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Further Properties of $h\alpha$ −Open Sets in Ideal Topological Spaces

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ABSTRACT. Introducing and analyzing a new idea of open sets called $h\alpha$ −open sets in ideal topological spaces and examining some of the relationships between open sets, h −open sets, α −open sets, and $h\alpha$ −open sets are the main goals of the work. We examine various features and provide a new, more natural definition of the separation axioms in ideal topological spaces. We demonstrate a condition that is true in ideal topological theory but not in the classical theory of topology, as well as a property that is true in the classical theory but not in the ideal topological theory.

1. Introduction

A new type of open sets in topological space, known as $h\alpha$ −open sets were introduced by Abdullah et al. [1] in 2022. Because they serve as the foundation for topological spaces, which in turn pave the way for the development of ideas like continuity, connectedness, compactness, separation axioms, convergence, etc., open sets are essential to topology. Numerous studies on open sets have occasionally been carried out in order to gain a better understanding of the structure and properties of topological spaces. In order to comprehend spaces with more structure or weakening requirements, topologists have long been investigating more extended and refined versions of open sets. The standard concept of open sets is essential for many classical conclusions [2]. Levine's 1963 proposal of *semi* −open sets and *semi* −continuity, which offered a framework for examining lesser forms of openness and continuity in topological spaces, was one of the first developments. As an expansion of the conventional notion of open sets, Levine

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and others [3,4,22] later put out the concept of generalized open sets in 1970. The concept of $b-$open sets was established by Andrijevic [5] in 1996, and other types of continuity associated with this concept have been investigated in [6,7,8,23].

Sundaram and Pushpalatha [9] introduced the concept of strongly generalized open sets and expanded on the study of generalized open sets in 2001. The advent of generalized open sets has made it possible to define continuity more broadly. For example, a function $f:(X,\tau)\to(Y,\sigma)$ is called generalized continuous if $f^{-1}(A)$ is generalized open in (X,τ) for any generalized open set A in (Y,σ). The study of mappings that retain a lesser type of continuity but do not strictly preserve classical open sets has been made possible by generalized continuity. The concept of α-open sets was initially introduced by Njasted [10] in 1965. Abbas [11] proposed the concept of $h-$open sets in 2021. Abdullah et al. [12] introduced the concept of $h\alpha-$open sets in 2022, utilizing these ideas. Dontchev [13] first proposed the concept of a contra-continuous function in 1999. Nearly contra-continuous and contra $almost-\beta-$continuous functions were defined by Baker [14,21]. Dontchev et al. introduced a new family of functions called regular set-linked functions [15]. Singh and Singal introduced the class of nearly continuous functions in [16]. Strong continuity in topological spaces was first proposed by Levine [4] in 1960. The concept of a strongly $h-$continuous function was recently created by Sharma et al. [17]. Unlike compactness or connectedness, an aura topological space is not a conventional or well-known notion in general topology. The phrase can have several meanings depending on the author and is used in some specialized or modern research areas.

The triplet (X,τ,I), where (X,τ) is a topological space and I is an ideal (a group of subsets closed under finite unions and subsets) on, is an ideal topological space [18,19]. By enabling the categorization of "small" or "negligible" sets, frequently via a local function, it broadens the scope of topology. Kuratowski [24], Noiri [25], and numerous others later investigated and explored the ideas. $I-$Hausdorff space was first proposed by Dontchev [26] in 1995, while $quasi-I-$Hausdorff space was established in 2000 by Abd El-Monsef [27].

A non-empty collection of subsets of a space X is an ideal on X, if the following hold:

(i) If $L\in\tau$ and $K\subseteq L$, then $K\in\tau$.

(ii) $K,L\in\tau$, then $K\cup L\in\tau$.

2. $h\alpha-$Open Set in Ideal Topological Space.

Definition 2.1 Consider the space X and $K\subseteq X$, then:

(i) K is an $h-$open set [11] if $\forall W\subset X,\ W\in\tau,\ K\subseteq int(K\cup W)$.

(ii) K is $\alpha-$open [21] if $A\subseteq int(cl(int(K)))$.

(iii) The family of all $h-$open ($\alpha-$open) sets of τ is denoted by τ^{h} (τ^{α}).

(iv) K is an $h\alpha-$open [20] if $\forall W\subset X, W\in\tau^{h\alpha}, K\subseteq int_I(K\cup W)$.

(v) K is regular open [21] if $K = int(cl(K))$ and it is regular closed if $K = cl(int(K))$.

(vi) K is a $semi$ −open set (s −open) [4] if $\exists O \in \tau$: $O \subseteq K \subseteq cl(K)$.

(vii) If K is open and closed, then it is clopen.

The family of $h\alpha$ −open is denoted by $\tau^{h\alpha}$.

Example 2.2 If $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$, and $I = \{X, \phi, \{a, c\}\}$, then

$$\tau^{h\alpha} = \{X, \phi, \{a\}, \{a, c\}, \{b, c\}\}.$$

Definition 2.3 Consider the ideal space X and $K \subseteq X$, then K is an h −open set modulo the ideal I if $\forall W \subset X$, $W \in \tau, K \subseteq int_I(K \cup W)$ and K is an $h\alpha$ −open modulo the ideal I if $\forall W \subset X$, $W \in \tau^{h\alpha}, K \subseteq int_I(K \cup W)$.

Lemma 2.4 Consider the ideal space X, and K is an open subset of X, then:

(i) K is α −open modulo an an ideal I.

(ii) K is an h −open set modulo an ideal I.

(iii) If K is an h −open set modulo an ideal I, then K is an $h\alpha$ −open set modulo I.

Proof. (iii) Let K be an h −open set modulo an ideal I. Hence, $\forall V \subset X$, and $V \in \tau$, $K \subseteq int_I(K \cup V)$. Using (i), V is an α −open set modulo I, so K is an $h\alpha$ −open set modulo I.

Theorem 2.5 Each open set modulo an ideal is an $h\alpha$ −open set modulo ideal.

The converse of the previous theorem is not true.

Example 2.6 If $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $\tau^{h\alpha} = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, and $I = \{X, \phi, \{a, b\}, (a, c\}, \{b, c\}\}$ then $\{c\}$ is an $h\alpha$ −open set modulo I, but it is not an open set.

Definition 2.7 Consider the ideal space X and $K \subseteq X$, then [20]:

(i) $h\alpha$ −interior of K modulo I is the union of all $h\alpha$ −open sets contained in K modulo I (denoted by $int_I^{h\alpha}(K)$).

(ii) K is $h\alpha$ −neighbourhood of $b \in X$ if $\exists U \in \tau^{h\alpha}$: $U \subseteq K$.

(iii) $h\alpha$ −closure of K modulo I is the intersection of $h\alpha$ −closed subsets of X modulo I containing K (denoted by $cl_I^{h\alpha}(K)$).

The collection of all $h\alpha$ −neighbourhood of $b \in X$ is called the $h\alpha$ −neighbourhood system of b and denoted by $N_I^{h\alpha}(b)$.

Clearly, $int_I^{h\alpha}(K)$ is an $h\alpha$ −open set modulo I.

Theorem 2.8 Consider the ideal space X, the arbitrary union of $h\alpha$ −open sets modulo I is $h\alpha$ −open.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a family of $h\alpha$ −open sets of X modulo I. If $U = \bigcup_{\alpha\in\Lambda} U_\alpha$, then $\exists A \in \tau^{\alpha}$: $U_\alpha \subseteq int_I^{h\alpha}(U_\alpha \cup A)$.

$$\begin{aligned} \text{Now, } U = \textstyle\bigcup_{\alpha\in\Lambda} U_\alpha &\subseteq \textstyle\bigcup_{\alpha\in\Lambda} int_I^{h\alpha}(U_\alpha \cup A) \\ &\subseteq int_I^{h\alpha} \bigcup_{\alpha\in\Lambda} (U_\alpha \cup A) \\ &= int_I^{h\alpha}(U \cup A). \end{aligned}$$

Hence, U is $h\alpha$ −open modulo I.

Remark 2.9 Consider the space X and $K \subseteq X$, then:

(i) $cl_I^{h\alpha}(K)$ is the smallest closed set containing K modulo I.

(ii) $cl_I^{h\alpha}(K)$ is $h\alpha$ −closed.

Theorem 2.10 Finite intersection of $h\alpha$ −open sets modulo I needs not to be $h\alpha$ −open modulo I.

Example 2.11 Consider $(\mathbb{R}, \tau_u)$, if $M = (-1,1) \cup (1,3)$ and $N = (1,3) \cup (3,5)$ are both $h\alpha$ −open sets modulo $I = \mathbb{R}$, then $M \cap N = (1,3)$ which is not $h\alpha$ −open.

Corollary 2.12 Consider the ideal space X and $K \subseteq X$, then $int_I^{h\alpha}(K) = \{U : U \subseteq K \ is \ h\alpha - \text{open}\}$

Theorem 2.13: Finite union of any collection of $h\alpha$ −closed sets modulo I is $h\alpha$ −closed.

Theorem 2.14 The family of $h\alpha$ −open sets modulo I is a generalized topology but not a topology [20].

Theorem 2.15 Consider the ideal space X and $K, L \subseteq X$, then:

(i) $int_I^{h\alpha}(X) = X$ and $int_I^{h\alpha}(\phi) = \phi$.

(ii) If $K \subseteq L$, then $int_I^{h\alpha}(K) \subseteq int_I^{h\alpha}(L)$.

(iii) $int_I^{h\alpha}(K) \subseteq K$.

(iv) For the $h\alpha$ −open set K modulo I such that $K \subseteq L$, $K \subseteq int_I^{h\alpha}(L)$.

Proof. (i) X and ϕ are both $h\alpha$ −open sets, so $int_I^{h\alpha}(X) = \{U : U \text{ is } h\alpha - \text{open}, U \subseteq X\} = X$.

In addition, ϕ is the only $h\alpha$ −open set contained in ϕ, so $int_I^{h\alpha}(\phi) = \phi$.

(ii) If $K \subseteq L$, then $\forall b \in int_I^{h\alpha}(K)$, then K is an $h\alpha$ −neighbourhood of b.

But $K \subseteq L$, then L is an $h\alpha$ −neighbourhood of b, so $b \in int_I^{h\alpha}(L)$.

Therefore, $int_I^{h\alpha}(K) \subseteq int_I^{h\alpha}(L)$.

(iii) $\forall b \in int_I^{h\alpha}(K)$, K is an $h\alpha$ −neighbourhood of b , then $b \in int_I^{h\alpha}(K)$.

Hence, $int_I^{h\alpha}(K) \subseteq K$.

(iv) If K is an $h\alpha$ −open set such that $K \subseteq L$, and $b \in K \subseteq L$, then K is an $h\alpha$ −neighbourhood of b contained in L, so by (iii), $b \in int_I^{h\alpha}(K)$. Thus, $K \subseteq int_I^{h\alpha}(L)$.

Theorem 2.16 Consider the ideal space X and $K, L \subseteq X$, then:

(i) $cl_I^{h\alpha}(X) = X$ and $cl_I^{h\alpha}(\phi) = \phi$.

(ii) If $K \subseteq L$, then $cl_I^{h\alpha}(K) \subseteq cl_I^{h\alpha}(L)$.

(iii) K is $h\alpha$ −closed iff $K = cl_I^{h\alpha}(K)$.

(iv) If L is an $h\alpha$ −closed set containing K moduo I, then $cl_I^{h\alpha}(K) \subseteq L$.

Proof. (i) Since the $h\alpha$ −closed set containing X is X itself, so $X \cap cl_I^{h\alpha}(X) = X$.

Hence $cl_I^{h\alpha}(X) = X$.

(ii) If $K \subseteq L$, then $L \subseteq \ cl_I^{h\alpha}(L)$. But $K \subseteq L$, so $K \subseteq \ cl_I^{h\alpha}(L)$. Since $cl_I^{h\alpha}(L)$ is closed, and $cl_I^{h\alpha}(K)$ is the smallest $h\alpha$ −closed set containing K, therefore, $cl_I^{h\alpha}(K) \subseteq cl_I^{h\alpha}(L)$.

(iii) K be an $h\alpha$ −closed set iff K is the smallest $h\alpha$ −closed set containing K iff

$K = cl_I^{h\alpha}(K)$.

(v) If L is an $h\alpha$ −closed set containing K, then $cl_I^{h\alpha}(K) = \cap\,\{F: K \subseteq F\}$ where F is an $h\alpha$ −closed subset of X. So, $cl_I^{h\alpha}(K) \subseteq L$.

Theorem 2.17 Consider the ideal space X and $K, L \subseteq X$, then $cl_I^{h\alpha}(K \cap L) \subseteq cl_I^{h\alpha}(K) \cap cl_I^{h\alpha}(L)$

Proof. If $K, L \subseteq X$, then $K \cap L \subseteq K$ and $K \cap L \subseteq L$. Hence, $cl_I^{h\alpha}(K \cap L) \subseteq cl_I^{h\alpha}(K)$ and $cl_I^{h\alpha}(K \cap L) \subseteq cl_I^{h\alpha}(L)$. Therefore, $cl_I^{h\alpha}(K \cap L) \subseteq cl_I^{h\alpha}(K) \cap cl_I^{h\alpha}(L)$.

Corollary 2.18 If $K, L \subseteq X$, then $cl_I^{h\alpha}(K \cap L) \subseteq K \cap cl_I^{h\alpha}(L)$ for each $h\alpha$ −closed set K.

Theorem 2.19 Consider the ideal space X and $F \subseteq X$ an $h\alpha$ −closed set, then $\forall b \in X - F, \exists U$ a neighbourhood of b: $U \cap F = \phi$.

Proof. If $F \subseteq X$ an $h\alpha$ −closed set, then $\forall b \in X - F$, $X - F$ has an $h\alpha$ −neighbourhood.
So, $\exists U$ a neighbourhood of b: $U \subseteq X - F$. Hence $U \cap F = \phi$.

Theorem 2.20 Consider the space X and $K, L \subseteq X$, then:

(i) $int_I^{h\alpha}(int_I(K)) = int_I^{h\alpha}(K)$.

(ii) $int_I^{h\alpha}(int_I(K)) = int_I(K)$.

(iii) $int_I^{h\alpha}(K) \cap int_I^{h\alpha}(L) = int_I^{h\alpha}(K \cap L)$

(iv) $int_I^{h\alpha}(K) \cup int_I^{h\alpha}(L) \subseteq int_I^{h\alpha}(K \cup L)$

(v) $int_I(K) \subseteq int_I^{h\alpha}(K)$.

Proof. (i) Since $int_I^{h\alpha}(K) \in \tau_I^{\alpha}$, so $int_I^{h\alpha}(K) = int_I^{h\alpha}(int_I^{h\alpha}(K))$.

(ii) Since $int_I(K)$ is an open set modulo I, so $int_I(K)$ is $h\alpha$ −open.
Therefore, $int_I^{h\alpha}(int_I(K)) = int_I(K)$.

(iii) Since $K \cap L \subseteq K$ and $K \cap L \subseteq L$, so $int_I^{h\alpha}(K \cap L) \subseteq int_I^{h\alpha}(K)$ and $int_I^{h\alpha}(K \cap L) \subseteq int_I^{h\alpha}(L)$.
Therefore, $int_I^{h\alpha}(K \cap L) \subseteq int_I^{h\alpha}(K) \cap int_I^{h\alpha}(L)$.
On the other hand, if $b \in int_I^{h\alpha}(K \cap L)$, so $b \in int_I^{h\alpha}(K)$ and $b \in int_I^{h\alpha}(L)$.
Hence, both of K and L are $h\alpha$ −neighbourhoods of b.
Thus, their intersection is also $h\alpha$ −neighbourhood of b. Consequently, $b \in int_I^{h\alpha}(K \cap L)$.
Hence the result.

(iv) Since $K \subseteq K \cup L$ or $L \subseteq K \cup L$, so $int_I^{h\alpha}(K) \subseteq int_I^{h\alpha}(K \cup L)$ or $int_I^{h\alpha}(L) \subseteq int_I^{h\alpha}(K \cup L)$.
Hence, $int_I^{h\alpha}(K) \cup int_I^{h\alpha}(L) \subseteq int_I^{h\alpha}(K \cup L)$.

(v) $\forall b \in int_I(K), \exists U \subseteq K$ an $h\alpha$ −open set : $b \in U$. So, $int_I(K) \subseteq int_I^{h\alpha}(K)$.

Theorem 2.21 Consider the ideal space X and $K \subseteq X$, K is $h\alpha$ −open modulo I iff it is an $h\alpha$ −neighbourhood of each of its points.

Proof. Let $b \in K$, so $\exists U \subseteq K$ an $h\alpha$ −neighbourhood such that $\forall K \in \tau^{h\alpha}$ an $h\alpha$ −open set modulo I, K is an $h\alpha$ −neighbourhood of b.

Theorem 2.22 Consider the ideal space X and $b \in X$. If U_1 and U_2 are two $h\alpha$ −neighbourhoods of b, then their intersection is an $h\alpha$ −neighbourhood of b.

Proof. Let U_1 and U_2 be two $h\alpha$ −neighbourhoods of b, then $\forall b \in X, \exists V_1, V_2 \in \tau^{h\alpha} : V_1 \subseteq U_1$ and $V_2 \subseteq U_2$. Now, $b \in V_1 \cap V_2 \subseteq U_1 \cap U_2$. Hence the result.

Theorem 2.23 Consider the ideal space $X, K \subseteq X$ and $b \in X$, then $b \in cl_I^{h\alpha}(K)$ iff $\forall W$ an $h\alpha$ −open set containing $b, W \cap K \neq \phi$.

Proof. If $b \in cl_I^{h\alpha}(K)$ $\forall b \in X$, then $\exists W$ an $h\alpha$ −open set containing b: $W \cap K = \phi$. That is, $K \subseteq X - W$ and $X - W$ is an $h\alpha$ −closed subset of X. Hence, $cl_I^{h\alpha}(K) \subseteq X - K$.

Thus, $b \notin cl_I^{h\alpha}(K)$ which contradicts the assumption. Therefore, $W \cap K \neq \phi$.

On the other hand, consider W an hα-open set containing b: $W \cap K \neq \phi$, and let $b \notin cl_I^{h\alpha}(K)$. Then $\exists L \subseteq X$: $K \subseteq L$ and $b \notin L$. So, $b \in X - L$ and $X - L$ is an $h\alpha$ −open set.

Thus, $(X - L) \cap K = \phi$ which contradicts the assumption. Consequently, $b \in cl_I^{h\alpha}(K)$.

Corollary 2.24 Consider the space X, and $K \subseteq X$, $cl_I^{h\alpha}(cl_I^{h\alpha}(K)) = cl_I^{h\alpha}(K)$.

Definition 2.25 Consider the ideal space X, $b \in X$ and $K \subseteq X, b$ is an $h\alpha$ −limit point of K modulo I if $\forall U$ an $h\alpha$ −open set containing $b, \exists k \in K : U \cap (K - \{b\}) \neq \phi$.

$D_I^{\mathrm{h\alpha}}(K)$ denotes the set of all $h\alpha$ −limit points of K modulo I.

Theorem 2.26 Consider the ideal space X, and $K \subseteq X$,then $\forall U \in \tau_I^{h\alpha}$ and $b \in U, U \cap K \neq \varphi$ iff $b \in cl_I^{h\alpha}(K)$.

Proof. If $b \notin cl_I^{h\alpha}(K)$, then $\exists F$ an $h\alpha$ −closed set: $K \subseteq F$ and $b \notin F$. Therefore, the $h\alpha$ −open set $U = X - F$ contains b and $U \cap K = \phi$ which contradicts the assumption.

Theorem 2.27 Consider the ideal space X, and K is an $h\alpha$ −closed subset of X modulo I, then $cl_I^{h\alpha}(K) = K \cup D_I^{\mathrm{h\alpha}}(K)$ iff $D_I^{\mathrm{h\alpha}}(K) \subseteq K$.

Proof. If $b \notin cl_I^{h\alpha}(K)$, then $\exists F$ an $h\alpha$ −closed subset of X: $K \subseteq F$ and $b \notin F$. By the previous theorem, $(X - F) \cap K = \phi$. Hence, $b \notin K$ and $b \notin D_I^{\mathrm{h\alpha}}(K)$.

Thus, $K \cup D_I^{\mathrm{h\alpha}}(K) \subseteq cl_I^{h\alpha}(K)$. Now, if $b \notin K \cup D_I^{\mathrm{h\alpha}}(K), \exists U$ an $h\alpha$ −open subset of X disjoint from K: $b \in U$. So, $L = X - U$ is $h\alpha$ −closed: $K \subseteq L$ and $b \notin L$.

Hence $b \notin cl_I^{h\alpha}(K)$ and $b \notin D_I^{\mathrm{h\alpha}}(K)$. So, $b \notin cl_I^{h\alpha}(K) \cap D_I^{\mathrm{h\alpha}}(K)$.

Thus, $cl_I^{h\alpha}(K) \subseteq K \cup D_I^{\mathrm{h\alpha}}(K)$. Consequently, $cl_I^{h\alpha}(K) = K \cup D_I^{\mathrm{h\alpha}}(K)$.

On the other hand, let K be an $h\alpha$ −closed set, hence $\forall b \in (X - K), \exists V$ an $h\alpha$ −neighbourhood of b: $V \subseteq X - K$ and $X - K$. Thus, b is an $h\alpha$ −limit point of K modulo I.

That is, K contains all its limit points. Therefore, $D_I^{\mathrm{h\alpha}}(K) \subseteq K$. Conversely, if $D_I^{\mathrm{h\alpha}}(K) \subseteq K$ and $b \in X - K$, then $b \notin D_I^{\mathrm{h\alpha}}(K) \subseteq K$ because $D_I^{\mathrm{h\alpha}}(K) \subseteq K$. So, $\exists W$ an $h\alpha$ −neighbourhood of b: $W \subseteq X - K$. Therefore, $X - K$ contains an $h\alpha$ −neighbourhood of all its points.

Thus $X - K$ is an $h\alpha$ −open set modulo I.

Theorem 2.28 Consider the ideal space X and K is an $h\alpha$ −closed subset of X modulo I,then $D_I^{\mathrm{h}\alpha}(K)$ is empty if X is a discrete ideal space.

Proof. If $b \in X$, then $\{b\}$ is an $h\alpha$ −closed subset of X modulo I. If O is an $h\alpha$ −neighbourhood of b, then $O \cap K \subseteq \{b\}$. That is; $b \notin K$. Thus, $D_I^{\mathrm{h}\alpha}(K) = \phi$.

Theorem2.29 Consider the space X and K is an $h\alpha$ −open subset of X modulo I, if $\tau_1^{h\alpha}$ is finer than $\tau_2^{h\alpha}$ for X, then each $h\alpha$ −limit point of K w.r.t $\tau_2^{h\alpha}$ is an $h\alpha$ −limit point of K w.r.t $\tau_1^{h\alpha}$.

Proof. If $b \in X$ is an $h\alpha$ −limit point of K w.r.t $\tau_2^{h\alpha}$, then $\exists W \in \tau_2^{h\alpha}$ an $h\alpha$ −neighbourhood of b: $W \cap (K - \{b\})$ is non-empty. But $\tau_1^{h\alpha}$ is finer than $\tau_2^{h\alpha}$ for X, so, $W \cap (K - \{b\})$ is non-empty too. Thus, b is an $h\alpha$ −limit point of K w.r.t $\tau_1^{h\alpha}$.

3. Separation Axioms of Ideal Topological Spaces

Definition 3.1 The ideal space X is called $T_\circ^{h\alpha}$ if $\forall a \neq b \in X$, $\exists U \in I$ an $h\alpha$ −open subset of X: U contains one point but not the other.

Definition 3.2 The ideal space X is called $T_1^{h\alpha}$ if $\forall a \neq b \in X, \exists U, V \in I$ the $h\alpha$ −open subsets of X: $a \in U, b \notin U$ *and* $b \in V, b \notin U$.

Theorem 3.3 Every $T_\circ$ −ideal space is $T_\circ^{h\alpha}$ −ideal space.

Proof. If the ideal space X is $T_\circ$, then $\forall a \neq b \in X, \exists U \in I$ an open subset of X: U contains one point but not the other. But U is an $h\alpha$ −open subset of X.

Hence X is a $T_\circ^{h\alpha}$ −ideal space.

The converse of the previous theorem is not true.

Example 3.4 Let $X = \{a, b, c\}, \tau = \{X, \phi, \{b, c\}\}$, and $I = \{X, \phi, \{b, c\}\}$, then X is a $T_\circ^{h\alpha}$ −ideal space but not a $T_\circ$ −ideal space.

Theorem 3.5 Every T_1-ideal space is $T_1^{h\alpha}$ −ideal space.

Proof. Let the ideal space X be a T_1, then $\forall a \neq b \in X, \exists U, V \in I$ the open subsets of X: $a \in U, b \notin U$ and $b \in V, b \notin U$. But U, V are $h\alpha$ −open subset of X, so X is a $T_1^{h\alpha}$ −ideal space.

The converse of the previous theorem is not true.

Example 3.6 If $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}, \{a, c\}\}$, and $I = \{X, \phi, \{a, c\}\}$, then X is a $T_1^{h\alpha}$ −ideal space but not a T_1-ideal space.

Definition 3.7 The ideal space X is called $T_2^{h\alpha}$ if $\forall a \neq b \in X, \exists U, V \in I$ disjoint $h\alpha$ −open subsets of X: $a \in U$, *and* $b \in V$.

Remark 3.8 If the ideal $I = \wp(X)$, then the ideal space X is not T_2 even if the space X is.

Proof. Let $K \subseteq X$, then $K \in I$, and so $K^* = \phi$. Now, no set other than ϕ is an h −open.

Hence, no disjoint $h\alpha$ −open subset of X containing two distinct points.

Definition 3.9 Consider the ideal space X, for $K \subseteq X$,then:

(i) $K^* = \{a \in X - (U \cap K) \in I: U \in T^{h\alpha}\}$.

(ii) $K = K \cup K^*$ is the closure operator.

Theorem 3.10 The ideal space X is a $T_2^{h\alpha}$ iff $\forall a, b \in X$ with $a \neq b, \exists U, V \in \tau_I^{h\alpha}$ and $I_1, I_2 \in I: \ a \in U - I_1, b \in V - I_2$, and $(U - I_1) \cap (V - I_2) \in I$.

Proof. Consider the $T_2^{h\alpha}$ ideal space X, Let $a, b \in X$ such that $a \neq b$.

Since X is $T_2^{h\alpha}$, then $\exists U, V \in \tau_I^{h\alpha}: a \in U, b \in V$ and $U \cap V \in I$. Considering $I_1 = I_2 = \phi$, then $(U - I_1) \cap (V - I_2) \in I$ holds.

On the other hand,

$(U - I_1) \cap (V - I_2) = (U \cap V) - (I_1 \cup I_2) = [(U - I_1) \cap (U - I_2)] \cup [(V - I_1) \cap (V - I_2)] \in I$.

Hence, $U \cap V \in I$. Therefore, X is $T_2^{h\alpha}$.

Definition 3.11 Consider the ideal space X, if $\forall a \in X$ and K an hα-open set not containing a, $\exists U, V \in \tau_I^{h\alpha}: a \in U, K \subseteq V$ and $U \cap V \in I$, then X is $T_3^{h\alpha}$.

Lemma 3.12 Consider the ideal space X, if each singleton is $h\alpha$ −closed, then X is $T_3^{h\alpha}$.

Theorem 3.13 Consider the ideal space X, if $\forall a \in X$ and K an $h\alpha$ −open set not containing a, $\exists U \in \tau_I^{h\alpha}: a \in U$, and $cl_I^{h\alpha}(K) \cap U = \phi$, then X is $T_3^{h\alpha}$.

Proof. If $F \in \tau_I^{h\alpha}$ is an $h\alpha$ −closed and $\forall a \in X$ such that $a \notin F, \exists U, V \in \tau_I^{h\alpha}: a \in U, F \subseteq V$ and $U \cap V \in I$. Let $U \cap V = I, W_1 = U$ and $W_2 = U - I$. Now, $F \cap I = \phi, \ F \subseteq V, W_2$, and $U \cap W_2 = \phi$.

So, X is $T_3^{h\alpha}$.

Corollary 3.14 Consider the ideal space X, if $\forall a \in X$ and K an $h\alpha$ −open set not containing a, $\exists U, V \in \tau_I^{h\alpha}$ such that $a \in U, K \subseteq V$ and $U \cap V = \phi$.

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