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Generalized Hyers–Ulam Stability of an n-Dimensional Cubic Functional Equation in Banach Spaces with an Application via Fixed Point Method

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Abstract. In this work, we investigate the generalized Hyers–Ulam stability of an n-dimensional cubic functional equation

$$f\left(\sum_{i=1}^{n} ia_i\right) = (3-n)\sum_{1\le i<j\le n} f(ia_i + ja_j) + \sum_{1\le i<j<k\le n} f(ia_i + ja_j + ka_k) + \left(\frac{n^2-5n+6}{2}\right)\sum_{i=0}^{n-1}(i+1)^3 f(a_{i+1}),$$

in Banach spaces. The problem is studied by employing both direct analytical technique and fixed point method, particularly Banach's Contraction Principle and the alternative fixed point theorem. We establish the existence and uniqueness of exact cubic mappings that approximate the given functional equation under suitable control functions. Explicit stability estimates are obtained, which ensure that the deviation of approximate solutions remains bounded. To demonstrate the applicability of the theoretical results, an illustrative example is provided. A perturbed cubic mapping is constructed and shown to satisfy the stability conditions. Furthermore, a numerical comparison supported by tabular data is included to verify the theoretical bounds and to highlight the effectiveness of the proposed approach.

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1. Introduction

In 1940, Ulam [36] posed a question about the stability of group homomorphisms, which led to the emergence of the stability problem in functional equations during a mathematical colloquium. He provided a comprehensive discussion, listing several open questions. Among them, the following hold significant importance in the field of functional equation theory: When can we assert that the solution to an inequality lies close to the solution to an equation? An equation is said to be stable when a problem admits a unique solution. Hyers [15] provided an affirmative response to Ulam's question in the context of Banach spaces in 1941. Hyer's theorem was subsequently generalized in several ways. Aoki [2] generalized Hyers' theorem. This outcome gave rise to what is now referred to as the Hyers-Ulam stability theory. Rassias created the well-known Hyers-Ulam-Rassias stability, which he later expanded by allowing unbounded perturbations and making a significant generalization [26]. Aoki [2], Gavruta [13], and Jung [18], among others, added more general functional equations and larger groups of control functions to the theory.

Non-linear functional equations that display polynomial-like characteristics have attracted significant interest. Cubic functional equations are important because they are used in approximation theory, operator theory, and information theory. Classical cubic equations naturally emerge in the characterization of cubic mappings, which adhere to identities similar to third-degree polynomials [16]. Many researchers have investigated the stability of cubic equations and concluded that approximation cubic mappings converge to exact solutions given particular norm conditions or contractive assumptions [12, 25]. Numerous generalizations using supplementary parameters or adjusted coefficients have significantly enhanced the discipline [5, 14, 17, 22, 24, 27].

The Banach spaces are among the ideal places to study the stability of functional equations because completeness makes sure that sequences made by iterative methods converge. The interaction between the structure of Banach spaces and functional stability has produced numerous findings related to additive, quadratic, cubic, quartic, and mixed-type functional equations [7, 21]. Diaz and Margolis [9] introduced a fixed point theorem in 1968 was a significant methodological advancement. It has since become one of the most useful tools for proving Ulam-type stability. Numerous researchers have utilized fixed point methods to derive existence, uniqueness, and stability results for an extensive range of nonlinear functional equations [3, 4, 8, 19, 20, 30, 34].

In this paper, we consider the Hyers–Ulam stability of a generalized cubic functional equation of the form

$$f\left(\sum_{i=1}^{n} ia_i\right) = (3-n)\sum_{1\le i<j\le n} f(ia_i + ja_j) + \sum_{1\le i<j<k\le n} f(ia_i + ja_j + ka_k) + \left(\frac{n^2-5n+6}{2}\right)\sum_{i=0}^{n-1}(i+1)^3 f(a_{i+1}), \tag{1.1}$$

where $n \geq 3$ is an integer. Equation (1.1) generalizes several well-known cubic equations by incorporating weighted indices, higher-order interactions, and multi-variable structures. This formulation captures the essence of generalized cubic behaviors more effectively than classical forms and thus provides a broad framework for studying stability phenomena.

Recent advancements in functional analysis, nonlinear operator theory, and extended normed spaces influence the stability of equation (1.1). The stability of nonlinear functional equations has been extensively analyzed in fuzzy normed spaces (see. [6,28]), intuitionistic fuzzy spaces (see. [31]), non-Archimedean spaces (see. [23]), and generalized metric spaces (see. [9]). Generalized cubic equations emerge inherently in approximation theory, discrete mathematics, and the analysis of nonlinear difference equations. Recent studies, notably those by Rassias (see. [25]), demonstrate that generalized cubic equations are crucial in the formulation of non-linear approximation models, underscoring the importance of rigorous stability outcomes.

In a wide range of contexts, including Banach algebras and C^*-algebras, recent studies have emphasized the structural properties and stability behavior of additive functional equations [10, 11]. Furthermore, direct and fixed-point approaches have been used to study the Hyers-Ulam stability of additive equations in fuzzy normed spaces [33,35].

In addition, quartic functional equations have attracted attention due to their difficulty and potential applications. In modular and quasi-β-normed spaces, their stability has been studied using techniques based on the Fatou property [1,37]. Fixed-point methods have recently been used to extend stability results to Banach and non-Archimedean fuzzy spaces [32].

The main goal of this work is to show that equation (1.1) has unique solutions, is generally stable, and is Hyers-Ulam stable in Banach spaces. The problem is studied by employing both direct analytical technique and fixed point method, particularly Banach's Contraction Principle and the alternative fixed point theorem. We establish the existence and uniqueness of exact cubic mappings that approximate the given functional equation under suitable control functions. Explicit stability estimates are obtained, which ensure that the deviation of approximate solutions remains bounded. To demonstrate the applicability of the theoretical results, an illustrative example is provided. A perturbed cubic mapping is constructed and shown to satisfy the stability conditions. Furthermore, a numerical comparison supported by tabular data is included to verify the theoretical bounds and to highlight the effectiveness of the proposed approach.

Theorem 1.1. *[29]*

Let (G, d) be a generalized complete metric space, and let $\Omega : G \to G$ be a contraction mapping with a Lipschitz constant $L < 1$. Then, for any initial point $\mathsf{a}_1 \in G$, one of the following statements is true:

$$d\Big(\Omega^m \mathsf{a}_1, \Omega^{m+1}\mathsf{a}_1\Big) = \infty \quad \textit{for every } m \geq m_0,$$

or there exists an integer $m_0 > 0$ such that:

(1) $d\Big(\Omega^m \mathsf{a}_1, \Omega^{m+1}\mathsf{a}_1\Big) < \infty$ *for each $m_0 \leq m$;*
(2) *The sequence $\{\Omega^m \mathsf{a}_1\}_{m\in\mathbb{N}} \to \mathsf{a}_1^*$ to a fixed point of Ω;*
(3) a_1^* *is the precisely one fixed point of Ω in*

$$G^* = \Big\{\mathsf{a}_2 \in G \;\Big|\; d(\Omega^{m_0}\mathsf{a}_1, \mathsf{a}_2) < \infty\Big\};$$

(4) *for every* $\mathsf{a}_2 \in G^*$*, the inequality holds*

$$d(\mathsf{a}_2, \mathsf{a}_1^*) \leq \frac{1}{1-L}\, d(\Omega \mathsf{a}_2, \mathsf{a}_2).$$

It is defined as $Q : P \to V$ for the sake of notational simplicity:

$$\begin{aligned} Q(a_1, a_2, \cdots, a_n) &= f\left(\sum_{i=1}^{n} i a_i\right) - (3-n) \sum_{1 \leq i < j \leq n} f(i a_i + j a_j) \\ &\quad - \sum_{1 \leq i < j < k \leq n} f(i a_i + j a_j + k a_k) \\ &\quad - \left(\frac{n^2 - 5n + 6}{2}\right) \sum_{i=0}^{n-1} (i+1)^3 f(a_{i+1}) \end{aligned}$$

for every $a_1, a_2, \cdots, a_n \in P$.

2. Stability Results: Direct Method

This section will use the direct technique to investigate the variations of Hyers-Ulam stability associated with the n-dimensional cubic functional equation (1.1) in Banach space.

Theorem 2.1. *Let* $\kappa \in \{-1, 1\}$ *and* $\eta : P^n \to [0, \infty)$ *be a function such that*

$$\sum_{i=0}^{\infty} \frac{\eta\left(2^{i\kappa}\mathsf{a}_1, 2^{i\kappa}\mathsf{a}_2, \ldots, 2^{i\kappa}\mathsf{a}_n\right)}{2^{3i\kappa}}$$

converges in $\mathbb{R}$ *and*

$$\lim_{i \to \infty} \frac{\eta\left(2^{i\kappa}\mathsf{a}_1, 2^{i\kappa}\mathsf{a}_2, \ldots, 2^{i\kappa}\mathsf{a}_n\right)}{2^{3i\kappa}} = 0 \tag{2.1}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n \in P$ *. If* $Q : P \to V$ *is a function satisfying the inequality*

$$\left\| Q(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n) \right\| \leq \eta(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n) \tag{2.2}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n \in P$*, then there exists a unique cubic mapping* $W : P \to V$ *which satisfies the functional equation (1.1) and*

$$\left\| f(\mathsf{a}) - W(\mathsf{a}) \right\| \leq \frac{1}{8(n^2 - 5n + 6)} \sum_{i=\frac{1-\kappa}{2}}^{\infty} \frac{\eta(2^{i\kappa}\mathsf{a}, 2^{i\kappa}\mathsf{a}, 0, \ldots, 0)}{2^{3i\kappa}} \tag{2.3}$$

for all $\mathsf{a} \in P$*. The mapping* $W(\mathsf{a})$ *is defined by*

$$W(\mathsf{a}) = \lim_{i \to \infty} \frac{f(2^{i\kappa}\mathsf{a})}{2^{3i\kappa}} \tag{2.4}$$

for all $\mathsf{a} \in P$*.*

Proof. Assume that $\kappa = 1$. Replacing $(a_1, a_2, \ldots, a_n)$ by $(a, a, 0, \ldots, 0)$ in (2.2), we get

$$\left\|(n^2-5n+6)f(2a)-8(n^2-5n+6)f(a)\right\| \leq \eta(a,a,0,\ldots,0) \tag{2.5}$$

for all $a \in P$. It follows from (2.5) that

$$\left\|\frac{f(2a)}{2^3}-f(a)\right\| \leq \frac{\eta(a,a,0,\ldots,0)}{8(n^2-5n+6)} \tag{2.6}$$

for all $a \in P$. Replacing a by $2a$ in (2.6), we have

$$\left\|\frac{f(2^2a)}{2^3}-f(2a)\right\| \leq \frac{\eta(2a,2a,0,\ldots,0)}{8(n^2-5n+6)} \tag{2.7}$$

for all $a \in P$. From (2.7), we get

$$\left\|\frac{f(2^2a)}{2^6}-\frac{f(2a)}{2^3}\right\| \leq \frac{1}{2^3(n^2-5n+6)}\frac{\eta(2a,2a,0,\ldots,0)}{2^3} \tag{2.8}$$

for all $a \in P$. Adding (2.6) and (2.8), we obtain

$$\left\|\frac{f(2^2a)}{2^6}-f(a)\right\| \leq \frac{1}{8(n^2-5n+6)}\left[\eta(a,a,0,\ldots,0)+\frac{\eta(2a,2a,0,\ldots,0)}{2^3}\right] \tag{2.9}$$

for all $a \in P$. It follows from (2.6), (2.8) and (2.9) and generalizing that

$$\begin{aligned}\left\|\frac{f(2^na)}{2^{3n}}-f(a)\right\| &\leq \frac{1}{8(n^2-5n+6)}\sum_{i=0}^{n-1}\frac{\eta(2^ia,2^ia,0,\ldots,0)}{2^{3i}}\\ &\leq \frac{1}{8(n^2-5n+6)}\sum_{i=0}^{\infty}\frac{\eta(2^ia,2^ia,0,\ldots,0)}{2^{3i}}\end{aligned} \tag{2.10}$$

for all $a \in P$. In order to prove convergence of the sequence $\left\{\frac{f(2^ia)}{2^{3i}}\right\}$, replace a by 2^la and dividing by 2^{3l} in (2.10). We then deduce that, for any $k, l > 0$,

$$\begin{aligned}\left\|\frac{f(2^{k+l}a)}{2^{3(k+l)}}-\frac{f(2^la)}{2^{3l}}\right\| &= \frac{1}{2^{3l}}\left\|\frac{f(2^{k+l}a)}{2^{3k}}-f(2^la)\right\|\\ &\leq \frac{1}{8(n^2-5n+6)}\sum_{k=0}^{n-1}\frac{\eta(2^{k+l}a,0,\ldots,0)}{2^{4(k+l)}}\\ &\leq \frac{1}{8(n^2-5n+6)}\sum_{k=0}^{\infty}\frac{\eta(2^{k+l}a,2^{k+1}a,0,\ldots,0)}{2^{3(k+l)}} \to 0 \text{ as } \quad l\to\infty\end{aligned}$$

for all $a \in P$. Hence the sequence $\left\{\frac{f(2^ka)}{2^{4k}}\right\}$ is a Cauchy sequence. Since V is complete, there exists a mapping $W : P \to V$ such that

$$W(a) = \lim_{k\to\infty}\frac{f(2^ka)}{2^{3k}}$$

for all $a \in P$. Letting $i \to \infty$ in (2.10), we get the result that (2.3) holds for all $a \in P$. To prove that W satisfies (1.1), replacing $(a_1, a_2, \ldots, a_n)$ by $(2^k a_1, 2^k a_2, \ldots, 2^k a_n)$ and dividing by 2^{3k} in (2.2), we have

$$\frac{1}{2^{3k}} \left\| Q(2^k a_1, 2^k a_2, \ldots, 2^k a_n) \right\| \leq \frac{1}{2^{3k}} \eta(2^k a_1, 2^k a_2, \ldots, 2^k a_n)$$

for all $a_1, a_2, \ldots, a_n \in P$. Letting $k \to \infty$ in the above inequality and using the definition of $W(a)$, we see that

$$W(a_1, a_2, \ldots, a_n) = 0$$

for all $a_1, a_2, \ldots, a_n \in P$. Hence W satisfies (1.1). To show that W is unique, let $H(a)$ be an another cubic mapping satisfying (1.1) and (2.3). Then

$$\begin{aligned} \left\| W(a) - H(a) \right\| &\leq \frac{1}{2^{3l}} \left\| W(2^l a) - f(2^l a) \right\| + \left\| f(2^l a) - H(2^l a) \right\| \\ &\leq \frac{1}{8(n^2 - 5n + 6)} \sum_{k=0}^{\infty} \frac{\eta(2^{k+l} a, 2^{k+1} a, 0, \ldots, 0)}{2^{3(k+l)}} \to 0 \quad \text{as} \quad l \to \infty \end{aligned}$$

for all $a \in P$. Hence W is unique. Now, replacing a by $\frac{a}{2}$ in (2.5), we have

$$\left\| (n^2 - 5n + 6) f(a) - 8(n^2 - 5n + 6) f\left(\frac{a}{2}\right) \right\| \leq \eta\left(\frac{a}{2}, \frac{a}{2}, 0, \ldots, 0\right) \tag{2.11}$$

for all $a \in P$. The rest of the proof is similar to that when $\kappa = 1$. This completes the proof of the theorem. □

The following corollary about the stability of (1.1) is obtained as an immediate application of Theorem 2.1.

Corollary 2.1. *Let β and γ be non-negative real numbers. If there exists a function $Q : P \to V$ satisfying the inequality*

$$\left\| Q(a_1, a_2, \ldots, a_n) \right\| \leq \begin{cases} \beta, \\ \beta \left\{ \sum_{i=1}^{n} ||a_i||^{\gamma} \right\}, \\ \beta \left\{ \prod_{i=1}^{n} ||a_i||^{\gamma} + \sum_{i=1}^{n} ||a_i||^{n\gamma} \right\} \end{cases} \tag{2.12}$$

for all $a_1, a_2, \ldots, a_n \in P$, then there exists a unique cubic function $W : P \to V$ such that

$$\left\| f(a) - W(a) \right\| \leq \begin{cases} \frac{\beta}{|7|} \\ \frac{2\beta ||a||^{\gamma}}{(n^2 - 5n + 6)|2^3 - 2^{\gamma}|}; & \gamma \neq 3 \\ \frac{2\beta ||a||^{n\gamma}}{(n^2 - 5n + 6)|2^3 - 2^{n\gamma}|}; & \gamma \neq \frac{3}{n} \end{cases} \tag{2.13}$$

for all $a \in P$.

3. Stability Results: Fixed Point Method

In this section, we use Banach's Contraction Principle and the alternative fixed point theorem to study the generalized Hyers-Ulam stability of the n-dimensional cubic functional equation (1.1) in Banach spaces using a fixed point approach.

Theorem 3.1. *Let $Q : P \to V$ be a mapping for which there exists a function $\eta : P^n \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\eta(\varsigma_i^k \mathsf{a}_1, \varsigma_i^k \mathsf{a}_2, \ldots, \varsigma_i^k \mathsf{a}_n)}{\varsigma_i^{3k}} = 0, \tag{3.1}$$

where $\varsigma_i = \begin{cases} 2, & \text{if } i = 0, \\ \frac{1}{2}, & \text{if } i = 1, \end{cases}$ *and such that it satisfies the functional inequality*

$$\left\| Q(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n) \right\| \le \eta(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n) \tag{3.2}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n \in P$. If there exists $L = L(i)$ such that the function

$$\mathsf{a} \to \xi(\mathsf{a}) = \frac{1}{(n^2 - 5n + 6)} \eta\left(\frac{\mathsf{a}}{2}, \frac{\mathsf{a}}{2}, 0, \ldots, 0\right)$$

has the property

$$\frac{\xi(\varsigma_i \mathsf{a})}{\varsigma_i^3} = L\xi(\mathsf{a}) \tag{3.3}$$

for all $\mathsf{a} \in P$, then there exists a unique cubic function $W : P \to V$ satisfying the functional equation (1.1) and such that

$$\left\| f(\mathsf{a}) - W(\mathsf{a}) \right\| \le \frac{L^{1-i}}{1-L}\, \xi(\mathsf{a}) \tag{3.4}$$

for all $\mathsf{a} \in P$.

Proof. Consider the set $\Omega = \{\varkappa_1 / \varkappa_1 : P \to V, \varkappa_1(0) = 0\}$ and introduce the generalized metric on Ω,

$$d(\varkappa_1, \varkappa_2) = \inf\{k \in (0, \infty) : \|\varkappa_1(\mathsf{a}) - \varkappa_2(\mathsf{a})\| \le k\xi(\mathsf{a}), \mathsf{a} \in W\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $T\varkappa_1(\mathsf{a}) = \frac{1}{\varsigma_i^3}\varkappa_1(\varsigma_i \mathsf{a})$, for all $\mathsf{a} \in P$. For $\varkappa_1, \varkappa_2 \in \Omega$ and $\mathsf{a} \in P$, we have that

$$\begin{aligned} d(\varkappa_1, \varkappa_2) &= k \\ \Rightarrow \|\varkappa_1(\mathsf{a}) - \varkappa_2(\mathsf{a})\| &\le k\xi(\mathsf{a}) \\ \Rightarrow \left\| \frac{\varkappa_1(\varsigma_i \mathsf{a})}{\varsigma_i^3} - \frac{\varkappa_2(\varsigma_i \mathsf{a})}{\varsigma_i^3} \right\| &\le \frac{1}{\varsigma_i^3} k\xi(\varsigma_i \mathsf{a}) \\ \Rightarrow \left\| T\varkappa_1(\mathsf{a}) - T\varkappa_2(\mathsf{a}) \right\| &\le \frac{1}{\varsigma_i^3} k\xi(\varsigma_i \mathsf{a}) \\ \Rightarrow d(T\varkappa_1, T\varkappa_2) &\le kL. \end{aligned}$$

That is, $d(T\varkappa_1, T\varkappa_2) \leq Ld(\varkappa_1, \varkappa_2)$. Therefore T is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (2.5) that

$$\|(n^2-5n+6)f(2\mathsf{a})-8(n^2-5n+6)f(\mathsf{a})\| \leq \eta(\mathsf{a},\mathsf{a},0,\dots,0) \tag{3.5}$$

for all $\mathsf{a} \in P$. It follows from (3.5) that

$$\left\|\frac{f(2\mathsf{a})}{2^3}-f(\mathsf{a})\right\| \leq \frac{\eta(\mathsf{a},\mathsf{a},0,\dots,0)}{2^3(n^2-5n+6)} \tag{3.6}$$

for all $\mathsf{a} \in P$. Using (3.3) for the case $i=0$, our result is reduced to

$$\begin{aligned} \left\|f(\mathsf{a})-\frac{f(2\mathsf{a})}{2^3}\right\| &\leq \frac{1}{2^3(n^2-5n+6)}\xi(\mathsf{a}) \\ \Rightarrow \left\|f(\mathsf{a})-T\varkappa_1(\mathsf{a})\right\| &\leq L\xi(\mathsf{a}) \end{aligned}$$

for all $\mathsf{a} \in P$. Hence, we obtain

$$d(Tf(\mathsf{a}), f(\mathsf{a})) \leq L = L^{1-i} < \infty \tag{3.7}$$

for all $\mathsf{a} \in P$. Replacing a by $\frac{\mathsf{a}}{2}$ in (3.6), we have

$$\left\|\frac{f(\mathsf{a})}{2^3}-f\left(\frac{\mathsf{a}}{2}\right)\right\| \leq \frac{\eta\left(\frac{\mathsf{a}}{2},\frac{\mathsf{a}}{2},0,\dots,0\right)}{8(n^2-5n+6)} \tag{3.8}$$

for all $\mathsf{a} \in P$. Using (3.3) for the case $i=1$, our result is reduced to

$$\begin{aligned} \left\|8f\left(\frac{\mathsf{a}}{2}\right)-f(\mathsf{a})\right\| &\leq \xi(\mathsf{a}) \\ \Rightarrow \|Tf(\mathsf{a})-f(\mathsf{a})\| &\leq \xi(\mathsf{a}) \end{aligned}$$

for all $\mathsf{a} \in P$. Hence, we get

$$d(f(\mathsf{a})-, Tf(\mathsf{a})) \leq \frac{1}{2^3} = L^{1-i} \tag{3.9}$$

for all $\mathsf{a} \in P$. From (3.7) and (3.9), we conclude

$$d(f(\mathsf{a})-Tp(\mathsf{a})) \leq L^{1-i} < \infty \tag{3.10}$$

for all $\mathsf{a} \in P$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point W of T in Ω such that

$$W(\mathsf{a}) = \lim_{k\to\infty}\frac{f(\varsigma_i^k\mathsf{a})}{\varsigma_i^{3k}} \tag{3.11}$$

for all $\mathsf{a} \in P$. In order to prove $W : P \to V$ satisfies the functional equation (1.1), we use an argument similar to that in the proof of Theorem 3.1. Since W is a unique fixed point of T in the set

$$\Lambda = \{f \in \Omega \,/\, d(f, W) < \infty\},$$

W is a unique function such that

$$d(f, W) \leq \frac{1}{1-L}d(f, Tf)$$

$$\begin{aligned}\Rightarrow d(f,W) &\leq \frac{L^{1-i}}{1-L}\\ \Rightarrow \|f(\mathsf{a})-W(\mathsf{a})\| &\leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a})\end{aligned}$$

for all $\mathsf{a} \in P$. This completes the proof of the Theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.1. *Let β and γ be non-negative real numbers. If a function $Q : P \to V$ satisfies the inequality (2.12) for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n \in P$, then there exists a unique cubic function such that (2.13) for all $\mathsf{a} \in P$.*

Proof. We set

$$\eta(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n) \leq \begin{cases}\beta, \\ \beta\left\{\sum_{i=1}^{n} \|\mathsf{a}_i\|^{\gamma}\right\}, \\ \beta\left\{\prod_{i=1}^{n} \|\mathsf{a}_i\|^{\gamma} + \sum_{i=1}^{n} \|\mathsf{a}_i\|^{n\gamma}\right\},\end{cases}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n \in P$. Now

$$\begin{aligned}\frac{\eta(\varsigma_i^k \mathsf{a}_1, \varsigma_i^k \mathsf{a}_2, \ldots, \varsigma_i^k \mathsf{a}_n)}{\varsigma_i^{3k}} &= \begin{cases}\frac{\beta}{\varsigma_i^{3k}}, \\ \frac{\beta}{\varsigma_i^{3k}}\left\{\sum_{i=1}^{n} \|\varsigma_i \mathsf{a}_i\|^{\gamma}\right\}, \\ \frac{\beta}{\varsigma_i^{3k}}\left\{\prod_{i=1}^{n} \|\varsigma_i \mathsf{a}_i\|^{\gamma} + \sum_{i=1}^{n} \|\varsigma_i \mathsf{a}_i\|^{n\gamma}\right\},\end{cases}\\ &= \begin{cases}\to 0 \quad \text{as } k \to \infty, \\ \to 0 \quad \text{as } k \to \infty, \\ \to 0 \quad \text{as } k \to \infty,\end{cases}\end{aligned}$$

i.e., (3.1) holds. Since we have

$$\xi(\mathsf{a}) = \frac{1}{(n^2-5n+6)}\eta\left(\frac{\mathsf{a}}{2}, \frac{\mathsf{a}}{2}, 0, \ldots, 0\right),$$

then

$$\xi(\mathsf{a}) = \frac{1}{(n^2-5n+6)}\eta\left(\frac{\mathsf{a}}{2}, \frac{\mathsf{a}}{2}, 0, \ldots, 0\right) = \begin{cases}\frac{\beta}{7}, \\ \frac{2\beta\|\mathsf{a}\|^{\gamma}}{2^{\gamma}}, \\ \frac{2\beta\|\mathsf{a}\|^{n\gamma}}{2^{n\gamma}}.\end{cases}$$

Also,

$$\frac{1}{\varsigma_i^3}\xi(\varsigma_i \mathsf{a}) = \begin{cases}\frac{\beta}{\varsigma_i^3}, \\ \frac{\beta}{\varsigma_i^3(n^2-5n+6)}\frac{\|\mathsf{a}\|^{\gamma}\varsigma_i^{\gamma}}{2^{\gamma}}, \\ \frac{\beta}{\varsigma_i^3(n^2-5n+6)}\frac{\|\mathsf{a}\|^{n\gamma}\varsigma_i^{n\gamma}}{2^{n\gamma}},\end{cases} = \begin{cases}\varsigma_i^{-3}\xi(\mathsf{a}), \\ \varsigma_i^{\gamma-3}\xi(\mathsf{a}), \\ \varsigma_i^{n\gamma-3}\xi(\mathsf{a}),\end{cases}$$

for all $\mathsf{a} \in P$. Hence the equality (1.1) holds for the following cases:
$L = 2^{-3}$ if $i = 0$ and $L = 2^{3}$ if $i = 1$;
$L = 2^{\gamma-3}$ for $\gamma < 3$ if $i = 0$ and $L = 2^{3-\gamma}$ for $\gamma > 3$ if $i = 1$;
$L = 2^{n\gamma-3}$ for $\gamma < \frac{3}{n}$ if $i = 0$ and $L = 2^{3-n\gamma}$ for $\gamma > \frac{3}{n}$ if $i = 1$.
Now from (3.4), we prove the following cases:
Case 1: $L = 2^{-3}$ if $i = 0$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{2^{-3}\beta}{1-2^{-3}} = \frac{\beta}{7}$.
Case 2: $L = 2^{3}$ if $i = 1$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{\beta}{1-2^{3}} = \frac{\beta}{-7}$.
Case 3: $L = 2^{\gamma-3}$ for $\gamma < 3$ if $i = 0$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{2^{\gamma-3}}{1-2^{\gamma-3}}\frac{2\beta\|\mathsf{a}\|^{\gamma}}{2^{\gamma}(n^2-5n+6)} = \frac{2\beta\|\mathsf{a}\|^{\gamma}}{(n^2-5n+6)(2^3-2^{\gamma})}$.
Case 4: $L = 2^{3-\gamma}$ for $\gamma > 3$ if $i = 1$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{1}{1-2^{3-\gamma}}\frac{2\beta\|\mathsf{a}\|^{\gamma}}{2^{\gamma}(n^2-5n+6)} = \frac{2\beta\|\mathsf{a}\|^{\gamma}}{(n^2-5n+6)(2^{\gamma}-2^3)}$.
Case 5: $L = 2^{n\gamma-3}$ for $\gamma < \frac{3}{n}$ if $i = 0$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{2^{n\gamma-3}}{1-2^{n\gamma-3}}\frac{2\beta\|\mathsf{a}\|^{n\gamma}}{(n^2-5n+6)2^{n\gamma}} = \frac{2\beta\|\mathsf{a}\|^{n\gamma}}{(n^2-5n+6)(2^3-2^{n\gamma})}$.
Case 6: $L = 2^{3-n\gamma}$ for $\gamma > \frac{3}{n}$ if $i = 1$.
$\|f(\mathsf{a}) - W(\mathsf{a})\| \leq \frac{L^{1-i}}{1-L}\xi(\mathsf{a}) = \frac{1}{1-2^{3-n\gamma}}\frac{2\beta\|\mathsf{a}\|^{n\gamma}}{(n^2-5n+6)2^{n\gamma}} = \frac{2\beta\|\mathsf{a}\|^{n\gamma}}{(n^2-5n+6)(2^{n\gamma}-2^3)}$.
Hence the proof is complete. □

4. Application and Illustrative Example

In this section, we provide an application to illustrate the generalized Hyers–Ulam stability of the n-dimensional cubic functional equation (1.1). We construct an approximate cubic mapping and verify the stability conditions established in the previous section.

4.1. **Example.** Let $X = \mathbb{R}$ and $Y = \mathbb{R}$ be Banach spaces equipped with the usual norm. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(a) = a^3 + \epsilon a,$$

where $\epsilon > 0$ is a small perturbation parameter.

Define the deviation operator corresponding to (1.1) by $Q(a_1, a_2, \ldots, a_n)$.

Then, by substituting $f(a)$ into the functional equation and simplifying, we obtain

$$|Q(a_1, a_2, \ldots, a_n)| \leq C\epsilon \sum_{i=1}^{n} |a_i|,$$

for some constant $C > 0$ depending only on n.

Thus, the inequality

$$\|Q(a_1, a_2, \ldots, a_n)\| \leq \eta(a_1, a_2, \ldots, a_n)$$

holds with

$$\eta(a_1, a_2, \ldots, a_n) = C\epsilon \sum_{i=1}^{n} \|a_i\|.$$

Since η satisfies the conditions of Theorem 2.1, there exists a unique cubic mapping $W : \mathbb{R} \to \mathbb{R}$ such that

$$|f(a) - W(a)| \leq K\epsilon|a|,$$

for some constant $K > 0$.

Clearly, the exact cubic mapping is

$$W(a) = a^3.$$

4.2. **Numerical Illustration.** We compare the approximate function $f(a)$ and the exact cubic mapping $W(a) = a^3$ for selected values of a.

a	$f(a) = a^3 + \epsilon a$	$W(a) = a^3$	$\lvert f(a) - W(a)\rvert$
0.5	$0.125 + 0.5\epsilon$	0.125	0.5ϵ
1	$1 + \epsilon$	1	ϵ
1.5	$3.375 + 1.5\epsilon$	3.375	1.5ϵ
2	$8 + 2\epsilon$	8	2ϵ

The above table confirms that the deviation is controlled linearly by $\epsilon|a|$, consistent with the theoretical estimate.

4.3. **Graphical Representation.**

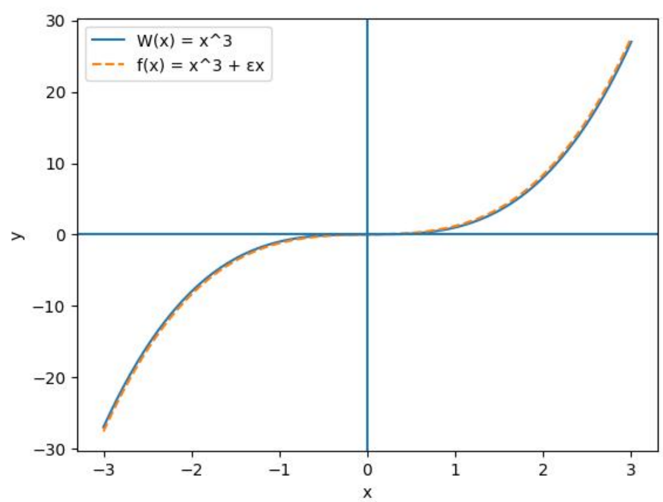

Figure 1. Comparison between $f(a) = a^3 + \epsilon a$ and $W(a) = a^3$.

The Figure 1 illustrates that the perturbed function remains close to the exact cubic mapping, demonstrating the Hyers–Ulam stability.

This example demonstrates that an approximate cubic mapping satisfying a bounded deviation from the functional equation is close to an exact cubic solution. Hence, the generalized Hyers–Ulam stability results obtained via the fixed point method are validated in a concrete setting.

5. Conclusion

This study proves the generalized Hyers-Ulam stability of a n-dimensional cubic functional equation in Banach spaces. By using both direct methods and fixed point approaches, we were able to provide sufficient criteria for the existence and uniqueness of accurate cubic mappings fitting the given functional equation.

We were able to get explicit stability bounds under suitable control functions by applying Banach's Contraction Principle and the alternative fixed point theorem. These findings verify that, in a clear sense, approximation solutions stay close to exact cubic mappings.

Apart from the theoretical progress, an application was showcased to demonstrate the reliability of the outcomes. It was demonstrated that the calculated stability estimations were satisfied by the departure of a perturbed cubic function from the exact cubic mapping. The theoretical results were validated by the numerical demonstration and tabular data, which further confirmed that the deviation is effectively managed.

The approach developed in this work not only extends existing results in the literature but also demonstrates practical applicability. Future work may focus on extending these results to non-Archimedean spaces, fuzzy normed spaces, and higher-order functional equations.

The methodology established in this paper shows practical application in addition to extending previous findings in the literature. Future research might concentrate on expanding these findings to higher-order functional equations, fuzzy normed spaces, and non-Archimedean spaces.

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