Int. J. Anal. Appl. (2026), 24:276

International Journal of Analysis and Applications

A Two-Step Inertial Picard–CR Framework for Fixed Point Problems with Applications to Optimization and Image Recovery

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Abstract. This paper proposes a two-step inertial Picard–CR iterative framework for approximating common fixed points of a countable family of nonexpansive mappings in real Hilbert spaces. The method combines a viscosity approximation with two inertial corrections and a Picard–CR-type fixed-point update. Under suitable control conditions and an asymptotic compatibility requirement on the intermediate iterates, strong convergence to the viscosity-selected common fixed point is established. The abstract fixed-point result is then specialized to a convex bilevel optimization problem through a family of forward–backward operators. An image-restoration experiment based on a regularized inverse problem is also presented. Numerical comparisons with several forward–backward and accelerated schemes illustrate the computational behavior of the proposed approach.

1. Introduction

Fixed point theory for nonlinear operators in Hilbert spaces is a fundamental component of modern nonlinear analysis and has important applications in optimization, inverse problems, signal processing, and related computational models [3,8,31]. Iterative methods for approximating fixed points of nonlinear mappings remain central because they provide constructive tools for solving a wide class of mathematical problems [6, 13]. Let H be a real Hilbert space and let $T : H \to H$. The fixed point problem consists of finding $x \in H$ such that $Tx = x$. This formulation is closely related to convex feasibility problems, equilibrium problems, and variational inequalities [10, 11, 15].

Received: Apr. 5, 2026.

2020 *Mathematics Subject Classification.* 47H10; 65Y05; 90C25; 90C30.

Key words and phrases. fixed point theory; nonexpansive mappings; two-step inertia; Picard–CR method; viscosity approximation; strong convergence; bilevel optimization; image recovery.

https://doi.org/10.28924/2291-8639-24-2026-276
ISSN: 2291-8639

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Among the most studied operators in this framework are nonexpansive mappings, characterized by

$$\|Tx - Ty\| \leq \|x - y\|, \qquad x, y \in H. \tag{1.1}$$

Such mappings arise naturally in projection methods and monotone operator theory [3,17]. However, classical fixed-point iterations may yield only weak convergence in infinite-dimensional Hilbert spaces, which motivates the development of strongly convergent modifications [16,19].

Viscosity approximation methods provide one important mechanism for obtaining strong convergence. The basic idea is to combine a nonexpansive mapping with a contraction so that the generated sequence converges to a distinguished fixed point satisfying an associated variational inequality [20,21,37].

Inertial techniques provide another important line of development. Inspired by Polyak's heavy-ball method [29] and Nesterov-type acceleration [22], inertial schemes use information from previous iterates to introduce momentum into the iterative process. Two-step and double-inertial constructions exploit information from several previous iterates and have been successfully incorporated into variational-inequality, fixed-point and image-recovery algorithms; see, for example, [9,18,23,25,26].

Picard-type hybrid iterations have also been developed to improve the computational behavior of classical fixed-point schemes. For example, the Picard–S^* method studied by Lamba and Panwar [30] combines several fixed-point evaluations within a recursive process, while related Picard–S constructions for common fixed-point approximation were investigated in [24]. Motivated by these developments, we consider a two-step inertial Picard–CR-type framework that combines viscosity approximation, two inertial terms, and repeated nonexpansive updates.

The main contributions of this paper are as follows. First, we introduce a two-step inertial Picard–CR algorithm for a countable family of nonexpansive mappings. Second, we establish strong convergence under suitable control conditions together with an asymptotic compatibility condition required by the viscosity argument. Third, we specialize the general fixed-point scheme to a convex bilevel optimization problem using forward–backward operators. Finally, we illustrate the computational behavior of the proposed framework on an image-restoration problem.

The remainder of the paper is organized as follows. Section 2 presents the notation and preliminary results. Section 3 introduces the proposed fixed-point algorithm and establishes its convergence. Section 4 gives an application to convex bilevel optimization. Section 5 discusses the image-restoration experiment. Section 6 concludes the paper.

2. Preliminaries

Throughout the paper, H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. We write

$$\mathrm{Fix}(T) := \{x \in C : Tx = x\}.$$

Definition 2.1. *Let $T : C \to C$. The mapping T is called L-Lipschitz continuous if there exists $L \geq 0$ such that*

$$\|Tu - Tv\| \leq L\|u - v\|, \qquad u, v \in C.$$

If $L \in [0, 1)$, then T is a contraction. If $L = 1$, then T is nonexpansive.

Definition 2.2 ([37]). *Let $T : C \to C$ be nonexpansive and let $\{T_n\}_{n\geq 1}$ be a family of nonexpansive mappings such that*

$$\varnothing \neq \mathrm{Fix}(T) \subseteq \Gamma := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n).$$

The family $\{T_n\}$ is said to satisfy the NST-condition (I) with respect to T if, for every bounded sequence $\{x_n\} \subset C$ satisfying

$$\|x_n - T_n x_n\| \to 0,$$

one has

$$\|x_n - Tx_n\| \to 0.$$

Definition 2.3. *Let $h : H \to (-\infty, +\infty]$ be proper, lower semicontinuous and convex, and let $\lambda > 0$. The proximity operator of h is defined by*

$$\mathrm{prox}_{\lambda h}(x) := \arg\min_{y\in H} \left\{ h(y) + \frac{1}{2\lambda}\|y - x\|^2 \right\}.$$

If $f : H \to \mathbb{R}$ is convex and differentiable, the associated forward–backward operator is

$$T_\lambda := \mathrm{prox}_{\lambda h}(I - \lambda \nabla f).$$

See [3, 4, 7, 8].

Remark 2.1. *If ∇f is L_f-Lipschitz continuous and $0 < \lambda < 2/L_f$, then the forward–backward operator T_λ is averaged, and hence nonexpansive; see [3].*

Definition 2.4 ([13]). *Let $\{T_n\}_{n\geq 1}$ be a sequence of nonexpansive self-mappings of C with*

$$\Gamma := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing.$$

The family $\{T_n\}$ is said to satisfy Condition (Z) if, whenever $\{u_n\} \subset C$ is bounded and

$$\|u_n - T_n u_n\| \to 0,$$

every weak cluster point of $\{u_n\}$ belongs to Γ.

Remark 2.2. *If $\{T_n\}$ satisfies the NST-condition (I) with respect to a nonexpansive mapping T and $\mathrm{Fix}(T) = \Gamma$, then Condition (Z) follows from the demiclosedness of $I - T$ at zero.*

Lemma 2.1. *For all $x, y \in H$ and $\kappa \in [0,1]$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.1}$$

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2, \tag{2.2}$$

$$\|\kappa x + (1-\kappa)y\|^2 = \kappa\|x\|^2 + (1-\kappa)\|y\|^2 - \kappa(1-\kappa)\|x-y\|^2. \tag{2.3}$$

Lemma 2.2. *Let $\{a_n\} \subset [0,\infty)$, $\{\theta_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy*

$$a_{n+1} \le (1-\theta_n)a_n + \theta_n\delta_n, \qquad n \ge 1,$$

and

$$\sum_{n=1}^{\infty} \theta_n = \infty.$$

Suppose that, for every subsequence $\{a_{n_j}\}$ satisfying

$$\liminf_{j\to\infty}(a_{n_j+1} - a_{n_j}) \ge 0,$$

one has

$$\limsup_{j\to\infty} \delta_{n_j} \le 0.$$

Then $a_n \to 0$.

Lemma 2.3. *Let $f : H \to \mathbb{R}$ be convex with L_f-Lipschitz continuous gradient and let $h : H \to (-\infty, +\infty]$ be proper, lower semicontinuous and convex. Let*

$$T_n = \operatorname{prox}_{c_n h}(I - c_n\nabla f), \qquad T = \operatorname{prox}_{ch}(I - c\nabla f),$$

where $c_n \to c$ and $c_n, c \in (0, 2/L_f)$. Then $\operatorname{Fix}(T_n) = \operatorname{Fix}(T) = \arg\min(f+h)$. Moreover, in the standard forward–backward setting the family $\{T_n\}$ satisfies the NST-condition (I) with respect to T; see [3,8].

3. Main Results

We now introduce the proposed two-step inertial Picard–CR algorithm.

Algorithm 1 Two-Step Inertial Picard–CR Algorithm (TIPCRA)

Initialization. Choose $x_{-1}, x_0, x_1 \in H$. Let $\{\eta_n\}, \{\zeta_n\} \subset (0,1]$, $\{\gamma_n\} \subset (0,1)$, $\{\sigma_n\} \subset [0,\infty)$, and let $\{\tau_n\}, \{\xi_n\} \subset (0,\infty)$ be bounded sequences. Set $n = 1$.

Iterative steps. Given x_n, x_{n-1}, x_{n-2}, calculate $\phi_n, \psi_n, y_n, k_n, z_n, w_n$ and x_{n+1} as follows.

$$\begin{cases} \phi_n = \begin{cases} \min\left\{\tau_n, \dfrac{\eta_n\zeta_n\sigma_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \tau_n, & x_n = x_{n-1}, \end{cases} \\ \psi_n = \begin{cases} \max\left\{-\xi_n, -\dfrac{\eta_n\zeta_n\sigma_n}{\|x_{n-1} - x_{n-2}\|}\right\}, & x_{n-1} \neq x_{n-2}, \\ -\xi_n, & x_{n-1} = x_{n-2}, \end{cases} \\ y_n = x_n + \phi_n(x_n - x_{n-1}) + \psi_n(x_{n-1} - x_{n-2}). \end{cases} \tag{3.1}$$

$$k_n = (1-\eta_n)y_n + \eta_n g(y_n). \tag{3.2}$$

$$z_n = (1-\zeta_n)T_n y_n + \zeta_n T_n k_n. \tag{3.3}$$

$$w_n = (1-\gamma_n)z_n + \gamma_n T_n z_n. \tag{3.4}$$

$$x_{n+1} = T_n w_n. \tag{3.5}$$

Set $n \leftarrow n+1$ and repeat the iteration.

Theorem 3.1. *Let $\{T_n\}$ be a family of nonexpansive self-mappings of H satisfying Condition (Z), and assume*

$$\Gamma := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing.$$

Let $g : H \to H$ be a κ-contraction with $\kappa \in [0,1)$, and let $\{x_n\}$ be generated by Algorithm 1. Assume that

(B1) $\sigma_n \to 0$;

(B2) $\eta_n\zeta_n \to 0$ *and* $\sum_{n=1}^{\infty} \eta_n\zeta_n = \infty$;

(B3) *there exist constants* $0 < d_1 \le d_2 < 1$ *such that* $d_1 \le \gamma_n \le d_2$ *for all* n;

(B4) $\|k_n - z_n\| \to 0$.

Then $\{x_n\}$ converges strongly to the unique point $z^* \in \Gamma$ satisfying

$$z^* = P_\Gamma g(z^*). \tag{3.6}$$

Proof. Since Γ is a nonempty closed convex subset of H and g is a contraction, $P_\Gamma g$ is also a contraction. Hence (3.6) has a unique solution $z^* \in \Gamma$.

Set

$$\alpha_n := \eta_n \zeta_n \qquad \text{and} \qquad e_n := \phi_n(x_n - x_{n-1}) + \psi_n(x_{n-1} - x_{n-2}).$$

From (**??**) and (**??**),

$$\phi_n \|x_n - x_{n-1}\| \le \alpha_n \sigma_n, \qquad |\psi_n| \|x_{n-1} - x_{n-2}\| \le \alpha_n \sigma_n.$$

Consequently,

$$\|e_n\| \le 2\alpha_n \sigma_n, \qquad \frac{\|e_n\|}{\alpha_n} \to 0. \tag{3.7}$$

Because $z^* \in \mathrm{Fix}(T_n)$, the nonexpansivity of T_n and the contractivity of g give

$$\begin{aligned} \|z_n - z^*\| &\le (1 - \zeta_n)\|y_n - z^*\| + \zeta_n \|k_n - z^*\| \\ &\le \Big[1 - \alpha_n(1 - \kappa)\Big]\|y_n - z^*\| + \alpha_n \|g(z^*) - z^*\|. \end{aligned}$$

Moreover,

$$\|w_n - z^*\| \le \|z_n - z^*\|, \qquad \|x_{n+1} - z^*\| \le \|w_n - z^*\|.$$

Using $y_n = x_n + e_n$, we obtain

$$\|x_{n+1} - z^*\| \le \Big[1 - \alpha_n(1 - \kappa)\Big]\|x_n - z^*\| + \Big[1 - \alpha_n(1 - \kappa)\Big]\|e_n\| + \alpha_n \|g(z^*) - z^*\|. \tag{3.8}$$

Since $\sigma_n \to 0$, the sequence $\{\sigma_n\}$ is bounded. Hence (3.7) and (3.8) imply, by induction, that $\{x_n\}$ is bounded. It follows from the definitions and the boundedness of the control sequences that $\{y_n\}$, $\{k_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded.

We next derive the basic squared-norm estimate. By Lemma 2.1,

$$\|k_n - z^*\|^2 \le \Big[1 - \eta_n(1 - \kappa)\Big]\|y_n - z^*\|^2 + 2\eta_n \langle g(z^*) - z^*, k_n - z^* \rangle. \tag{3.9}$$

Using (**??**), (**??**), (**??**), Lemma 2.1, and the nonexpansivity of T_n, we obtain

$$\begin{aligned} \|x_{n+1} - z^*\|^2 \le \Big[1 - \alpha_n(1 - \kappa)\Big]\|y_n - z^*\|^2 &+ 2\alpha_n \langle g(z^*) - z^*, k_n - z^* \rangle \\ &- \gamma_n(1 - \gamma_n)\|T_n z_n - z_n\|^2. \end{aligned} \tag{3.10}$$

Since the sequences involved are bounded and (3.7) holds, there exists a real sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$\|y_n - z^*\|^2 \le \|x_n - z^*\|^2 + \alpha_n \varepsilon_n. \tag{3.11}$$

Indeed, one may take

$$\varepsilon_n = 4M\sigma_n + 4\alpha_n \sigma_n^2,$$

where $M > 0$ is a bound for $\{\|x_n - z^*\|\}$.

Let

$$a_n := \|x_n - z^*\|^2, \qquad c := 1 - \kappa > 0.$$

Combining (3.10) and (3.11), and enlarging ε_n if necessary by a null sequence, gives

$$a_{n+1} \le (1 - c\alpha_n)a_n + c\alpha_n \beta_n - \gamma_n(1 - \gamma_n)\|T_n z_n - z_n\|^2, \tag{3.12}$$

where

$$\beta_n := \frac{1}{c}\left[\varepsilon_n + 2\langle g(z^*) - z^*, k_n - z^*\rangle\right]. \tag{3.13}$$

Because $\alpha_n \to 0$ and $\sum_n \alpha_n = \infty$, we have $c\alpha_n \in (0,1)$ for all sufficiently large n and

$$\sum_{n=1}^{\infty} c\alpha_n = \infty.$$

It remains to verify the critical-subsequence condition of Lemma 2.2. Let $\{a_{n_j}\}$ satisfy

$$\liminf_{j\to\infty}(a_{n_j+1} - a_{n_j}) \geq 0.$$

The boundedness established above implies that $\{\beta_n\}$ is bounded. From (3.12), $\alpha_{n_j} \to 0$, and (B3), we obtain

$$\|T_{n_j} z_{n_j} - z_{n_j}\| \to 0.$$

Choose a further subsequence, not relabeled, such that $z_{n_j} \rightharpoonup \bar{z}$ and

$$\limsup_{j\to\infty}\langle g(z^*) - z^*, k_{n_j} - z^*\rangle = \lim_{j\to\infty}\langle g(z^*) - z^*, k_{n_j} - z^*\rangle.$$

Condition (Z) yields $\bar{z} \in \Gamma$. By (B4), $k_{n_j} - z_{n_j} \to 0$, hence $k_{n_j} \rightharpoonup \bar{z}$. Since $z^* = P_\Gamma g(z^*)$, the characterization of the metric projection gives

$$\langle g(z^*) - z^*, \bar{z} - z^*\rangle \leq 0.$$

Therefore,

$$\limsup_{j\to\infty}\langle g(z^*) - z^*, k_{n_j} - z^*\rangle \leq 0.$$

Together with $\varepsilon_n \to 0$, this implies

$$\limsup_{j\to\infty} \beta_{n_j} \leq 0.$$

Lemma 2.2 applied to (3.12) now gives $a_n \to 0$. Hence

$$x_n \to z^*$$

strongly, as required. □

Remark 3.1. *Condition (B4) is the asymptotic compatibility requirement needed to transfer the viscosity selection inequality from the intermediate point k_n to a weak cluster point controlled by Condition (Z). It should be verified for any concrete specialization of Algorithm 1.*

4. Application to a Convex Bilevel Optimization Problem

Bilevel optimization arises in areas such as hyperparameter optimization, meta-learning, and hierarchical decision models [5, 12, 28]. Related first-order, inertial and hierarchical approaches can be found in [1, 9, 32, 33]. We consider the convex bilevel problem

$$\min_{x \in S^*} \vartheta(x), \tag{4.1}$$

where

$$S^* = \arg\min_{x \in \mathbb{R}^n} \{f(x) + h(x)\}. \tag{4.2}$$

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable with L_f-Lipschitz continuous gradient, that $h : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous, and that $\vartheta : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with parameter $\mu > 0$ and has L_ϑ-Lipschitz continuous gradient.

A point $z^* \in S^*$ solves (4.1) if and only if

$$\langle \nabla\vartheta(z^*), z - z^* \rangle \geq 0, \qquad z \in S^*. \tag{4.3}$$

See, for example, [15, 32].

For $c \in (0, 2/L_f)$, define the forward–backward operator

$$T_c = \operatorname{prox}_{ch}(I - c\nabla f). \tag{4.4}$$

Then

$$\operatorname{Fix}(T_c) = S^*.$$

This fixed-point formulation is standard in composite convex optimization [3, 8, 27].

We impose the following assumptions:

(C1) f and h satisfy the assumptions stated above, and $c_n \to c$ with $c_n, c \in (0, 2/L_f)$.

(C2) ϑ is μ-strongly convex and $\nabla\vartheta$ is L_ϑ-Lipschitz continuous. Choose

$$0 < s < \frac{2}{\mu + L_\vartheta}$$

and define

$$g_s := I - s\nabla\vartheta.$$

Algorithm 2 Two-Step Inertial Bilevel Gradient Picard–CR Algorithm (TIBiG-PCRA)

Initialization. Choose $x_{-1}, x_0, x_1 \in \mathbb{R}^n$. Let $\{\eta_n\}, \{\zeta_n\} \subset (0,1]$, $\{\gamma_n\} \subset (0,1)$, $\{\sigma_n\} \subset [0,\infty)$, and let $\{\tau_n\}, \{\xi_n\} \subset (0,\infty)$ be bounded sequences. Choose

$$c_n \in \left(0, \frac{2}{L_f}\right), \qquad c_n \to c \in \left(0, \frac{2}{L_f}\right),$$

and set

$$g_s = I - s\nabla\vartheta, \qquad 0 < s < \frac{2}{\mu + L_\vartheta}.$$

Set $n = 1$.

Iterative steps. Given x_n, x_{n-1}, x_{n-2} and c_n, calculate $\phi_n, \psi_n, y_n, k_n, u_n, v_n, z_n, r_n, w_n$ and x_{n+1} as follows.

$$\begin{cases} \phi_n = \begin{cases} \min\left\{\tau_n, \dfrac{\eta_n\zeta_n\sigma_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \tau_n, & x_n = x_{n-1}, \end{cases} \\ \psi_n = \begin{cases} \max\left\{-\xi_n, -\dfrac{\eta_n\zeta_n\sigma_n}{\|x_{n-1} - x_{n-2}\|}\right\}, & x_{n-1} \neq x_{n-2}, \\ -\xi_n, & x_{n-1} = x_{n-2}, \end{cases} \\ y_n = x_n + \phi_n(x_n - x_{n-1}) + \psi_n(x_{n-1} - x_{n-2}). \end{cases} \tag{4.5}$$

$$k_n = (1-\eta_n)y_n + \eta_n g_s(y_n) = y_n - \eta_n s\nabla\vartheta(y_n). \tag{4.6}$$

$$\begin{cases} u_n = \operatorname{prox}_{c_n h}\Big(y_n - c_n\nabla f(y_n)\Big), \\ v_n = \operatorname{prox}_{c_n h}\Big(k_n - c_n\nabla f(k_n)\Big), \\ z_n = (1-\zeta_n)u_n + \zeta_n v_n. \end{cases} \tag{4.7}$$

$$\begin{cases} r_n = \operatorname{prox}_{c_n h}\Big(z_n - c_n\nabla f(z_n)\Big), \\ w_n = (1-\gamma_n)z_n + \gamma_n r_n, \\ x_{n+1} = \operatorname{prox}_{c_n h}\Big(w_n - c_n\nabla f(w_n)\Big). \end{cases} \tag{4.8}$$

Set $n \leftarrow n+1$ and repeat the iteration.

Proposition 4.1. *Under (C2), the mapping $g_s = I - s\nabla\vartheta$ is a contraction. More precisely,*

$$\|g_s(a) - g_s(b)\| \leq \sqrt{1 - \frac{2s\mu L_\vartheta}{\mu + L_\vartheta}}\,\|a - b\|, \qquad a, b \in \mathbb{R}^n. \tag{4.9}$$

Proof. For a μ-strongly convex function with L_ϑ-Lipschitz continuous gradient,

$$\langle \nabla\vartheta(a) - \nabla\vartheta(b), a-b\rangle \geq \frac{\mu L_\vartheta}{\mu + L_\vartheta}\|a-b\|^2 + \frac{1}{\mu + L_\vartheta}\|\nabla\vartheta(a) - \nabla\vartheta(b)\|^2.$$

Consequently,

$$\begin{aligned}\|g_s(a) - g_s(b)\|^2 &= \|a-b\|^2 - 2s\langle \nabla\vartheta(a) - \nabla\vartheta(b), a-b\rangle \\ &\quad + s^2\|\nabla\vartheta(a) - \nabla\vartheta(b)\|^2 \\ &\leq \left(1 - \frac{2s\mu L_\vartheta}{\mu + L_\vartheta}\right)\|a-b\|^2,\end{aligned}$$

because $0 < s < 2/(\mu + L_\vartheta)$. Taking square roots yields (4.9). □

Theorem 4.1 (Strong convergence of TIBiG-PCRA)**.** *Let $\{x_n\}$ be generated by Algorithm 2. Assume (C1)–(C2), conditions (B1)–(B3) of Theorem 3.1, and the asymptotic compatibility condition*

$$\|k_n - z_n\| \to 0.$$

Then $\{x_n\}$ converges strongly to the unique solution z^* of the bilevel problem (4.1)–(4.2)*. Moreover,*

$$z^* = P_{S^*}(I - s\nabla\vartheta)(z^*). \tag{4.10}$$

Proof. Define

$$T_n = \operatorname{prox}_{c_n h}(I - c_n\nabla f) \qquad \text{and} \qquad T = \operatorname{prox}_{ch}(I - c\nabla f).$$

By (C1), each T_n is nonexpansive and

$$\operatorname{Fix}(T_n) = \operatorname{Fix}(T) = S^*.$$

By Lemma 2.3, $\{T_n\}$ satisfies the NST-condition (I) with respect to T, and hence Condition (Z). Proposition 4.1 shows that $g_s = I - s\nabla\vartheta$ is a contraction. Algorithm 2 is precisely Algorithm 1 with these choices of T_n and g_s. Therefore, Theorem 3.1 gives

$$x_n \to z^* \in S^*, \qquad z^* = P_{S^*}g_s(z^*).$$

The characterization of the metric projection implies

$$\langle g_s(z^*) - z^*, z - z^*\rangle \leq 0, \qquad z \in S^*.$$

Since $g_s(z^*) - z^* = -s\nabla\vartheta(z^*)$ and $s > 0$, we obtain

$$\langle \nabla\vartheta(z^*), z - z^*\rangle \geq 0, \qquad z \in S^*.$$

This is exactly the optimality condition (4.3), so z^* is the solution of the bilevel problem. □

Remark 4.1. *The specialization above corrects the forward–backward implementation by using the same operator*

$$T_n = \operatorname{prox}_{c_n h}(I - c_n\nabla f)$$

at every occurrence of T_n in the abstract TIPCRA scheme. This avoids mixing the upper-level contraction with the proximal operator of the lower-level function.

5. Application to Image Restoration

The image-restoration problem can be written as the linear inverse model

$$b = Ax + \omega, \tag{5.1}$$

where $x \in \mathbb{R}^N$ is the unknown image, A is a blurring operator, b is the observed degraded image, and ω represents additive noise. A basic least-squares reconstruction solves

$$\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2. \tag{5.2}$$

Because image-recovery problems are frequently ill posed, regularization is commonly introduced [14, 36]. Inertial and extragradient-type techniques have also been applied successfully to image-recovery models [23, 26]. In particular, ℓ^1 regularization is closely related to the LASSO principle [35]. We consider the regularized model

$$\min_{x\in\mathbb{R}^N} \left\{\|Ax - b\|_2^2 + \gamma\|Wx\|_1\right\}, \tag{5.3}$$

where $\gamma > 0$ is a regularization parameter and W is the transform used in the numerical implementation.

Thus, in the composite formulation,

$$f(x) = \|Ax - b\|_2^2, \qquad h(x) = \gamma\|Wx\|_1.$$

The methods TIBiG-PCRA, FBMIA (I), FBMIA (II), FISTA, NAGA and IFBS are compared in the numerical experiment [2, 4, 22, 34].

For TIBiG-PCRA, we choose the upper-level function

$$\vartheta(x) = \frac{1}{2}\|x\|^2. \tag{5.4}$$

Then $\nabla\vartheta(x) = x$, so ϑ is 1-strongly convex and $\nabla\vartheta$ is 1-Lipschitz continuous. Hence $\mu = L_\vartheta = 1$, and the admissible interval in assumption (C2) becomes $0 < s < 1$. We take

$$s = \frac{1}{2}. \tag{5.5}$$

Consequently,

$$g_s(x) = (I - s\nabla\vartheta)(x) = \frac{1}{2}x,$$

which is a contraction with coefficient $1/2$. With this choice, the upper-level problem selects the minimum-norm solution of the lower-level regularized image-recovery problem.

In the experiment, $\gamma = 5\times10^{-5}$ and the Peppers image has resolution 256×256 pixels. The degraded image is generated using a Gaussian blur with kernel size 9×9 and standard deviation 5, together with a motion blur of angle 10°.

The peak signal-to-noise ratio is defined by

$$\mathrm{PSNR}(x_n) = 10\log_{10}\left(\frac{255^2}{\mathrm{MSE}(x_n)}\right), \tag{5.6}$$

where

$$\mathrm{MSE}(x_n) = \frac{1}{N}\|x_n - z^*\|_2^2. \tag{5.7}$$

The numerical parameters are summarized in Table 1.

TABLE 1. Algorithms and parameter settings used in the image-restoration experiment.

Method	**Parameter setting**
TIBiG-PCRA	$\vartheta(x) = \frac{1}{2}\|x\|^2$, $s = \frac{1}{2}$, $\eta_n = \gamma_n = \frac{0.99n}{n+1}$, $\zeta_n = \frac{1}{50n}$, $c_n = \frac{1}{L_f}$, $\sigma_n = \frac{10^{14}}{n^2}$, $\tau_n = \frac{n}{n+1}$, $\xi_n = \frac{1}{n+1}$.
FBMIA (I)	$\alpha_n = \frac{1}{100}$, $\beta_n = 0.5$, $\mu_n = \kappa_n = \frac{n}{L(n+1)}$.
FBMIA (II)	$\theta_n = \begin{cases} \frac{n}{n+1}, & 1 \le n < N, \\ \frac{1}{2n}, & n \ge N, \end{cases}$ where N is the stopping iteration.
FISTA	$\mu = \frac{1}{L}$, $\theta_n = \frac{t_{n-1}}{t_{n+1}}$, $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$.
NAGA	$\beta_n = 0.5$, $\mu_n = \frac{n}{L(n+1)}$, $\theta_n = \frac{t_{n-1}}{t_{n+1}}$, $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$.
IFBS	$\mu_n = \frac{n}{L(n+1)}$, $\theta_n = \begin{cases} \frac{1}{n^2\|x_n - x_{n-1}\|^2}, & x_n \ne x_{n-1}, \\ 0, & x_n = x_{n-1}. \end{cases}$

The TIBiG-PCRA control sequences satisfy conditions (B1)–(B3). The strong-convergence conclusion is obtained under the additional asymptotic condition (B4) in Theorem 3.1.

All experiments were carried out in MATLAB on an Intel Core i7-9700 CPU with 32 GB RAM. The PSNR values are presented in Table 2.

TABLE 2. PSNR values of the compared methods at selected iterations.

Method	**Iteration**				
	10	20	40	60	100
TIBiG-PCRA	32.25	31.90	29.92	27.37	22.74
FBMIA (I)	30.64	30.85	29.71	27.33	22.74
FBMIA (II)	28.46	29.55	29.46	27.29	22.74
FISTA	32.25	31.90	29.92	27.37	22.74
NAGA	24.65	26.98	28.84	27.18	22.73
IFBS	23.12	25.82	28.49	27.12	22.73

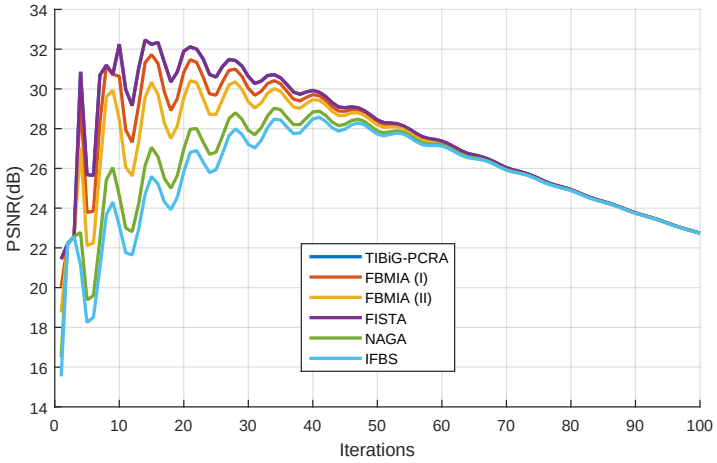

FIGURE 1. PSNR curves for the Peppers image over 100 iterations.

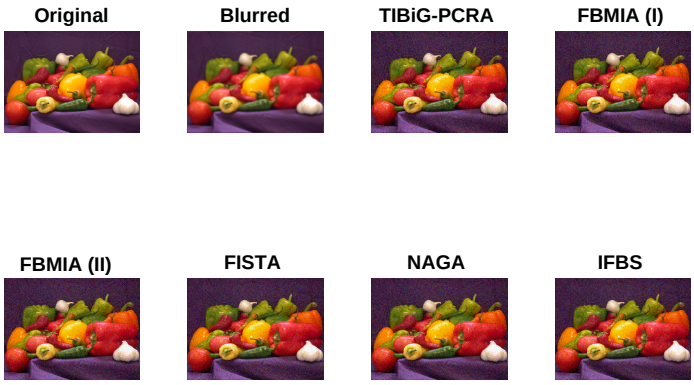

FIGURE 2. Image-restoration results at the 100th iteration.

The numerical results show that TIBiG-PCRA attains the highest PSNR at iteration 10 and matches FISTA at all tabulated checkpoints. The decrease in PSNR after the early iterations indicates a semi-convergence effect; hence the best reconstruction quality is attained before the final iteration rather than increasing monotonically with the iteration count.

6. Conclusion

We introduced a two-step inertial Picard–CR framework for approximating common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. The method combines two inertial corrections with a viscosity approximation and repeated nonexpansive evaluations. A strong convergence result was established under suitable control conditions and an explicit asymptotic compatibility assumption on the intermediate iterates. The abstract method was

then specialized to a convex bilevel optimization problem through forward–backward operators, yielding the TIBiG-PCRA scheme.

The image-restoration example illustrates the computational use of the proposed framework in a regularized inverse problem. The PSNR values show competitive early-iteration reconstruction quality and reveal semi-convergence behavior, which emphasizes the importance of an appropriate stopping criterion in imaging applications. The framework can be extended to adaptive parameter rules, stochastic models and nonconvex settings.

Author Contributions: All authors contributed to the conception, mathematical analysis, interpretation of the results, and preparation of the manuscript. All authors reviewed the manuscript and approved the final version.

Data Availability: The numerical data in this study are generated computationally from the problem specifications given in the manuscript. The MATLAB codes and generated numerical data are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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