

Next, we prove $D(A_1)$ is dense in Y . If we define

$$\Theta = \left\{ \pi(y) = (\pi_0, \pi_1(y), \pi_2(y), \pi_3(y)) \mid \pi_0 \in \mathbb{R}, \pi_i \in C_0^\infty[0, \infty), i = 1, 2, 3 \right\},$$

then, we get that Θ is dense in $Y^{[22]}$. If we define

$$\bar{\Theta} = \left\{ \pi(y) = (\pi_0, \pi_1(y), \pi_2(y), \pi_3(y)) \mid \begin{array}{l} \pi_i \in C_0^\infty[0, \infty) \text{ and there exist} \\ \text{numbers } \kappa_i > 0, \text{ such that} \\ \pi_i(y) = 0, \text{ for } y \in [0, \kappa_i], i = 1, 2, 3 \end{array} \right\},$$

then the set $\bar{\Theta}$ is dense in Θ . Hence, to show that $D(A_1)$ is dense in Y , it suffices to show that $D(A_1)$ is dense in $\bar{\Theta}$. Based on the definition of $\bar{\Theta}$, $\exists \kappa_i > 0$, such that $\pi_i(y) = 0$ ($i = 1, 2, 3$), for $y \in [0, \kappa_i]$ ($i = 1, 2, 3$), which means that if we take $0 < \kappa = \min_{i=1,2,3} \kappa_i$, then it follows that $\pi_i(y) = 0$ ($i = 1, 2, 3$), for $y \in [0, \kappa]$. From which we can introduce

$$\begin{aligned} \ell^\kappa(0) &= (\pi_0, \ell_1^\kappa(0), \ell_2^\kappa(0), \ell_3^\kappa(0)) \\ &= \left(\pi_0, \delta_1 \pi_0, \delta_2 \pi_0, \delta_1 \int_\kappa^\infty \pi_2(y) dy + \delta_2 \int_\kappa^\infty \pi_1(y) dy \right), \\ \ell^\kappa(y) &= (\ell_0^\kappa, \ell_1^\kappa(y), \ell_2^\kappa(y), \ell_3^\kappa(y)), \end{aligned}$$

where

$$\ell_i^\kappa(y) = \begin{cases} \ell_i^\kappa(0) \left(1 - \frac{y}{\kappa}\right)^2 & \text{if } y \in [0, \kappa), \\ \pi_i(y) & \text{if } y \in [\kappa, \infty), \end{cases} \quad i = 1, 2, 3,$$

then, it is easy to check that $\ell^\kappa(y) \in D(A_1)$. Moreover,

$$\begin{aligned} \|\pi - \ell^\kappa\| &= \sum_{i=1}^3 \int_0^\infty |\pi_i(y) - \ell_i^\kappa(y)| dy = \sum_{i=1}^3 \int_0^\kappa |\pi_i(y) - \ell_i^\kappa(y)| dy \\ &= \sum_{i=1}^3 \int_0^\kappa |\ell_i^\kappa(0)| \left(1 - \frac{y}{\kappa}\right)^2 dy = \frac{\kappa}{3} \sum_{i=1}^3 |\ell_i^\kappa(0)| \rightarrow 0, \kappa \rightarrow 0. \end{aligned}$$

The aforementioned equation demonstrates that $D(A_1)$ is dense in $\bar{\Theta}$, which means $D(A_1)$ is dense in Y . Drawing upon the outcomes from the first and second steps, along with the application of the Hille-Yosida theorem^[23], we can conclude that A_1 generates a C_0 -semigroup.

In the following, we verify that operator A_2 is a bounded linear operator. By the definition of A_2 , for any given $\pi \in Y$, we derive

$$\|A_2 \pi\| \leq \sum_{i=1}^3 \int_0^\infty |\pi_i(y)| |\beta_i(y)| dy \leq \max_{i=1,2,3} \bar{\beta}_i \sum_{i=1}^3 \|\pi_i\|_{L^1[0, \infty)} \leq \max_{i=1,2,3} \bar{\beta}_i \|\pi\|.$$

The aforementioned expression shows that A_2 is a bounded operator. It is easy to see that A_2 is also linear, therefore, we can use the perturbation theorem of C_0 -semigroup^[23], and obtain that the $A_1 + A_2$ generates a C_0 -semigroup $T(t)$.

Finally, we prove that $A_1 + A_2$ is a dispersive operator. For $\forall \pi \in D(A_1)$, we consider

$$\theta = \left(\frac{[\pi_0]^+}{\pi_0}, \frac{[\pi_1(y)]^+}{\pi_1(y)}, \frac{[\pi_2(y)]^+}{\pi_2(y)}, \frac{[\pi_3(y)]^+}{\pi_3(y)} \right)$$

where

$$[\pi_0]^+ = \begin{cases} \pi_0 & \text{if } \pi_0 > 0 \\ 0 & \text{if } \pi_0 \leq 0 \end{cases}, [\pi_i(y)]^+ = \begin{cases} \pi_i(y) & \text{if } \pi_i(y) > 0 \\ 0 & \text{if } \pi_i(y) \leq 0 \end{cases}, i = 1, 2, 3.$$

If we define $z_i^+ = \{y \in (0, \infty) \mid \pi_i(y) > 0\}$, $z_i^- = \{y \in [0, \infty) \mid \pi_i(y) \leq 0\}$, then we get

$$\begin{aligned} \int_0^\infty \frac{d\pi_i(y)}{dy} \cdot \frac{[\pi_i(y)]^+}{\pi_i(y)} dy &= \int_{z_i^+} \frac{d\pi_i(y)}{dy} \frac{[\pi_i(y)]^+}{\pi_i(y)} dy + \int_{z_i^-} \frac{d\pi_i(y)}{dy} \frac{[\pi_i(y)]^+}{\pi_i(y)} dy \\ &= \int_{z_i^+} \frac{d\pi_i(y)}{dy} = \int_0^\infty \frac{d[\pi_i(y)]^+}{dy} = -[\pi_i(0)]^+, i = 1, 2, 3. \end{aligned} \quad (3.20)$$

By Equation (3.20), inequalities $\int_0^\infty \beta_i(y) \pi_i(y) dy \leq \int_0^\infty \beta_i(y) [\pi_i(y)]^+ dy$ ($i = 1, 2, 3$) and the boundary conditions, we calculate

$$\begin{aligned} &\langle (A_1 + A_2) \pi, \theta \rangle \\ &= \left[-(\delta_1 + \delta_2) \pi_0 + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \right] \cdot \frac{[\pi_0]^+}{\pi_0} \\ &\quad + \int_0^\infty \left[-\frac{d\pi_1(y)}{dy} - (\delta_2 + \beta_1(y)) \pi_1(y) \right] \cdot \frac{[\pi_1(y)]^+}{\pi_1(y)} dy \\ &\quad + \int_0^\infty \left[-\frac{d\pi_2(y)}{dy} - (\delta_1 + \beta_2(y)) \pi_2(y) \right] \cdot \frac{[\pi_2(y)]^+}{\pi_2(y)} dy \\ &\quad + \int_0^\infty \left[-\frac{d\pi_3(y)}{dy} - \beta_3(y) \pi_3(y) \right] \cdot \frac{[\pi_3(y)]^+}{\pi_3(y)} dy \\ &= -(\delta_1 + \delta_2) [\pi_0]^+ + \frac{[\pi_0]^+}{\pi_0} \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \\ &\quad - \sum_{i=1}^3 \int_0^\infty \frac{d\pi_i(y)}{dy} \frac{[\pi_i(y)]^+}{\pi_i(y)} dy - \int_0^\infty (\delta_2 + \beta_1(y)) [\pi_1(y)]^+ dy \\ &\quad - \int_0^\infty (\delta_1 + \beta_2(y)) [\pi_2(y)]^+ dy - \int_0^\infty \beta_3(y) [\pi_3(y)]^+ dy \\ &= -(\delta_1 + \delta_2) [\pi_0]^+ + \frac{[\pi_0]^+}{\pi_0} \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \\ &\quad + \sum_{i=1}^3 [\pi_i(0)]^+ - \delta_1 \int_0^\infty [\pi_2(y)]^+ dy - \delta_2 \int_0^\infty [\pi_1(y)]^+ dy \\ &\quad - \sum_{i=1}^3 \int_0^\infty \beta_i(y) [\pi_i(y)]^+ dy \end{aligned}$$

$$\begin{aligned}
&= -(\delta_1 + \delta_2) [\pi_0]^+ + \frac{[\pi_0]^+}{\pi_0} \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy + (\delta_1 + \delta_2) [\pi_0]^+ \\
&+ \delta_1 \int_0^\infty [\pi_2(y)]^+ dy + \delta_2 \int_0^\infty [\pi_1(y)]^+ dy \\
&- \delta_1 \int_0^\infty [\pi_2(y)]^+ dy - \delta_2 \int_0^\infty [\pi_1(y)]^+ dy \\
&- \sum_{i=1}^3 \int_0^\infty [\pi_i(y)]^+ \beta_i(y) dy \\
&\leq \frac{[\pi_0]^+}{\pi_0} \sum_{i=1}^3 \int_0^\infty [\pi_i(y)]^+ \beta_i(y) dy - \sum_{i=1}^3 \int_0^\infty [\pi_i(y)]^+ \beta_i(y) dy \\
&= \left(\frac{[\pi_0]^+}{\pi_0} - 1 \right) \sum_{i=1}^3 \int_0^\infty [\pi_i(y)]^+ \beta_i(y) dy \leq 0.
\end{aligned}$$

The aforementioned result shows that $A_1 + A_2$ is dispersive. Hence, from the fourth step, together with the first and second steps, we obtain that $A_1 + A_2$ generates a positive contraction C_0 -semigroup. By the uniqueness theorem of C_0 -semigroup^[23], we know that this nonnegative contraction C_0 -semigroup is just $T(t)$. Thus, we complete the proof of Theorem 3.1. $\square$

Based on the definition of the state space Y , we easily get the dual space of Y as follows

$$Y^* = \left\{ \pi^* \in \mathbb{R} \times (L^\infty[0, \infty))^3 \mid \|\pi^*\| = \sup \left\{ |\pi_0|, \sup_{i=1,2,3} \|\pi_i^*\|_{L^\infty[0, \infty)} \right\} < \infty \right\}.$$

Clearly, Y^* is a Banach space. If we consider a set J in Y as

$$J = \{ \pi \in Y \mid \pi_0 \geq 0, \pi_i(y) \geq 0, y \in [0, \infty), i = 1, 2, 3 \},$$

then J is a cone in Y . For any $\pi \in D(A_1) \cap J$, we consider $\pi^* = \|\pi\|(1, 1, 1, 1) \in Y^*$. Thus, using the boundary conditions, we obtain

$$\begin{aligned}
&\langle (A_1 + A_2) \pi, \pi^* \rangle \\
&= \left[-(\delta_1 + \delta_2) \pi_0 + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \right] \|\pi\| \\
&+ \int_0^\infty \left\{ -\frac{d\pi_1(y)}{dy} - (\delta_2 + \beta_1(y)) \pi_1(y) \right\} \|\pi\| dy \\
&+ \int_0^\infty \left\{ -\frac{d\pi_2(y)}{dy} - (\delta_1 + \beta_2(y)) \pi_2(y) \right\} \|\pi\| dy \\
&+ \int_0^\infty \left\{ -\frac{d\pi_3(y)}{dy} - \beta_3(y) \right\} \|\pi\| dy \\
&= -(\delta_1 + \delta_2) \pi_0 \|\pi\| - \|\pi\| \sum_{i=1}^3 \int_0^\infty \frac{d\pi_i(y)}{dy} dy
\end{aligned}$$

$$\begin{aligned}
& -\delta_2 \|\pi\| \int_0^\infty \pi_1(y) dy - \delta_1 \|\pi\| \int_0^\infty \pi_2(y) dy \\
& = -(\delta_1 + \delta_2) \pi_0 \|\pi\| + \|\pi\| \sum_{i=1}^3 \pi_i(0) \\
& \quad -\delta_2 \|\pi\| \int_0^\infty \pi_1(y) dy - \delta_1 \|\pi\| \int_0^\infty \pi_2(y) dy \\
& = -(\delta_1 + \delta_2) \pi_0 \|\pi\| + (\delta_1 + \delta_2) \|\pi\| \pi_0 \\
& \quad + \delta_1 \|\pi\| \int_0^\infty \pi_2(y) dy + \delta_2 \|\pi\| \int_0^\infty \pi_1(y) dy \\
& \quad -\delta_2 \|\pi\| \int_0^\infty \pi_1(y) dy - \delta_1 \|\pi\| \int_0^\infty \pi_2(y) dy = 0.
\end{aligned}$$

This result shows that $A_1 + A_2$ is a conservative operator. Because the initial value $\pi(0) \in D(A_1^2) \cap J$, using Fattorini theorem^[24], we obtain the following results.

Theorem 3.2. $T(t)$ is isometric for the initial value of System (2.1), which is

$$\|T(t)\pi(0)\| = \|\pi(0)\|, \quad \forall t \in [0, \infty). \quad (3.21)$$

From Theorems 3.1 and 3.2, we obtain the well-posedness of the System (2.1).

Theorem 3.3. If $\overline{\beta_i} = \sup_{y \in [0, \infty)} \beta_i(y) < \infty$ ($i = 1, 2, 3$), then the System (2.1) has a unique nonnegative time-dependent solution $\pi(t, y)$, that satisfies

$$\|\pi(t, \cdot)\| = 1, \quad \forall t \in (0, \infty). \quad (3.22)$$

Proof. Because $\pi(0) \in D(A_1^2) \cap J$, from Theorem 3.1 and Theorem 1.81 in Gupur [23], we know that System (2.1) has a unique nonnegative time-dependent solution $\pi(t, y)$ which can be expressed as

$$\pi(t, y) = T(t)\pi(0), \quad \forall t \in (0, \infty).$$

From the above equation, together with the Theorem 3.2, we obtain

$$\|\pi(t, \cdot)\| = \|T(t)\pi(0)\| = \|\pi(0)\| = \|(1, 0, 0, 0)\| = 1, \quad \forall t \in [0, \infty). \quad (3.23)$$

□

(3.23) just reflects the physical meaning of the $\pi(t, y)$.

4. SPECTRAL PROPERTIES OF THE UNDERLYING OPERATOR GIVEN BY (2.1)

According to Theorem 1.96 in reference [23], to ascertain the asymptotic behavior of the time-dependent solution for System (2.1), it is necessary to confirm that zero serves as an eigenvalue for both the operator $A_1 + A_2$ and its adjoint operator, each with a geometric multiplicity of one, and to understand the spectral distribution of $A_1 + A_2$ along the imaginary axis. Utilizing Greiner's boundary perturbation concept in [25], Haji and Radl [26] determined the resolvent set on the

imaginary axis for the operator related to a queueing model defined by a finite number of PDEs, and Gupur [27] later extended this to PDEs with nonconstant parameters. Thus, by referring to the methods in [26,27], we study the resolvent set of $A_1 + A_2$ on the imaginary axis and use the results to find the asymptotic behavior of the system's time-dependent solution. Thus, we discuss the spectral properties of the operator of the heterogeneous hot standby renewable reliability model.

Lemma 4.1. *Zero is an eigenvalue of $A_1 + A_2$ with geometric multiplicity one.*

Proof. Consider the equation $(A_1 + A_2) \pi = 0$, which is equivalent to

$$(\delta_1 + \delta_2) \pi_0 = \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy, \quad (4.1)$$

$$\frac{d\pi_1(y)}{dy} = -(\delta_2 + \beta_1(y)) \pi_1(y), \quad (4.2)$$

$$\frac{d\pi_2(y)}{dy} = -(\delta_1 + \beta_2(y)) \pi_2(y), \quad (4.3)$$

$$\frac{d\pi_3(y)}{dy} = -\beta_3(y) \pi_3(y), \quad (4.4)$$

$$\pi_1(0) = \delta_1 \pi_0, \quad (4.5)$$

$$\pi_2(0) = \delta_2 \pi_0, \quad (4.6)$$

$$\pi_3(0) = \delta_1 \int_0^\infty \pi_2(y) dy + \delta_2 \int_0^\infty \pi_1(y) dy. \quad (4.7)$$

By solving (4.2)-(4.4) we obtain

$$\pi_1(y) = j_1 e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi}, \quad (4.8)$$

$$\pi_2(y) = j_2 e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi}, \quad (4.9)$$

$$\pi_3(y) = j_3 e^{-\int_0^y \beta_3(\xi) d\xi}. \quad (4.10)$$

By combining (4.8)-(4.9) with (4.5)-(4.6), and employing the equation $j_i = \pi_i(0)$ ($i = 1, 2, 3$), we obtain

$$\pi_1(y) = \delta_1 \pi_0 e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi}, \quad (4.11)$$

$$\pi_2(y) = \delta_2 \pi_0 e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi}. \quad (4.12)$$

By inserting (4.11) and (4.12) into (4.7), we obtain

$$\begin{aligned} j_3 = \pi_3(0) &= \delta_1 \int_0^\infty \pi_2(y) dy + \delta_2 \int_0^\infty \pi_1(y) dy \\ &= \delta_1 \delta_2 \pi_0 \left(\int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \right), \end{aligned} \quad (4.13)$$

which imply $\pi_3(y)$ can be written as follows from (4.10)

$$\pi_3(y) = \delta_1 \delta_2 \pi_0 \left(\int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \right) e^{-\int_0^y \beta_2(\xi) d\xi}. \quad (4.14)$$

(4.11), (4.12), and (4.14) yield

$$\begin{aligned} \|\pi\| &= |\pi_0| + \sum_{i=1}^3 \|\pi_i\|_{L^1[0,\infty)} \\ &= |\pi_0| + \delta_1 |\pi_0| \int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \delta_2 |\pi_0| \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \\ &\quad + \delta_1 \delta_2 |\pi_0| \left(\int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \right) \times \int_0^\infty e^{-\int_0^y \beta_2(\xi) d\xi} dy \\ &= \left[1 + \delta_1 \int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \delta_2 \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \right. \\ &\quad \left. + \delta_1 \delta_2 \left(\int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} dy + \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} dy \right) \times \int_0^\infty e^{-\int_0^y \beta_2(\xi) d\xi} dy \right] |\pi_0| < \infty. \end{aligned}$$

This shows that zero is an eigenvalue of $A_1 + A_2$. Moreover, (4.11), (4.12), and (4.14) indicate that the eigenvectors corresponding to zero span a one dimensional linear space, i.e., the geometric multiplicity of zero is one. $\square$

Lemma 4.2. $(A_1 + A_2)^*$, the adjoint operator of $A_1 + A_2$ is given by

$$\begin{aligned} (A_1 + A_2)^* \begin{pmatrix} \pi_0^* \\ \pi_1^*(y) \\ \pi_2^*(y) \\ \pi_3^*(y) \end{pmatrix} &= \begin{pmatrix} 0 & \delta_1 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & 0 & \delta_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_0^* \\ \pi_1^*(0) \\ \pi_2^*(0) \\ \pi_3^*(0) \end{pmatrix} \\ &+ \begin{pmatrix} -(\delta_1 + \delta_2) & 0 & 0 & 0 \\ \beta_1(y) & \frac{d}{dy} - (\beta_1(y) + \delta_2) & 0 & 0 \\ \beta_2(y) & 0 & \frac{d}{dy} - (\beta_2(y) + \delta_1) & 0 \\ \beta_3(y) & 0 & 0 & \frac{d}{dy} - \beta_3(y) \end{pmatrix} \begin{pmatrix} \pi_0^* \\ \pi_1^*(y) \\ \pi_2^*(y) \\ \pi_3^*(y) \end{pmatrix}, \\ D((A_1 + A_2)^*) &= \left\{ \pi^* \in Y^* \left| \frac{d\pi_i^*(y)}{dy} \text{ exist and } \pi_i^*(\infty) = M, i = 1, 2, 3 \right. \right\}. \end{aligned}$$

Here, M is a constant independent of the variable i .

Proof. For $\pi^* \in D((A_1 + A_2)^*)$, using integration by parts and the boundary conditions on $\pi \in D(A_1 + A_2)$, we obtain

$$\begin{aligned} \langle (A_1 + A_2) \pi, \pi^* \rangle &= \left[-(\delta_1 + \delta_2) \pi_0 + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \right] \pi_0^* \end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \left[-\frac{d\pi_1(y)}{dy} - (\delta_2 + \beta_1(y)) \pi_1(y) \right] \pi_1^*(y) dy \\
& + \int_0^\infty \left[-\frac{d\pi_2(y)}{dy} - (\delta_1 + \beta_2(y)) \pi_2(y) \right] \pi_2^*(y) dy \\
& + \int_0^\infty \left[-\frac{d\pi_3(y)}{dy} - \beta_3(y) \pi_3(y) \right] \pi_3^*(y) dy \\
& = \left[-(\delta_1 + \delta_2) \pi_0 + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy \right] \pi_0^* \\
& \quad - \sum_{i=1}^3 \int_0^\infty \frac{d\pi_i(y)}{dy} \pi_i^*(y) dy - \delta_2 \int_0^\infty \pi_1(y) \pi_1^*(y) dy \\
& \quad - \delta_1 \int_0^\infty \pi_2(y) \pi_2^*(y) dy - \sum_{i=1}^3 \int_0^\infty \beta_i(y) \pi_i(y) \pi_i^*(y) dy \\
& = \pi_0 \left[-(\delta_1 + \delta_2) \pi_0^* \right] + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \left[\beta_i(y) (\pi_0^* - \pi_i^*(y)) \right] dy \\
& \quad + \sum_{i=1}^3 \pi_i(0) \pi_i^*(0) + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \frac{d\pi_i^*(y)}{dy} dy \\
& \quad - \delta_2 \int_0^\infty \pi_1(y) \pi_1^*(y) dy - \delta_1 \int_0^\infty \pi_2(y) \pi_2^*(y) dy \\
& = \pi_0 \left[-(\delta_1 + \delta_2) \pi_0^* \right] + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \left[\beta_i(y) (\pi_0^* - \pi_i^*(y)) \right] dy \\
& \quad + \sum_{i=1}^2 \delta_i \pi_0 \pi_i^*(0) + \left[\delta_1 \int_0^\infty \pi_2(y) dy + \delta_2 \int_0^\infty \pi_1(y) dy \right] \pi_3^*(0) \\
& \quad + \sum_{i=1}^3 \int_0^\infty \pi_i(y) \frac{d\pi_i^*(y)}{dy} dy - \delta_2 \int_0^\infty \pi_1(y) \pi_1^*(y) dy - \delta_1 \int_0^\infty \pi_2(y) \pi_2^*(y) dy \\
& = \pi_0 \left[-(\delta_1 + \delta_2) \pi_0^* + \sum_{i=1}^2 \delta_i \pi_i^*(0) \right] \\
& \quad + \int_0^\infty \pi_1(y) \left[\beta_1(y) (\pi_0^* - \pi_1^*(y)) + \delta_2 \pi_3^*(0) + \frac{d\pi_1^*(y)}{dy} - \delta_2 \pi_1^*(y) \right] dy \\
& \quad + \int_0^\infty \pi_2(y) \left[\beta_2(y) (\pi_0^* - \pi_2^*(y)) + \delta_1 \pi_3^*(0) + \frac{d\pi_2^*(y)}{dy} - \delta_1 \pi_2^*(y) \right] dy \\
& \quad + \int_0^\infty \pi_3(y) \left[\beta_3(y) (\pi_0^* - \pi_3^*(y)) + \frac{d\pi_3^*(y)}{dy} \right] dy \\
& = \pi_0 \left[-(\delta_1 + \delta_2) \pi_0^* + \sum_{i=1}^2 \delta_i \pi_i^*(0) \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \pi_1(y) \left\{ \beta_1(y) \pi_0^* + \left[\frac{d}{dy} - (\beta_1(y) + \delta_2) \right] \pi_1^*(y) + \delta_2 \pi_3^*(0) \right\} dy \\
& + \int_0^\infty \pi_2(y) \left\{ \beta_2(y) \pi_0^* + \left[\frac{d}{dy} - (\beta_2(y) + \delta_1) \right] \pi_2^*(y) + \delta_1 \pi_3^*(0) \right\} dy \\
& + \int_0^\infty \pi_3(y) \left\{ \beta_3(y) \pi_0^* + \left(\frac{d}{dy} - \beta_3(y) \right) \pi_3^*(y) \right\} dy \\
& = \langle \pi, (A_1 + A_2)^* \pi^* \rangle.
\end{aligned}$$

From the above result, together with the definition of the adjoint operator, we know that the assertion of this lemma is true. $\square$

Lemma 4.3. *Zero is an eigenvalue of $(A_1 + A_2)^*$ with geometric multiplicity one.*

Proof. We consider the equation $(A_1 + A_2)^* \pi^* = 0$, which is equivalent to

$$-(\delta_1 + \delta_2) \pi_0^* + \delta_1 \pi_1^*(0) + \delta_2 \pi_2^*(0) = 0, \quad (4.15)$$

$$\frac{d\pi_1^*(y)}{dy} = (\beta_1(y) + \delta_2) \pi_1^*(y) - \beta_1(y) \pi_0^* - \delta_2 \pi_3^*(0), \quad (4.16)$$

$$\frac{d\pi_2^*(y)}{dy} = (\beta_2(y) + \delta_1) \pi_2^*(y) - \beta_2(y) \pi_0^* - \delta_1 \pi_3^*(0), \quad (4.17)$$

$$\frac{d\pi_3^*(y)}{dy} = \beta_3(y) \pi_3^*(y) - \beta_3(y) \pi_0^*, \quad (4.18)$$

$$\pi_i^*(\infty) = M, \quad i = 1, 2, 3. \quad (4.19)$$

By (4.16)-(4.18), we get

$$\pi_1^*(y) = \tau_1 e^{\delta_2 y + \int_0^y \beta_1(\eta) d\eta} - e^{\delta_2 y + \int_0^y \beta_1(\eta) d\eta} \int_0^y [\beta_1(\eta) \pi_0^* + \delta_2 \pi_3^*(0)] e^{-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi} d\eta, \quad (4.20)$$

$$\pi_2^*(y) = \tau_2 e^{\delta_1 y + \int_0^y \beta_2(\eta) d\eta} - e^{\delta_1 y + \int_0^y \beta_2(\eta) d\eta} \int_0^y [\beta_2(\eta) \pi_0^* + \delta_1 \pi_3^*(0)] e^{-\delta_1 \eta - \int_0^\eta \beta_2(\xi) d\xi} d\eta, \quad (4.21)$$

$$\pi_3^*(y) = \tau_3 e^{\int_0^y \beta_3(\eta) d\eta} - e^{\int_0^y \beta_3(\eta) d\eta} \int_0^y \beta_3(\eta) \pi_0^* e^{-\int_0^\eta \beta_3(\xi) d\xi} d\eta. \quad (4.22)$$

By multiplying $e^{-\delta_2 y - \int_0^y \beta_1(\eta) d\eta}$, $e^{-\delta_1 y - \int_0^y \beta_2(\eta) d\eta}$, and $e^{-\int_0^y \beta_3(\eta) d\eta}$ with two sides of (4.20), (4.21), and (4.22) and taking the limit $y \rightarrow \infty$, we obtain

$$\tau_1 = \int_0^\infty [\beta_1(\eta) \pi_0^* + \delta_2 \pi_3^*(0)] e^{-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi} d\eta, \quad (4.23)$$

$$\tau_2 = \int_0^\infty [\beta_2(\eta) \pi_0^* + \delta_1 \pi_3^*(0)] e^{-\delta_1 \eta - \int_0^\eta \beta_2(\xi) d\xi} d\eta, \quad (4.24)$$

$$\tau_3 = \int_0^\infty \beta_3(\eta) \pi_0^* e^{-\int_0^\eta \beta_3(\xi) d\xi} d\eta. \quad (4.25)$$

Inserting (4.23)-(4.25) into (4.20)-(4.22), we obtain

$$\pi_1^*(y) = e^{\delta_2 y + \int_0^y \beta_1(\eta) d\eta} \int_y^\infty [\beta_1(\eta) \pi_0^* + \delta_2 \pi_3^*(0)] e^{-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi} d\eta, \quad (4.26)$$

$$\pi_2^*(y) = e^{\delta_1 y + \int_0^y \beta_2(\eta) d\eta} \int_y^\infty [\beta_2(\eta) \pi_0^* + \delta_1 \pi_3^*(0)] e^{-\delta_1 \eta - \int_0^\eta \beta_2(\xi) d\xi} d\eta, \quad (4.27)$$

$$\begin{aligned} \pi_3^*(y) &= e^{\int_0^y \beta_3(\eta) d\eta} \int_y^\infty \beta_3(\eta) \pi_0^* e^{-\int_0^\eta \beta_3(\xi) d\xi} d\eta \\ &= -\pi_0^* e^{\int_0^y \beta_3(\eta) d\eta} \int_y^\infty e^{-\int_0^\eta \beta_3(\xi) d\xi} d\left(-\int_0^\eta \beta_3(\xi) d\xi\right) \\ &= -\pi_0^* e^{\int_0^y \beta_3(\eta) d\eta} \left[e^{-\int_0^\eta \beta_3(\xi) d\xi} \right]_{\eta=y}^{\eta=\infty} = \pi_0^*. \end{aligned} \quad (4.28)$$

Substituting (4.28) into (4.26) and (4.27) we derive

$$\begin{aligned} \pi_1^*(y) &= e^{\delta_2 y + \int_0^y \beta_1(\eta) d\eta} \int_y^\infty [\beta_1(\eta) \pi_0^* + \delta_2 \pi_0^*] e^{-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi} d\eta \\ &= -\pi_0^* e^{\delta_2 y + \int_0^y \beta_1(\eta) d\eta} \int_y^\infty e^{-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi} d\left(-\delta_2 \eta - \int_0^\eta \beta_1(\xi) d\xi\right) \\ &= \pi_0^*, \end{aligned} \quad (4.29)$$

$$\begin{aligned} \pi_2^*(y) &= \pi_0^* e^{\delta_1 y + \int_0^y \beta_2(\eta) d\eta} \int_y^\infty [\delta_1 + \beta_2(\eta)] e^{-\delta_1 \eta - \int_0^\eta \beta_2(\xi) d\xi} d\eta \\ &= \pi_0^*. \end{aligned} \quad (4.30)$$

(4.28)-(4.30) give

$$\|\pi\| = \sup \left\{ |\pi_0^*|, \sup_{i=1,2,3} \|\pi_i^*\|_{L^\infty[0,\infty)} \right\} = |\pi_0^*| < \infty. \quad (4.31)$$

(4.31) shows that zero is an eigenvalue of $(A_1 + A_2)^*$. Moreover, it follows from (4.28)-(4.30) that the eigenvector space corresponding to zero is one dimensional, i.e., the geometric multiplicity of zero is one. $\square$

In the following section, we investigate the resolvent set of the underlying operator. First, we introduce the maximal operator A_m , the boundary operator G , Ω , and their respective domains. Subsequently, we define the operator A_0 along with its domain, and delve into a discussion on the resolvent set of A_0 . Finally, we examine the kernel of $\tilde{\gamma}I - A_m$. Based on this examination, we introduce the Dirichlet operator $D_{\tilde{\gamma}}$, and determine the expression for $\Omega D_{\tilde{\gamma}}$. by analyzing the spectral radius of $\Omega D_{\tilde{\gamma}}$, and utilizing the results from [26,28], we obtain the resolvent set of $A_1 + A_2$.

The maximal operator $(A_m, D(A_m))$ (see [25]) is defined as

$$\begin{aligned} A_m \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix} &= \begin{pmatrix} -(\delta_1 + \delta_2) & \Phi_1 & \Phi_2 & \Phi_3 \\ 0 & \psi_1 & 0 & 0 \\ 0 & 0 & \psi_2 & 0 \\ 0 & 0 & 0 & \psi_3 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix}, \\ D(A_m) &= \left\{ \pi \in Y \left| \begin{array}{l} \frac{d\pi_i(y)}{dy} \in L^1[0, \infty), \pi_i(y) \text{ are absolutely} \\ \text{continuous functions, } i = 1, 2, 3 \end{array} \right. \right\}, \end{aligned}$$

where

$$\begin{aligned}\Phi_i f(y) &= \int_0^\infty f(y) \beta_i(y) dy, \quad i = 1, 2, 3, \quad f \in L^1[0, \infty), \\ \psi_1 g(y) &= -\frac{dg(y)}{dy} - (\delta_2 + \beta_1(y)) g(y), \quad g \in W^{1,1}[0, \infty), \\ \psi_2 g(y) &= -\frac{dg(y)}{dy} - (\delta_1 + \beta_2(y)) g(y), \quad g \in W^{1,1}[0, \infty), \\ \psi_3 g(y) &= -\frac{dg(y)}{dy} - \beta_3(y) g(y), \quad g \in W^{1,1}[0, \infty).\end{aligned}$$

We take the boundary space of Y as $\partial Y := \mathbb{C}^3$, and introduce the boundary operators

$$G : D(A_m) \rightarrow \partial Y, \quad \Omega : D(A_m) \rightarrow \partial Y$$

as

$$G \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix} = \begin{pmatrix} \pi_1(0) \\ \pi_2(0) \\ \pi_3(0) \end{pmatrix}, \quad \Omega \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix} = \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ \delta_2 & 0 & 0 & 0 \\ 0 & \Phi_{\delta_2} & \Phi_{\delta_1} & 0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1(y) \\ \pi_2(y) \\ \pi_3(y) \end{pmatrix},$$

where

$$\Phi_{\delta_i} f(y) = \delta_i \int_0^\infty f(y) dy, \quad i = 1, 2, \quad f \in L^1[0, \infty).$$

Now, we define the underlying operator $((A_1 + A_2), D(A_1 + A_2))$ by

$$(A_1 + A_2) \pi = A_m \pi, \quad D(A_1 + A_2) = \{\pi \in D(A_m) \mid G\pi = \Omega\pi\}.$$

Thus, the system of Equations (1.1)-(1.8) can be written as the abstract Cauchy problem (2.1).

Next, we discuss the resolvent set of $A_1 + A_2$. To do this we define $(A_0, D(A_0))$ as

$$A_0 \pi = A_m \pi, \quad D(A_0) = \{\pi \in D(A_m) \mid G\pi = 0\},$$

and study the resolvent set of A_0 . for any given $v \in Y$, consider the equation $(\widetilde{\gamma}I - A_0) \pi = v$, that is

$$(\widetilde{\gamma} + \delta_1 + \delta_2) \pi_0 = \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy + v_0, \quad (4.32)$$

$$\frac{d\pi_1(y)}{dy} = -[\widetilde{\gamma} + (\delta_2 + \beta_1(y))] \pi_1(y) + v_1(y), \quad (4.33)$$

$$\frac{d\pi_2(y)}{dy} = -[\widetilde{\gamma} + (\delta_1 + \beta_2(y))] \pi_2(y) + v_2(y), \quad (4.34)$$

$$\frac{d\pi_3(y)}{dy} = -(\widetilde{\gamma} + \beta_3(y)) \pi_3(y) + v_3(y), \quad (4.35)$$

$$\pi_i(0) = 0, \quad i = 1, 2, 3. \quad (4.36)$$

From (4.32)-(4.36), we obtain

$$\pi_0 = \frac{v_0}{\bar{\gamma} + \delta_1 + \delta_2} + \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy, \quad (4.37)$$

$$\pi_1(y) = e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \int_0^y v_1(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta, \quad (4.38)$$

$$\pi_2(y) = e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\eta) d\eta} \int_0^y v_2(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta, \quad (4.39)$$

$$\pi_3(y) = e^{-\bar{\gamma}y - \int_0^y \beta_3(\eta) d\eta} \int_0^y v_3(\eta) e^{\bar{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} d\eta, \quad (4.40)$$

$\Rightarrow$

$$\begin{aligned} \pi_0 = \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} & \left\{ v_0 + \int_0^\infty \beta_1(y) e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \right. \\ & \int_0^y v_1(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta dy \\ & + \int_0^\infty \beta_2(y) e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\eta) d\eta} \int_0^y v_2(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta dy \\ & \left. + \int_0^\infty \beta_3(y) e^{-\bar{\gamma}y - \int_0^y \beta_3(\eta) d\eta} \int_0^y v_3(\eta) e^{\bar{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} d\eta dy \right\}, \end{aligned} \quad (4.41)$$

for $\forall f \in L^1[0, \infty)$, if we define

$$H_1 f(y) = e^{-(\bar{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \int_0^y f(\eta) e^{(\bar{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta,$$

$$H_2 f(y) = e^{-(\bar{\gamma} + \delta_1)y - \int_0^y \beta_2(\eta) d\eta} \int_0^y f(\eta) e^{(\bar{\gamma} + \delta_1)\eta + \int_0^\eta \beta_2(\xi) d\xi} d\eta,$$

$$H_3 f(y) = e^{-\bar{\gamma}y - \int_0^y \beta_3(\eta) d\eta} \int_0^y f(\eta) e^{\bar{\gamma}\eta + \int_0^\eta \beta_3(\xi) d\xi} d\eta,$$

then (4.38)-(4.41) can be written as

$$(\bar{\gamma}I - A_0)^{-1} v = \begin{pmatrix} \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} & \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} \Phi_1 H_1 & \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} \Phi_2 H_2 & \frac{1}{\bar{\gamma} + \delta_1 + \delta_2} \Phi_3 H_3 \\ 0 & H_1 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1(y) \\ v_2(y) \\ v_3(y) \end{pmatrix}.$$

From which, together with the definition of the resolvent set, we have the following result.

Lemma 4.4. Let $\beta_i(y)$ ($i = 1, 2, 3$) be measurable and

$$0 < \underline{\beta}_i = \inf_{y \in [0, \infty)} \beta_i(y) \leq \sup_{y \in [0, \infty)} \beta_i(y) = \bar{\beta}_i < \infty, \quad i = 1, 2, 3,$$

then

$$\left\{ \bar{\gamma} \in \mathbb{C} \left| \begin{array}{l} \operatorname{Re} \bar{\gamma} + \delta_1 + \delta_2 > 0, \operatorname{Re} \bar{\gamma} + \delta_2 + \underline{\beta}_1 > 0 \\ \operatorname{Re} \bar{\gamma} + \delta_1 + \underline{\beta}_2 > 0, \operatorname{Re} \bar{\gamma} + \underline{\beta}_3 > 0 \end{array} \right. \right\} \subset \rho(A_0).$$

Proof. By the definition of $H_i (i = 1, 2, 3)$ and using integration by parts, we estimate, for any $f \in L^1[0, \infty)$

$$\begin{aligned}
 \|H_1 f\|_{L^1[0, \infty)} &= \int_0^\infty |H_1 f(y)| dy \\
 &= \int_0^\infty \left| e^{-(\widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \int_0^y f(\eta) e^{(\widetilde{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta \right| dy \\
 &\leq \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \int_0^y |f(\eta)| e^{(\operatorname{Re} \widetilde{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} d\eta dy \\
 &= \int_0^\infty \frac{-1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \beta_1(y)} \left(\int_0^y e^{(\operatorname{Re} \widetilde{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} |f(\eta)| d\eta \right) dy e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \\
 &\leq \frac{-1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \beta_1} \left\{ e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \times \left(\int_0^y e^{(\operatorname{Re} \widetilde{\gamma} + \delta_2)\eta + \int_0^\eta \beta_1(\xi) d\xi} |f(\eta)| d\eta \right) \right\}_{y=0}^{y=\infty} \\
 &\quad - \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\eta) d\eta} \cdot e^{(\operatorname{Re} \widetilde{\gamma} + \delta_2)y + \int_0^y \beta_1(\eta) d\eta} |f(y)| dy \\
 &= \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \beta_1} \|f\|_{L^1[0, \infty)} \\
 \Rightarrow \|H_1\| &\leq \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \beta_1}. \tag{4.42}
 \end{aligned}$$

Similarly, we obtain

$$\|H_2\| \leq \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \beta_2}, \quad \|H_3\| \leq \frac{1}{\operatorname{Re} \widetilde{\gamma} + \beta_3}. \tag{4.43}$$

For any $v \in Y$, by using (4.42), (4.43), the expression of $(\widetilde{\gamma}I - A_0)^{-1}$, $\|\Phi_i\| \leq \overline{\beta}_i (i = 1, 2, 3)$, and the conditions of this lemma, we estimate

$$\begin{aligned}
 &\|(\widetilde{\gamma}I - A_0)^{-1} v\| \\
 &= \left| \frac{v_0}{\widetilde{\gamma} + \delta_1 + \delta_2} + \frac{\Phi_1 H_1 v_1}{\widetilde{\gamma} + \delta_1 + \delta_2} + \frac{\Phi_2 H_2 v_2}{\widetilde{\gamma} + \delta_1 + \delta_2} + \frac{\Phi_3 H_3 v_3}{\widetilde{\gamma} + \delta_1 + \delta_2} \right| \\
 &\quad + \sum_{i=1}^3 \|H_i v_i\|_{L^1[0, \infty)} \\
 &\leq \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \left(|v_0| + \sum_{i=1}^3 |\Phi_i H_i v_i| \right) + \sum_{i=1}^3 \|H_i v_i\|_{L^1[0, \infty)} \\
 &\leq \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} |v_0| + \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \sum_{i=1}^3 \|\Phi_i\| \|H_i\| \|v_i\|_{L^1[0, \infty)} \\
 &\quad + \sum_{i=1}^3 \|H_i\| \|v_i\|_{L^1[0, \infty)}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} |v_0| + \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \cdot \frac{\overline{\beta_1}}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta_1}} \|v_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \cdot \frac{\overline{\beta_2}}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta_2}} \|v_2\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \cdot \frac{\overline{\beta_3}}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta_3}} \|v_3\|_{L^1[0,\infty)} + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta_1}} \|v_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta_2}} \|v_2\|_{L^1[0,\infty)} + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta_3}} \|v_3\|_{L^1[0,\infty)} \\
&= \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} |v_0| + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta_1}} \left(\frac{\overline{\beta_1}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) \|v_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta_2}} \left(\frac{\overline{\beta_2}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) \|v_2\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta_3}} \left(\frac{\overline{\beta_3}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) \|v_3\|_{L^1[0,\infty)} \\
&\leq \sup \left\{ \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|}, \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta_1}} \left(\frac{\overline{\beta_1}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right), \right. \\
&\quad \left. \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta_2}} \left(\frac{\overline{\beta_2}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right), \right. \\
&\quad \left. \frac{1}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta_3}} \left(\frac{\overline{\beta_3}}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) \right\} \|v\| < \infty,
\end{aligned}$$

which shows that the assertion of this lemma holds true. $\square$

Lemma 4.5. Let $\beta_i(y) (i = 1, 2, 3)$ be measurable and

$$0 < \underline{\beta_i} = \inf_{y \in [0,\infty)} \beta_i(y) \leq \sup_{y \in [0,\infty)} \beta_i(y) = \overline{\beta_i} < \infty, \quad i = 1, 2, 3,$$

then for

$$\widetilde{\gamma} \in \left\{ \widetilde{\gamma} \in \mathbb{C} \left| \begin{array}{l} \operatorname{Re} \widetilde{\gamma} + \delta_1 + \delta_2 > 0, \operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta_1} > 0 \\ \operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta_2} > 0, \operatorname{Re} \widetilde{\gamma} + \underline{\beta_3} > 0 \end{array} \right. \right\} \subset \rho(A_0),$$

we have

$$\begin{aligned}
&\pi \in \ker(\widetilde{\gamma}I - A_m) \Leftrightarrow \\
&\pi_0 = \frac{1}{\widetilde{\gamma} + \delta_1 + \delta_2} \left[\tau_1 \int_0^\infty e^{-(\widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy \right. \\
&\quad + \tau_2 \int_0^\infty e^{-(\widetilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\
&\quad \left. + \tau_3 \int_0^\infty e^{-\widetilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \right], \tag{4.44}
\end{aligned}$$

$$\pi_1(y) = \tau_1 e^{-(\tilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi}, \quad |\tau_1| < \infty, \quad (4.45)$$

$$\pi_2(y) = \tau_2 e^{-(\tilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi}, \quad |\tau_2| < \infty, \quad (4.46)$$

$$\pi_3(y) = \tau_3 e^{-\tilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi}, \quad |\tau_3| < \infty. \quad (4.47)$$

Proof. If $\pi \in \ker(\tilde{\gamma}I - A_m)$, then $(\tilde{\gamma}I - A_m)\pi = 0$, that is

$$(\tilde{\gamma} + \delta_1 + \delta_2)\pi_0 = \sum_{i=1}^3 \int_0^\infty \pi_i(y) \beta_i(y) dy, \quad (4.48)$$

$$\frac{d\pi_1(y)}{dy} = -(\tilde{\gamma} + \delta_2 + \beta_1(y))\pi_1(y), \quad (4.49)$$

$$\frac{d\pi_2(y)}{dy} = -(\tilde{\gamma} + \delta_1 + \beta_2(y))\pi_2(y), \quad (4.50)$$

$$\frac{d\pi_3(y)}{dy} = -(\tilde{\gamma} + \beta_3(y))\pi_3(y). \quad (4.51)$$

By solving (4.49)-(4.51), we obtain

$$\pi_1(y) = \tau_1 e^{-(\tilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi}, \quad (4.52)$$

$$\pi_2(y) = \tau_2 e^{-(\tilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi}, \quad (4.53)$$

$$\pi_3(y) = \tau_3 e^{-\tilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi}. \quad (4.54)$$

By inserting (4.52)-(4.54) into (4.48), it follows that

$$\begin{aligned} \pi_0 = & \frac{1}{\tilde{\gamma} + \delta_1 + \delta_2} \left[\tau_1 \int_0^\infty e^{-(\tilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy \right. \\ & + \tau_2 \int_0^\infty e^{-(\tilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\ & \left. + \tau_3 \int_0^\infty e^{-\tilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \right]. \end{aligned} \quad (4.55)$$

Since $\pi \in \ker(\tilde{\gamma}I - A_m)$, $\pi \in D(A_m)$, which implies by the imbedding theorem in Adams^[22]

$$\begin{aligned} |\tau_i| & \leq \sum_{i=1}^3 |\tau_i| = \sum_{i=1}^3 \pi_i(0) \leq \sum_{i=1}^3 \|\pi_i\|_{L^\infty[0, \infty)} \\ & \leq \sum_{i=1}^3 \|\pi_i\|_{L^1[0, \infty)} + \sum_{i=1}^3 \left\| \frac{d\pi_i}{dy} \right\|_{L^1[0, \infty)} < \infty, \quad i = 1, 2, 3. \end{aligned}$$

From this, together with (4.52)-(4.55), we conclude that (4.44)-(4.47) hold.

Conversely, if (4.44)-(4.47) hold, then by direct calculation, we obtain

$$\begin{aligned} \|\pi_1\|_{L^1[0, \infty)} & = |\tau_1| \int_0^\infty \left| e^{-(\tilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \right| dy \\ & \leq |\tau_1| \int_0^\infty e^{-(\operatorname{Re} \tilde{\gamma} + \delta_2)y - \underline{\beta}_1 y} dy \end{aligned}$$

$$\begin{aligned}
&= |\tau_1| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta}_1)y} dy \\
&= \frac{|\tau_1|}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta}_1},
\end{aligned} \tag{4.56}$$

$$\|\pi_2\|_{L^1[0,\infty)} = \frac{|\tau_2|}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta}_2}, \tag{4.57}$$

$$\|\pi_3\|_{L^1[0,\infty)} = \frac{|\tau_3|}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta}_3}. \tag{4.58}$$

Using (4.56)-(4.58), and the lemma's condition, we derive

$$\begin{aligned}
\|\pi\| &= |\pi_0| + \sum_{i=1}^3 \|\pi_i\|_{L^1[0,\infty)} \\
&\leq \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \left[|\tau_1| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy \right. \\
&\quad + |\tau_2| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\
&\quad \left. + |\tau_3| \int_0^\infty e^{-\operatorname{Re} \widetilde{\gamma} y - \int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \right] + \sum_{i=1}^3 \|\pi_i\|_{L^1[0,\infty)} \\
&\leq \frac{1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} \left[\overline{\beta}_1 |\tau_1| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta}_1)y} dy \right. \\
&\quad + \overline{\beta}_2 |\tau_2| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta}_2)y} dy + \overline{\beta}_3 |\tau_3| \int_0^\infty e^{-(\operatorname{Re} \widetilde{\gamma} + \underline{\beta}_3)y} dy \Big] \\
&\quad + \frac{|\tau_1|}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta}_1} + \frac{|\tau_2|}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta}_2} + \frac{|\tau_3|}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta}_3} \\
&= \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_2 + \underline{\beta}_1} \left(\frac{\overline{\beta}_1}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) |\tau_1| \\
&\quad + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \delta_1 + \underline{\beta}_2} \left(\frac{\overline{\beta}_2}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) |\tau_2| \\
&\quad + \frac{1}{\operatorname{Re} \widetilde{\gamma} + \underline{\beta}_3} \left(\frac{\overline{\beta}_3}{|\widetilde{\gamma} + \delta_1 + \delta_2|} + 1 \right) |\tau_3| < \infty.
\end{aligned} \tag{4.59}$$

(4.45)-(4.47) give

$$\frac{d\pi_1(y)}{dy} = -(\widetilde{\gamma} + \delta_2 + \beta_1(y)) \pi_1(y), \tag{4.60}$$

$$\frac{d\pi_2(y)}{dy} = -(\widetilde{\gamma} + \delta_1 + \beta_2(y)) \pi_2(y), \tag{4.61}$$

$$\frac{d\pi_3(y)}{dy} = -(\widetilde{\gamma} + \beta_3(y)) \pi_3(y). \tag{4.62}$$

From (4.60)-(4.62), we obtain

$$\begin{aligned} \sum_{i=1}^3 \left\| \frac{d\pi_i}{dy} \right\|_{L^1[0,\infty)} &= \int_0^\infty |(\tilde{\gamma} + \delta_2 + \beta_1(y)) \pi_1(y)| dy \\ &\quad + \int_0^\infty |(\tilde{\gamma} + \delta_1 + \beta_2(y)) \pi_2(y)| dy + \int_0^\infty |(\tilde{\gamma} + \beta_3(y)) \pi_3(y)| dy \\ &\leq \left(|\tilde{\gamma}| + \sum_{i=1}^2 |\delta_i| + \sum_{i=1}^3 \tilde{\beta}_i \right) \sum_{i=1}^3 \|\pi_i\|_{L^1[0,\infty)} < \infty. \end{aligned} \quad (4.63)$$

(4.59) and (4.63) show that $\pi \in \ker(\tilde{\gamma}I - A_m)$. $\square$

Observe that the operator G is surjective. Hence

$$G|_{\ker(\tilde{\gamma}I - A_m)}: \ker(\tilde{\gamma}I - A_m) \rightarrow \partial Y$$

is invertible for $\tilde{\gamma} \in \rho(A_0)$. Thus, for $\tilde{\gamma} \in \rho(A_0)$, we introduce the Dirichlet operator as

$$D_{\tilde{\gamma}} := \left(G|_{\ker(\tilde{\gamma}I - A_m)} \right)^{-1}: \partial Y \rightarrow \ker(\tilde{\gamma}I - A_m).$$

By Lemma 4.5, we obtain the explicit expression of $D_{\tilde{\gamma}}$, for $\tilde{\gamma} \in \rho(A_0)$ as follows

$$D_{\tilde{\gamma}} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix},$$

where

$$\begin{aligned} \omega_i &= \frac{1}{\tilde{\gamma} + \delta_1 + \delta_2} \int_0^\infty \varepsilon_i \beta_i(y) dy, i = 1, 2, 3, \\ \varepsilon_1 &= e^{-(\tilde{\gamma} + \delta_2)y - \int_0^y \beta_1(\xi) d\xi}, \quad \varepsilon_2 = e^{-(\tilde{\gamma} + \delta_1)y - \int_0^y \beta_2(\xi) d\xi}, \quad \varepsilon_3 = e^{-\tilde{\gamma}y - \int_0^y \beta_3(\xi) d\xi}. \end{aligned}$$

So, by using the definition of Ω and the expression of $D_{\tilde{\gamma}}$, It is easy to write down the explicit form of $\Omega D_{\tilde{\gamma}}$ as follows

$$\begin{aligned} \Omega D_{\tilde{\gamma}} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} &= \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ \delta_2 & 0 & 0 & 0 \\ 0 & \Phi_{\delta_2} & \Phi_{\delta_1} & 0 \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \\ &= \begin{pmatrix} \delta_1 \omega_1 & \delta_1 \omega_2 & \delta_1 \omega_3 \\ \delta_2 \omega_1 & \delta_2 \omega_2 & \delta_2 \omega_3 \\ \Phi_{\delta_2} \varepsilon_1 & \Phi_{\delta_1} \varepsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}. \end{aligned} \quad (4.64)$$

The following result can be found in [26]:

Lemma 4.6. Suppose $\tilde{\gamma} \in \rho(A_0)$ and there exist $\tilde{\gamma}_0 \in \mathbb{C}$, such that $1 \notin \sigma(\Omega D_{\tilde{\gamma}_0})$, then

$$\tilde{\gamma} \in \sigma(A_1 + A_2) \Leftrightarrow 1 \in \sigma(\Omega D_{\tilde{\gamma}}).$$

By Lemma 4.6 and Nagel [28], we obtain the following result on the resolvent set of $A_1 + A_2$.

Lemma 4.7. *Let $\beta_i(y) (i = 1, 2, 3)$ be measurable and*

$$0 < \underline{\beta}_i = \inf_{y \in [0, \infty)} \beta_i(y) \leq \sup_{y \in [0, \infty)} \beta_i(y) = \overline{\beta}_i < \infty, \quad i = 1, 2, 3,$$

then all points on the imaginary axis except zero belong to the resolvent set of $A_1 + A_2$.

Proof. Let $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$, $|\tau_i| < \infty (i = 1, 2, 3)$, and $\tilde{\gamma} = ih, h \in \mathbb{R} \setminus \{0\}$, then by Riemann-Lebesgue lemma

$$\lim_{h \rightarrow \infty} \int_0^\infty v(y) \cos(hy) dy = 0, \lim_{h \rightarrow \infty} \int_0^\infty v(y) \sin(hy) dy = 0, v \in L^1[0, \infty), v(y) \geq 0,$$

we know that there exists a finite positive constant $C > 0$, such that when $|h| > C$, we have

$$\begin{aligned} \left| \int_0^\infty v(y) e^{ihy} dy \right|^2 &= \left| \int_0^\infty v(y) (\cos(hy) - i \sin(hy)) dy \right|^2 \\ &= \left(\int_0^\infty v(y) \cos(hy) dy \right)^2 + \left(\int_0^\infty v(y) \sin(hy) dy \right)^2 \\ &< \left(\int_0^\infty v(y) dy \right)^2 \\ &\Rightarrow \left| \int_0^\infty v(y) e^{ihy} dy \right| < \int_0^\infty v(y) dy. \end{aligned} \quad (4.65)$$

For such $|h| > C$, by (4.64), (4.65), and $\int_0^\infty g(y) e^{-\int_0^y g(\xi) d\xi} = 1$, we obtain

$$\begin{aligned} \|\Omega D_{\tilde{\gamma}} \vec{\tau}\| &= |\delta_1 \omega_1 \tau_1 + \delta_1 \omega_2 \tau_2 + \delta_1 \omega_3 \tau_3| \\ &\quad + |\delta_2 \omega_1 \tau_1 + \delta_2 \omega_2 \tau_2 + \delta_3 \omega_3 \tau_3| + |\Phi_{\delta_2} \varepsilon_1 \tau_1 + \Phi_{\delta_1} \varepsilon_2 \tau_2| \\ &\leq \frac{\delta_1 |\tau_1|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-(ih + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy \\ &\quad + \frac{\delta_1 |\tau_2|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-(ih + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\ &\quad + \frac{\delta_1 |\tau_3|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-ihy - \int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \\ &\quad + \frac{\delta_2 |\tau_1|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-(ih + \delta_2)y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy \\ &\quad + \frac{\delta_2 |\tau_2|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-(ih + \delta_1)y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\ &\quad + \frac{\delta_2 |\tau_3|}{\sqrt{h^2 + (\delta_1 + \delta_2)^2}} \int_0^\infty e^{-ihy - \int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \end{aligned}$$

$$\begin{aligned}
& + |\tau_1| \delta_2 \int_0^\infty e^{-(ih+\delta_2)y - \int_0^y \beta_1(\xi) d\xi} dy \\
& + |\tau_2| \delta_1 \int_0^\infty e^{-(ih+\delta_1)y - \int_0^y \beta_2(\xi) d\xi} dy \\
& < |\tau_1| \int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} \beta_1(y) dy + |\tau_2| \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} \beta_2(y) dy \\
& + |\tau_3| \int_0^\infty e^{-\int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy + |\tau_1| \int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} \delta_2 \beta_1(y) dy \\
& + |\tau_2| \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} \delta_1 \beta_2(y) dy \\
& = |\tau_1| \int_0^\infty e^{-\delta_2 y - \int_0^y \beta_1(\xi) d\xi} (\delta_2 + \beta_1(y)) dy \\
& + |\tau_2| \int_0^\infty e^{-\delta_1 y - \int_0^y \beta_2(\xi) d\xi} (\delta_1 + \beta_2(y)) dy + |\tau_3| \int_0^\infty e^{-\int_0^y \beta_3(\xi) d\xi} \beta_3(y) dy \\
& = |\tau_1| + |\tau_2| + |\tau_3| = \|\vec{\tau}\| \\
& \Rightarrow \|\Omega D_{\vec{\gamma}}\| < 1.
\end{aligned} \tag{4.66}$$

(4.66) imply the spectral radius $r(\Omega D_{\vec{\gamma}}) \leq \|\Omega D_{\vec{\gamma}}\| < 1$, when $|h| > C$. This means $1 \notin \sigma(\Omega D_{\vec{\gamma}})$, when $|h| > C$. which together with Lemma 4.6, we get $\vec{\gamma} = ih \notin \sigma(A_1 + A_2)$ for $|h| > C$, that is

$$\{ih \mid |h| > C\} \subset \rho(A_1 + A_2), \quad \{ih \mid |h| \leq C\} \subset \sigma(A_1 + A_2) \cap i\mathbb{R}. \tag{4.67}$$

Moreover, because $T(t)$ is a positive contraction C_0 -semigroup with spectral bound zero by Theorem 3.1 and Lemma 4.1, which means $\sigma(A_1 + A_2) \cap i\mathbb{R}$ is imaginary additively cyclic^[28], that is

$$ih \in \sigma(A_1 + A_2) \cap i\mathbb{R} \Rightarrow i h k \in \sigma(A_1 + A_2) \cap i\mathbb{R}, \quad k \in \mathbb{N},$$

from which, together with (4.67) and Lemma 4.1, we obtain that $\sigma(A_1 + A_2) \cap i\mathbb{R} = \{0\}$. $\square$

5. ASYMPTOTIC BEHAVIOR OF THE TIME-DEPENDENT SOLUTION OF THE SYSTEM (2.1)

Because Theorem 3.1, Lemma 4.1, Lemma 4.3, and Lemma 4.7 satisfy the conditions of Theorem 1.96 in [23], by combining them, we conclude the desired result in this paper.

Theorem 5.1. *Let $\beta_i(y)$ ($i = 1, 2, 3$) be measurable, and*

$$0 < \underline{\beta}_i = \inf_{y \in [0, \infty)} \beta_i(y) \leq \sup_{y \in [0, \infty)} \beta_i(y) = \overline{\beta}_i < \infty,$$

then the time-dependent solution of System (2.1) converges strongly to its steady state solution, i.e.,

$$\lim_{t \rightarrow \infty} \|\pi(\cdot, t) - c\pi(\cdot)\| = 0,$$

where $\pi(y)$ is the eigenvector in Lemma 4.1.

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