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GELFAND TRIPLE ISOMORPHISMS FOR WEIGHTED BANACH SPACES ON LOCALLY COMPACT GROUPS

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Abstract. As in [1], we use the concept of wavelet transform on a locally compact group G to construct weighted Banach spaces $\mathcal{H}_w^1(G)$, we being a submultiple weight function on G. The main result of this paper provides an extension of a unitary mapping $\mathcal{U}$ from $\mathcal{H}(G_1)$ to $\mathcal{H}(G_2)$ under suitable conditions to an isomorphism between the Gelfand triple $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$; where $G_1,\ G_2$ are any two locally compact groups, $\mathcal{H}$ a Hilbert space and $\mathcal{H}_w^{1\sim}$ is the space of all continuous-conjugate linear functional on $\mathcal{H}_w^1$. This paper paves the way for the study of some other properties of Gelfand triples.

1. Introduction

I.M. Gelfand introduced a triple of abstract space consisting of a Frechet space $\mathcal{F}$ of test functions continuously and densely embedded in Hilbert space $\mathcal{H}$ while $\mathcal{H}$ itself is continuously and densely embedded in the dual space $\mathcal{F}'$ of $\mathcal{F}$ (for details see [3]). Feichtinger and Kozek [2] have studied a number of properties of Gelfand triple replacing the Frechet space $\mathcal{F}$ by a suitable Banach space. In particular, they have discussed in detail the extensions of isomorphisms between L^2-spaces on elementary locally compact abelian groups G to Gelfand triples of the form $(S_0, L^2, S_0')(G)$ where $S_0(G)$ is the well known Feichtinger algebra, which has a number of highly useful functional properties ([2], p. 237). Also, they have studied some important properties of the operator Gelfand triple $(B,\ H,\ B')$, where B is the Banach space of all bounded linear operators from $S_0'(G)$ to $S_0(G)$ with respect to the operator norm $\|\cdot\|_{op}$.

In the present paper, following Feichtinger and Gröchenig ([1], p. 309), we define Wavelet transform $V_g f$ of a function f with respect to g, both as elements of a Hilbert space $\mathcal{H}$, on a locally compact group G. Using these wavelet transforms, we construct a weighted Banach spaces $\mathcal{H}_w^1(G)$ as in ([1], p. 317), where w is a submultiplicative weight function on G. We prove that any unitary map $\mathcal{U}$ from $\mathcal{H}(G_1)$ to $\mathcal{H}(G_2)$ extends as an isomorphism from the Gelfand triple $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ to $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$ if and only if the restrictions of $\mathcal{U}$ and $\mathcal{U}^*$ are bounded linear operators from $\mathcal{H}_w^1(G_1)$ to $\mathcal{H}_w^1(G_2)$ respectively, where $\mathcal{H}_w^{1\sim}(G_1)$ is the Banach space of all continuous-conjugate linear functionals on $\mathcal{H}_w^1(G)$.

This paper paves the way for the study of some properties associated with these Gelfand triples.

2. Notations and Basic Concepts.

Let G be a locally compact group and dx the normalized Haar measure on it. We assume that $w : G \to R_+$ is a submultiplicative weight function on G such that

$$w(x \circ y) \le w(x)\ w(y)$$

for all $x,\ y \in G$

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We suppose that all weight on G are symmetrical, i.e.,
$w(x) = w(-x), \quad \forall\ x \in G.$

We denote by $L^p_w(G),\ 1 \le p < \infty$, the Banach space of functions on G with respect to the norm

$$\|f\|_{p,w} = \left(\int_G |f(x)|^p \ w^p(x) \ dx \right)^{1/p} < \infty. \tag{2.1}$$

In case $p = \infty$, we define the space $L^\infty_w(G)$ as the Banach space of all measurable functions f on G such that

$$\|f\|_{\infty,w} = \operatorname{ess\,sup}\{|f(x)| \ w(x) : \ x \in G\} < \infty. \tag{2.2}$$

The conjugate space of $L^p_w(G)$ is the space $L^{p'}_{w^{-1}}(G)$, where $1/p + 1/p' = 1.$.

It is well known that $L^1_w(G)$ is a commutative Banach algebra under convolution , which is usually known as Beurling algebra, and we have the properties:

$$L^p_w(G) * L^1_w(G) \subseteq L^p_w(G)$$

and

$$\|f * g\| \le \|f\|_{p,w} \ \|g\|_{1,w}$$

for all $f \in L^p_w(G)$ and $g \in L^1_w(G)$.

3. Wavelet Transform on G.

Let $(\pi, \mathcal{H})$ be an irreducible continuous unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}(G)$. If $f, g \in \mathcal{H}$, then the wavelet transform of f with respect to g is given by ([1], p. 317):

$$V_g f : x \to \langle \pi(x) g, \ f \rangle,$$

where

$$V_g f(x) = \int_G \pi(x) \ g(x) \bar{f}(y) \ dy$$

and $\bar{f}(y)$ is the complex conjugate of $f(y)$.

The representation π is called square integrable provided $V_g f \in L^1(G), \quad \forall\ g \in \mathcal{H}$. It is known that if π is square-integrable, i.e., $V_g g \in L^2(G)$, then $\exists$ a unique, positive, self-adjoint and densely defined operator A on $\mathcal{H}$ satisfying the following orthogonality relation ([1], pp. 309-310):

$$\int_G V_{g_1} f_1(x) \ V_{g_2} f_2(x) \ dx = \langle A g_2, \ A g_1 \rangle \langle f_1, f_2 \rangle$$

for all $f_1, f_2 \in \mathcal{H}$ and $g_1, g_2 \in \operatorname{dom} A$.
In case $f_1 = f_2 = g_1 = g_2 = g \in \operatorname{dom} A,\ f_1 = f$ and $\|g_1\|_2 = 1$, then we have

$$\begin{aligned} V_g \ f * V_g \ g &= V_g \ f \langle g, g \rangle \\ &= V_g. \end{aligned}$$

Now, on the line of Feichtinger and Gröchenig ([1], p.317),we define the set of analyzing vector $h^1_w(G)$ by

$$h^1_w(G) = \{g : g \in \mathcal{H}, V_g \ g \in L^1_w(G)\}.$$

Since π is irreducible ,h^1_w is a dense linear subspace of $\mathcal{H}$. We suppose that $h^1_w(G)$ is non-trivial and g is a non-zero fixed element of $h^1_w(G)$. we define $\mathcal{H}^1_w(G)$ by

$$\mathcal{H}^1_w(G) = \{f ; f \in \mathcal{H}, V_g \ f \in L^1_w\},$$

which is a Banach space under the norm

$$\|f | \mathcal{H}^1_w\| = \|V_g \ f | L^1_w\|.$$

As mention by Feichtinger and Gröchenig (loc. cit.),$\mathcal{H}_w^1(G)$ is a π-invariant Banach space dense in $\mathcal{H}$ and the set $\{\pi(x)g, x \in G\}$ is a total subset of $\mathcal{H}_w^1(G)$

We denote by $\mathcal{H}_w^{1^\sim}(G)$ the Banach space of all continuous-conjugate linear functionals on $\mathcal{H}_w^1(G)$. Hence $\mathcal{H}_w^{1^\sim}(G)$ is a π-invariant Banach space with the continuous dense embeddings

$$\mathcal{H}_w^1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_w^{1^\sim}$$

Which insure that

$$(\mathcal{H}_w^1,\ \mathcal{H},\ \mathcal{H}_w^{1^\sim})$$

forms a Gabor triple (for detail see [3])

4. Extension of Unitary Gelfand Triple Isomorphisms

In a recent paper Feichtinger and Kozek ([2], pp.239-240) have shown that a unitary mapping U acting from $L^2(G_1)$ to $L^2(G_2)$ extends to an isomorphism between the Gelfand triples $(S_0,\ L^2,\ S_0')(G_1)$ and $(S_0,\ L^2,\ S_0')(G_2)$ if an only if the restrictions of U and U^* are bounded linear operators between $S_0(G_1))$ and $S_0(G_2)$, where $S_0(G_1)$ and $S_0(G_2)$ denote the Feichtinger algebras on elementary locally compact abelian groups G_1 and G_2 respectively and $S_0'(G_1))$ and $S_0'(G)1)$ their topological duals. Also, they have pointed out some applications of the above isomorphism ([2], p.239).

In this section, on the lines of Feichtinger and Kozek, we study an extension of a unitary mapping $\mathcal{U}$ acting from $\mathcal{H}$ to $\mathcal{H}$ to an isomorphism between the Gelfand triples $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$, where $G_1,\ G_2$ are any two locally compact groups.

Precisely, we prove the following :

Theorem 4.1. If $\mathcal{U}$ is an unitary operator from $\mathcal{H}(G_1)$ to $\mathcal{H}(G_2)$, then it extends isomorphism from the Gelfand triples where $G_1,\ G_2$ are any two locally compact groups $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$ if and only if there exists a positive constant c such that

$$\|\mathcal{U}f|\mathcal{H}_w^1(G_2)\| \leq c\ \|f|\mathcal{H}_w^1(G_1)\|, \quad \forall\ f \in \mathcal{H}_w^1(G_1) \tag{4.1}$$

and

$$\|\mathcal{U}^*f|\mathcal{H}_w^1(G_1)\| \leq c\ \|f|\mathcal{H}_w^1(G_2)\|, \quad \forall\ f \in \mathcal{H}_w^1(G_2) \tag{4.2}$$

where $\mathcal{U}^*$ is the adjoint operator of $\mathcal{U}$.

Proof: The proof follows on the lines of Feichtinger and Kozek ([2], p. 240). But, since our settings are different, it is necessary to give the proof.

Let us assume that (4.1) holds true. Then by virtue of the relation

$$\langle \bar{\mathcal{U}}g, f\rangle = \langle g, \mathcal{U}^* f\rangle,\ \ \forall\ g \in \mathcal{H}_w^{1\sim}(G_1) \text{ and } \ f \in \mathcal{H}_w^{1\sim}(G_2),$$

we see that $g \to \bar{\mathcal{U}}\ g$ is a bounded linear mapping, which extends the unitary map $\mathcal{U}$ on $\mathcal{H}(G_1)$. Next, since $\mathcal{H}(G_1)$ is boundedly dense in $\mathcal{H}_w^{1\sim}(G_1)$, $\bar{\mathcal{U}}$ is a continuous and bounded linear mapping from $\mathcal{H}_w^{1\sim}(G_1)$ to $\mathcal{H}_w^{1\sim}(G_2)$. Also, $\bar{\mathcal{U}}$ is unique and it coincides with $\mathcal{U}$ on $\mathcal{H}(G_1)$. In the same way it is clear that $\bar{\mathcal{U}}^*$ is a unique, continuous and bounded linear mapping from $\mathcal{H}_w^{1\sim}(G_2)$ to $\mathcal{H}_w^{1\sim}(G_1)$, which coincides with $\mathcal{U}^*$ on $\mathcal{H}(G_2)$.

Thus we infer that $\bar{\mathcal{U}}$ defines an isomorphism between $\mathcal{H}_w^{1\sim}(G_1)$ and $\mathcal{H}_w^{1\sim}(G_2)$ with respect to their norm topologies.

Conversely, we suppose that $\mathcal{U}$ is a unitary operator, which extends as an isomorphism from $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$. Hence the restrictions of $\mathcal{U}$ and $\mathcal{U}^*$ are bounded linear mapping on the spaces $\mathcal{H}_w^1(G_1)$ and $\mathcal{H}_w^1(G_2)$ respectively.

This completes the proof of the theorem.

As a corollary of the above theorem, we show that the Gelfand triples isomorphisms holds true provided there exists a bijection mapping V between $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ such that

$$\langle f_1\ f_2\rangle_{\mathcal{H}(G_1)} = \langle Vf_1,\ Vf_2\rangle_{\mathcal{H}(G_2)}$$

for all $f_1,\ f_2 \in \mathcal{H}(G_1)$.

As in [2], p.240), we prove the following :

Corollary: If $V : \mathcal{H}_w^1(G_1) \to \mathcal{H}_w^1(G_2)$ is an isomorphism, then it extends to a unitary Gelfand triples isomorphism between $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$ if and only if

$$\langle f_1,\ f_2\rangle_{\mathcal{H}(G_1)} = \langle Vf_1,\ Vf_2\rangle_{\mathcal{H}(G_2)} \tag{4.3}$$

for all $f_1,\ f_2 \in \mathcal{H}_w^1(G_1)$.

Proof: The isomorphism V as defined above is a unitary operator isomorphism from $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ to $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$ provided the condition (4.3) holds true.

Conversely, we suppose that the condition (4.3) holds. Then we have

$$\begin{aligned}
\|f\|^2_{\mathcal{H}(G_1)} &= \langle f,\ f\rangle_{\mathcal{H}(G_1)} \\
&= \langle Vf,\ Vf\rangle_{\mathcal{H}(G_2)} \\
&= \|Vf\|^2_{\mathcal{H}(G_2)}, \quad \forall\ f \in \mathcal{H}_w^1(G_1). \\
\Rightarrow\ \|f\|^2_{\mathcal{H}(G_1)} &= \|Vf\|_{\mathcal{H}(G_2)} \qquad \text{say}\, \mathcal{U},
\end{aligned}$$

$\Rightarrow$ V is extends to an isomorphism mapping, from $\mathcal{H}(G_1)$ to $\mathcal{H}(G_2)$.

Next, since $V(\mathcal{H}_w^1(G_1)) = \mathcal{H}_w^1(G_1)$ is dense in $\mathcal{H}(G_2)$ $\mathcal{U}$ has a dense range in $\mathcal{H}(G_2)$.

$\Rightarrow$ $\mathcal{U}$ is an isomorphism from $\mathcal{H}(G_1)$ to $\mathcal{H}(G_2)$.

Hence, by the duality condition, $\mathcal{U}$ is an isomorphism between the Gelfand triples $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_1)$ and $(\mathcal{H}_w^1, \mathcal{H}, \mathcal{H}_w^{1\sim})(G_2)$
This complete the proof the theorem

References

[1] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions I, *Journal of Functional analysis*, 86 (1989), 307-340.
[2] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In H.G. Feichtinger and T. Ströhmer, Editors, Gabor Analysis and Alorithms: Theory and Applications, Birkhaüser, Boston, 1998, 233-266.
[3] I.M. Gelfand and N.J. Wilenkin. Genralized Functions, Vol. IV: Some Applications of Harmonic Analysis. Rigged HIlbert Spaces. Academic Press, New York, 1964.

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