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ON THE DEGREE OF APPROXIMATION OF A FUNCTION BY $(C,1)(E,q)$ MEANS OF ITS FOURIER-LAGUERRE SERIES

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Abstract. In this note a theorem on the degree of approximation of a function by $(C,1)(E,q)$ means of its Fourier-Laguerre series at the frontier point $x=0$ is proved.

1. Introduction

Let us consider the infinite series $\sum_{n=0}^{\infty} u_n$ with the sequence of its n-th partial sums $s := \{s_n\}$.

If for $q > 0$

$$E_n^q(s) = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^k s_k \to s_1 \quad \text{as} \quad n\to\infty, \tag{1.1}$$

then it is said that $s := \{s_n\}$ is summable by (E,q) means (see Hardy [3]), and we write $s_n \to s_1(E,q)$.

The Fourier-Laguerre expansion of a function $f(x) \in L(0,\infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.2}$$

where

$$a_n = \frac{1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}}\int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha)}(y)dy, \tag{1.3}$$

$L_n^{(\alpha)}(x)$ denotes the n-th Laguerre polynomial of order $\alpha > -1$, defined by generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = \frac{e^{\frac{x\omega}{\omega-1}}}{(1-\omega)^{\alpha+1}}, \tag{1.4}$$

and it is assumed that the integral (1.3) exists.

In 1971, D. P. Gupta [2] estimated the order of the function by Cesáro means of series (1.2) at the point $x = 0$, after replacing the continuity condition in Szegö's theorem [6] by a much lighter condition. He proved the following theorem.

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Theorem 1.1 ([2]). *If*

$$F(t)=\int_0^t \frac{|f(y)|}{y}dy = o\left(\log\left(\frac{1}{t}\right)\right)^{1+p}, \quad t\to 0, -1<p<\infty,$$

and

$$\int_1^\infty e^{-y/2}y^{(3\alpha-3k-1)/3}|f(y)|dy<\infty,$$

are fulfilled, then

$$\sigma_n^k(0)=o\left(\log n\right)^{p+1}$$

provided that $k>\alpha+1/2$, $\alpha>-1$, *with* $\sigma_n^k(0)$ *being the* n*-th Cesàro mean of order* k.

Further, we use the notation

$$\phi(y)=\frac{e^{-y}y^{\alpha}[f(y)-f(0)]}{\Gamma(\alpha+1)}, \tag{1.5}$$

and denote by t_n harmonic means of the series (1.2). T. Singh [5] estimated the deviation $t_n(x)-f(x)$ at the point $x=0$ by some weaker conditions than those of Theorem 1.1. Namely, he verified the following theorem.

Theorem 1.2 ([5]). *For* $\alpha\in(-5/6,-1/2)$

$$t_n(0)-f(0)=o\left(\log n\right)^{p+1}$$

provided that

$$\int_t^\delta \frac{|\phi(y)|}{y^{\alpha+1}}dy=o\left(\log\left(\frac{1}{t}\right)\right)^{1+p}, \quad t\to 0, -1<p<\infty,$$

$$\int_\delta^n e^{y/2}y^{-(2\alpha+3)/4}|\phi(y)|dy=o\left(n^{-(2\alpha+1)/4}\left(\log n\right)^{p+1}\right), \tag{1.6}$$

and

$$\int_n^\infty e^{y/2}y^{-1/3}|\phi(y)|dy=o\left(\left(\log n\right)^{p+1}\right), \quad n\to\infty,$$

where δ *is a fixed positive constant.*

Very recently, Nigam and Sharma [4] proved a theorem of such type using $(E,1)$ means which is entirely different from (C,k) and harmonic means of the series (1.2), they employed a condition which is weaker than condition (1.6), and increased the range of α to $(-1,-1/2)$ which is more appropriate for applications. In their paper they established the following statement.

Theorem 1.3 ([4]). *If*

$$E_n^1=\frac{1}{2^n}\sum_{k=0}^n\binom{n}{k}s_k\to\infty \quad as \quad n\to\infty, \tag{1.7}$$

then the degree of approximation of Fourier-Laguerre expansion (1.2) at the point $x=0$ *by* $(E,1)$ *means* E_n^1 *is given by*

$$E_n^1(0)-f(0)=o\left(\xi(n)\right) \tag{1.8}$$

provided that

$$\Phi(t)=\int_0^t|\phi(y)|dy=o\left(t^{\alpha+1}\xi\left(\frac{1}{t}\right)\right), \quad t\to 0, \tag{1.9}$$

$$\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left(n^{-(2\alpha+1)/4} \xi(n)\right), \tag{1.10}$$

and

$$\int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o\left(\xi(n)\right), \quad n \to \infty, \tag{1.11}$$

where δ is a fixed positive constant, $\alpha \in (-1, -1/2)$, and $\xi(t)$ is a positive monotonic increasing function of t such that $\xi(n) \to \infty$ as $n \to \infty$.

As is pointed out in [1] the infinite series

$$1 - 4 \sum_{n=1}^{\infty} (-3)^{n-1}$$

is not $(E, 1)$ summable nor $(C, 1)$ summable. However, it is proved that the above series is $(C, 1)(E, 1)$ summable. Therefore the product summability $(C, 1)(E, 1)$ is more powerful than the individual methods $(C, 1)$ and $(E, 1)$. Thus, $(C, 1)(E, 1)$ mean gives an approximation for a wider class of Fourier-Laguerre series than the individual methods $(C, 1)$ and $(E, 1)$. The main aim of this paper is to prove the counterpart of the Theorem 1.3 using the product mean $(C, 1)(E, q)$, which obviously, based on what we discussed above, will give more general results. To achieve this aim we need an auxiliary result (see [6], page 175).

Lemma 1.1. *Let α be arbitrary and real, c and d be fixed positive constants, and let $n \to \infty$. Then*

$$L_n^{(\alpha)}(x) = O\left(n^{\alpha}\right), \quad \text{if} \quad 0 \le x \le \frac{c}{n} \tag{1.12}$$

and

$$L_n^{(\alpha)}(x) = O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right) \quad \text{if} \quad \frac{c}{n} \le x \le d. \tag{1.13}$$

2. Main Result

We prove the following theorem.

Theorem 2.1. *Te degree of approximation of Fourier-Laguerre expansion (1.2) at the point $x = 0$ by $(C, 1)(E, q)$, $q \ge 1$ means $[(C, 1)(E, q)]_n$ is given by*

$$[(C, 1)(E, q)]_n(0) - f(0) = o\left(\xi(n)\right)$$

provided that

$$\Phi(t) = \int_0^t |\phi(y)| dy = o\left(t^{\alpha+1} \xi\left(\frac{1}{t}\right)\right), \quad t \to 0, \tag{2.1}$$

$$\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left(n^{-(2\alpha+1)/4} \xi(n)\right), \tag{2.2}$$

and

$$\int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o\left(\xi(n)\right), \quad n \to \infty, \tag{2.3}$$

where δ is a fixed positive constant, $\alpha \in (-1, -1/2)$, and $\xi(t)$ is a positive monotonic increasing function of t such that $\xi(n) \to \infty$ as $n \to \infty$.

Proof. Based on the equality

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{\alpha}, \tag{2.4}$$

we obtain

$$\begin{aligned} s_n(0) &= \sum_{k=0}^{n} a_k L_n^{(\alpha)}(0) \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y) dy \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y) dy. \end{aligned} \tag{2.5}$$

Thus,

$$\begin{aligned} [(E,q)]_n(0) &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k s_k(0) \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{q^k}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha+1)}(y) dy, \end{aligned}$$

and

$$\begin{aligned} [(C,1)(E,q)]_n(0) &= \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k s_k(0) \\ &= \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} \frac{q^k}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha+1)}(y) dy. \end{aligned} \tag{2.6}$$

Therefore, using (1.5) we have

$$\begin{aligned} &(C,1)(E_n^q)(0) - f(0) = \\ &= \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k \int_0^\infty \phi(y) L_k^{(\alpha+1)}(y) dy \\ &= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} + \int_n^\infty \right) \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k \phi(y) L_k^{(\alpha+1)}(y) dy \\ &:= \sum_{m=0}^{4} r_m. \end{aligned} \tag{2.7}$$

Using the property of the orthogonality, condition (2.1) and Lemma 1.1, we obtain

$$\begin{aligned} r_1 &= \frac{1}{n+1}\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k\mathcal{O}\left(k^{\alpha+1}\right)\int_0^{1/n}|\phi(y)|dy \\ &= \frac{1}{n+1}\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k\mathcal{O}\left(n^{\alpha+1}\right)o\left(\frac{\xi(n)}{n^{\alpha+1}}\right) \\ &= o\left(\frac{1}{n+1}\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k\xi(n)\right) \\ &= o\left(\xi(n)\right), \qquad (2.8) \end{aligned}$$

since

$$\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k = n+1.$$

Again, using the property of the orthogonality and Lemma 1.1, we have

$$r_2 = \frac{1}{n+1}\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k\mathcal{O}\left(k^{(2\alpha+1)/4}\right)\int_{1/n}^{\delta}y^{(2\alpha+3)/4}|\phi(y)|dy.$$

Since

$$\begin{aligned} \sum_{k=0}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4} &= \sum_{k=0}^{[\frac{v}{2}]}\binom{v}{k}q^k k^{(2\alpha+1)/4} + \sum_{k=[\frac{v}{2}]+1}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4} \\ &\le \sum_{k=0}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4} + \binom{v}{[\frac{v}{2}]}\sum_{k=[\frac{v}{2}]+1}^{v}q^k k^{(2\alpha+1)/4} \\ &\le (1+q)^v v^{(2\alpha+1)/4} + \binom{v}{[\frac{v}{2}]}v^{(2\alpha+5)/4}q^v \\ &= (1+q)^v v^{(2\alpha+1)/4} + \binom{v}{[\frac{v}{2}]}v^{(2\alpha+1)/4}vq^v \quad q\ge 1. \end{aligned}$$

and

$$\begin{aligned} (1+q)^v &= \sum_{k=0}^{v}\binom{v}{k}q^k \\ &= \binom{v}{0}q^0 + \binom{v}{1}q^1 + \cdots + \binom{v}{[\frac{v}{2}]}q^{[\frac{v}{2}]} + \binom{v}{[\frac{v}{2}]+1}q^{[\frac{v}{2}]+1} + \cdots + \binom{v}{v}q^v \\ &\ge \binom{v}{[\frac{v}{2}]}q^{[\frac{v}{2}]} + \binom{v}{[\frac{v}{2}]+1}q^{[\frac{v}{2}]+1} + \cdots + \binom{v}{v}q^v \\ &\ge \left[\binom{v}{[\frac{v}{2}]} + \binom{v}{[\frac{v}{2}]} + \cdots + \binom{v}{[\frac{v}{2}]}\right]q^{[\frac{v}{2}]} \\ &\ge K\left(\left[\frac{v}{2}\right]+1\right)\binom{v}{[\frac{v}{2}]}q^v \ge \frac{K}{2}v\binom{v}{[\frac{v}{2}]}q^v, \quad (\text{for } K\le 1/q), \end{aligned}$$

then

$$\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4} \quad \leq \quad \left(1+\frac{2}{K}\right)v^{(2\alpha+1)/4}.$$

and moreover,

$$\frac{1}{n+1}\sum_{v=0}^{n}\frac{1}{(1+q)^v}\sum_{k=0}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4} = \mathcal{O}\left(n^{(2\alpha+1)/4}\right).$$

Using latter estimation, and doing the same reasoning as in [4] page 6, we obtain

$$r_2 = \mathcal{O}\left(n^{(2\alpha+1)/4}\right)\int_{1/n}^{\delta} y^{(2\alpha+3)/4}|\phi(y)|dy = \mathcal{O}\left(\xi(n)\right). \tag{2.9}$$

Further we estimate r_3:

$$\begin{aligned} r_3 \quad &\leq \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\int_{\delta}^{n} e^{y/2}y^{-(2\alpha+3)/4}|\phi(y)|e^{-y/2}y^{(2\alpha+3)/4}|L_k^{(\alpha+1)}(y)|dy \\ &= \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\mathcal{O}\left(k^{(2\alpha+1)/4}\int_{\delta}^{n} e^{y/2}y^{-(2\alpha+3)/4}|\phi(y)|dy\right) \\ &= \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\mathcal{O}\left(k^{(2\alpha+1)/4}o\left(n^{-(2\alpha+1)/4}\xi(n)\right)\right) \\ (2.10) \quad & o\left(\xi(n)\right). \end{aligned}$$

Finally, we have

$$\begin{aligned} r_4 \quad &\leq \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\int_{n}^{\infty} e^{y/2}y^{-(3\alpha+5)/6}|\phi(y)|e^{-y/2}y^{(3\alpha+5)/6}|L_k^{(\alpha+1)}(y)|dy \\ &= \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\mathcal{O}\left(k^{(\alpha+1)/4}\int_{n}^{\infty}\frac{e^{y/2}|\phi(y)|}{y^{(\alpha+1)/2+1/3}}dy\right) \\ &= \quad \frac{1}{n+1}\sum_{v=0}^{n}\frac{\sum_{k=0}^{v}\binom{v}{k}q^k}{(1+q)^v}\mathcal{O}\left(k^{(\alpha+1)/2}k^{-(\alpha+1)/2}o\left(\xi(n)\right)\right) \\ (2.11) \quad & o\left(\xi(n)\right). \end{aligned}$$

Now, putting estimations (2.8)-(2.11) into (2.7) we obtain

$$[(C,1)(E,q)]_n(0) - f(0) = o\left(\xi(n)\right).$$

The proof of the theorem is completed. □

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