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Some New Localization Results for Eigenvalues of Complex Matrices

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Abstract. The localization of spectrum of complex-valued matrices is a foundational tool in matrix analysis and applied mathematics which offers a concise and computable descriptions on the location of spectrum in the complex plane. Many effective localization bounds, for instance, Gorgerin-type regions, and pseudo-spectrum allows to study and analyze the stability, sensitivity to perturbations, and spectral separation without full eigenvalue computation. In this paper, we present some new results on the localization of the spectrum of complex-valued matrices. Our approach is based on collection of various tools from matrix theory and numerical linear algebra. We have used MATLAB to demonstrate the localization of spectrum contained in the Gershgorin disks in the complex plane.

1. Introduction

The problem of locating eigenvalues of a matrix has been a central question in linear algebra. Computing exact eigenvalues of a matrix can be challenging, especially for large or complex matrices, and often unnecessary when approximate locations suffice for applications such as stability analysis, control theory, or spectral estimation. For a given $M \in \mathbb{C}^{n,n}$, one of the fundamental and classical results was given by Gershgorin [4] on the localization of all eigenvalues in the complex plane. His result determines a collection of disks in the complex plane whose union contains all eigenvalues of a given n-dimensional square complex-valued matrix. Over the last

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several decades, this result has served as a basic and powerful tool for estimating the spectrum of a complex matrix.

According to the Gershgorin circle theorem, the spectrum

$$\sigma(M) := \{\lambda \in \mathbb{C} : \det(\lambda I_n - M) = 0\}$$

of a complex matrix $M = [m_{ij}]$ belongs to the Gershgorin set $D(M)$, which is the union of all Gershgorin disks

$$D_i(M) := \{z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M)\}, \quad i = 1, 2, \ldots, n,$$

with center m_{ii} and radius $R_i(M) := \sum_{j \in \mathbb{N}\setminus\{i\}} |m_{ij}|$. This result remains important since, in many applications, it is sufficient to determine a region containing the eigenvalues rather than computing them exactly.

Many important applications and extensions of the Gershgorin circle theorem are presented in the monograph by Varga [12]. A major refinement was introduced by Brauer in 1947, who rediscovered the non-singularity result of Ostrowski [5], leading to the well-known Brauer's Cassini ovals. Instead of constructing disks centered at diagonal entries, this approach considers pairs of diagonal elements to produce Cassini-type regions, which typically yield tighter spectral bounds. While there are n Gershgorin disks, there are $\binom{n}{2}$ Cassini ovals for eigenvalue inclusion. Both the Gershgorin set $D(M)$ and the Brauer set $\kappa(M)$ depend on the same quantities

$$\{m_{ii}\}_{i=1}^{n} \quad \text{and} \quad \{R_i(M)\}_{i=1}^{n}.$$

Moreover, it is known that $\kappa(M) \subseteq D(M)$, implying that the Cassini ovals provide a sharper estimate of $\sigma(M)$.

Further refinements have focused on the role of geometric multiplicity in eigenvalue localization. In particular, it was shown in [7] that if $M \in \mathbb{C}^{n,n}$ has an eigenvalue of geometric multiplicity k, then this eigenvalue lies in at least k Gershgorin disks. This idea was extended in subsequent works [6, 8], leading to improved understanding of spectral inclusion. More recently, Marsli and Hall [9] introduced Gershgorin disks of the second type, which incorporate multiplicity information to obtain sharper localization results.

New Gershgorin-type results based on Schur complement techniques have also been developed [1]. These methods provide improved inclusion regions and, in some cases, outperform classical Cassini-type estimates by incorporating block structure and operator-theoretic ideas.

Despite these developments, Gershgorin-type methods often yield conservative bounds, especially when the matrix is not diagonally dominant. This has motivated the development of alternative approaches aimed at improving localization accuracy. In this context, Wu, Wang, and Zhao [13] proposed refined localization techniques based on matrix norms and algebraic identities, leading to sharper bounds for eigenvalue regions and matrix spread. Furthermore, Wu and Zhao [14] provided a comprehensive survey highlighting the limitations of classical methods and the need for improved localization techniques.

Another important direction involves analytical characterization of spectral regions through matrix equations. In [3], localization of $\sigma(M)$ is linked to Lyapunov-type equations, providing necessary and sufficient conditions for spectral inclusion in regions bounded by conic sections. These ideas are further extended in [2] to study localization with respect to parabolic domains. Although these approaches are theoretically strong, they often require solving auxiliary equations, which may limit their practical use.

Structure-based methods have also contributed to improving eigenvalue localization. Peña [10] derived sharp inclusion intervals for symmetric positive Toeplitz matrices, while Solmaz and Ayyıldız [11] proposed a hybrid symbolic–numeric approach combining Gershgorin disks with algebraic techniques to obtain certified intervals for real eigenvalues.

Despite the extensive literature, several challenges remain. Many existing methods either provide conservative bounds, involve complex computations, or are restricted to special matrix classes. Therefore, developing localization techniques that are both accurate and computationally efficient remains an important problem.

Motivated by these observations, the main objective of this paper is to establish new results on the localization of eigenvalues of a matrix M. Our approach refines classical Gershgorin-type techniques and introduces new analytical tools to derive improved inclusion regions for $\sigma(M)$. The proposed results aim to reduce overestimation while preserving simplicity and general applicability. Furthermore, numerical experiments using MATLAB are provided to illustrate the effectiveness of the proposed methods.

2. Preliminaries

The notation $M \in \mathbb{C}^{n,n}$ represent a n-dimensional complex-valued matrix. The radius of Gresgorin disk with center m_{ii} is denoted with $R_i = \sum_{j\in\mathbb{N}\setminus\{i\}} |m_{ij}|$. The real part of spectrum (the set of eigenvalues) is represented by $Re(\lambda_i)$. Next, We recall definitions, propositions, theorems from [12].

Definition 1. *For any complex-valued matrix $M = [m_{ij}] \in \mathbb{C}^{n,n}$, the spectrum $\sigma(M)$ is the collection of all eigenvalues of M, i.e,*

$$\sigma(M) = \{\lambda \in \mathbb{C} \, : \, det(M - \lambda I) = 0\}.$$

where a scalar $\lambda \in \mathbb{C}$ is called an eigenvalue of M if $Mv = \lambda v$, i.e., there exists a vector $v \neq 0$ where the vector $v \in \mathbb{C}^n$. The vector v is called an eigenvector corresponding to the eigenvalue λ.

Definition 2. *Let $M = [m_{ij}] \in \mathbb{C}^{n,n}$. The ith deleted absolute row sum of a matrix or for the i-th row of M, the sum of absolute values of all entries of a matrix except the diagonal entry m_{ii} is defined as*

$$R_i(M) := \sum_{j\in\mathbb{N}\setminus\{i\}} |m_{ij}|, \quad \forall i \in \mathbb{N} \quad where \quad \mathbb{N} := \{1, 2, 3, \dots n\}.$$

Definition 3. *A matrix $M \in \mathbb{C}^{n,n}$ is said to be diagonally dominant if $|m_{ii}| \geq \sum_{j\neq i} |m_{ij}|$, $\forall i \in \mathbb{N}$.*

Definition 4. *A matrix $M = [m_{ij}] \in \mathbb{C}^{n,n}$ is said to be strictly diagonally dominant if $|m_{ii}| > \sum_{j\neq i} |m_{ij}|$, $\forall i \in \mathbb{N}$. If M is strictly diagonally dominant, then it is non-singular.*

Definition 5. *Let $M \in \mathbb{C}^{n,n}$. The matrix M is said to be Hermitian if $M = M^*$, where $M^* = \overline{M}^T$ denotes the conjugate transpose of M. Equivalently, $m_{ij} = \overline{m_{ji}} \quad \text{for all} i, j = 1, 2, \ldots, n$.*

Proposition 1. *Let $M \in \mathbb{C}^{n,n}$ be Hermitian. Then:*

(1) *The matrix has all real eigenvalues, that is, $\sigma(M) \subset \mathbb{R}$.*
(2) *The eigenvectors corresponding to distinct eigenvalues are orthogonal.*
(3) *For all $x \in \mathbb{C}^n$, the quadratic form x^*Mx is real.*

Definition 6. *Let $M \in \mathbb{C}^{n,n}$. The Gershgorin disk corresponding to the i-th row is defined as*

$$D(m_{ii}, R_i) = \{z \in \mathbb{C} : |z - m_{ii}| \leq R_i\},$$

Here $D(m_{ii}, R_i)$ can be written as $= D_i(M)$. The ith-Gershgorin disk in the complex plane $\mathbb{C}$ having center at m_{ii} and radius $R_i(M)$. The union of all these Gershgorin disks of M is called the Gershgorin set and is defined as

$$D(M) = \bigcup_{i=1}^{n} D(m_{ii}, R_i) = \bigcup_{i=1}^{n} D_i(M).$$

The following theorem shows that all eigenvalues of the matrix M must lie within the union of these disks, thereby providing a clear and useful localization of the spectrum.

Theorem 1. *Let $M \in \mathbb{C}^{n,n}$ and any $\lambda \in \sigma(M), \exists\ k \in \mathbb{N}$ such that $|\lambda - m_{kk}| \leq R_k(M)$. Consequently, $\lambda \in D_k(M)$, and thus $\lambda \in D(M)$. Since this is true for every $\lambda \in \sigma(M)$, then $\sigma(M) \subseteq D(M)$.*

Corollary 1. *Let $M = [m_{ij}] \in \mathbb{C}^{n,n}$. Then every eigenvalue λ of M satisfies*

$$|\lambda| \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |m_{ij}|.$$

Proposition 2. *Let $M = [m_{ij}] \in \mathbb{C}^{n,n}$. Then the spectral radius satisfies*

$$\rho(M) \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |m_{ij}| = \|M\|_\infty.$$

Definition 7. *Let $M = [m_{ij}] \in \mathbb{C}^{n,n}$ and $\vec{v} = (d_1, \ldots, d_n)^T$ with $d_i > 0$. The ith weighted deleted absolute row sum of M is defined by*

$$r_i^{\vec{v}}(M) = \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{ij}| \frac{d_j}{d_i}, \quad i = 1, \ldots, n.$$

Definition 8. *The ith weighted Gershgorin disk corresponding to $\vec{v}$ is defined by*

$$D_i^{r^{\vec{v}}}(M) := \{z \in \mathbb{C} : |z - m_{ii}| \leq r_i^{\vec{v}}(M)\}, \quad \forall i \in \mathbb{N}.$$

The weighted Gershgorin set is

$$D^{r^{\vec{v}}}(M) := \bigcup_{i\in\mathbb{N}} D_i^{r^{\vec{v}}}(M).$$

Theorem 2. *Let $M \in \mathbb{C}^{n,n}$ and $\vec{v} \in \mathbb{R}^n$ with $d_i > 0$. Then every eigenvalue of M lies in the weighted Gershgorin set:*

$$\sigma(M) \subseteq \bigcup_{i \in \mathbb{N}} D_i^{r^{\vec{v}}}(M).$$

3. New Results

3.1. Localization and Perturbation of Eigenvalues. In this section, we present some new results on the localization of the eigenvalues of $M \in \mathbb{C}^{n,n}$. We use different tools from matrix theory and linear algebra to derive our new results, and then we make use of MATLAB to support these theoretical results.

The following Theorem 3 is to show that the real part of all eigenvalues of $M \in \mathbb{C}^{n,n}$ is strictly positive. All of these eigenvalues are positive if $M = M^*$.

Theorem 3. *Let $M \in \mathbb{C}^{n,n}$ such that $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i$. Then*

(i) If for all $i = 1, 2, \cdots, n$ we have that $|m_{ii}| > 0$, then $Re\left(\lambda_i(M)\right) > 0$.

(ii) If $M^* = M$, and $|m_{ii}| > 0$ for all $i = 1, 2, \cdots, n$, then $\lambda_i(M) > 0, \ \forall i$, here $*$ stands for the complex conjugate transpose of a matrix.

Proof. (i) Given $M \in \mathbb{C}^{n,n}$ such that $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i \in \mathbb{N}$, where $N = 1, 2, \cdots, n$, and $|m_{ii}| > 0, \forall i \in \mathbb{N}$. We need to show that the real part of every eigenvalue of M is positive, that is, $Re\left(\lambda_i(M)\right) > 0$. The diagonal entries $m_{i,i}$ satisfies $|m_{i,i}| > 0$, and condition is $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i \in \mathbb{N}$. This condition suggests that M is a type of matrix which is dominant by it's diagonal entries, and often rise to property about the real part of its eigenvalues. We make use of the Gershgorin circle theorem to the eigenvalues $\lambda_i, \ \forall i$ of M, that is, $\lambda_i(M) \in D_i(M) = (z \in \mathbb{C} : |z - m_{i,i}| \leq \sum_{j \neq i} |m_{i,j}|)$ where $D_i(M)$ are Gershgorin disks of M. The condition $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i \in \mathbb{N}$ implies that the radius $R_i = \sum_{j \in \mathbb{N} \setminus \{i\}} |m_{ij}|$ is less than or equal to the distance from the center $m_{i,i}$ to the origin. Thus, we have that $|m_{ii}| \geq R_i$. Since $|m_{i,i}| > 0$, so disk D_i is centered at positive real number $m_{i,i}$ and its radius doesn't exceed the distance from center to origin. This means that all eigenvalues of M must lie in the origin where the real part is positive. So $D_i, \ \forall i$ is in the right half of the plane. It follows that all eigenvalues of M must have positive real part. Thus, we finally have $Re\left(\lambda_i(M)\right) > 0$.

(ii) Given $M \in \mathbb{C}^{n,n}$ such that $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i$. If $M = M^*$, where $*$ denotes the complex conjugate transpose of M and $|m_{i,i}| > 0$, we have to show that $\left(\lambda_i(M)\right) > 0 \ \forall i$. Since M is symmetric, it means that M is a Hermitian matrix. Gershgorin circle theorem implies that $\lambda_i(M)$ must lie within the union of these disks. The condition $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|, \ \forall i$, ensures that the center of the disk $m_{i,i} > 0, \ \forall i$, is at least as far from the origin as the radius of the disks. This implies that the disk is entirely to the right (or on) the real axis, and that the real part of each eigenvalue must be positive. Since the given matrix is Hermitian, the eigenvalues of M are always real, and the disk being centered on the real axis, will contain real numbers as its eigenvalues. Since $|m_{i,i}| > 0, \ \forall i$, so Gershgorin disks have center $m_{i,i} > 0, \ \forall i$, and have non-negative radii. Thus, the disk is entirely to the right of the real axis of $\mathbb{C}$ and all eigenvalues of M are strictly positive. Thus, we finally have $\lambda_i(M) > 0, \ \forall i.$ □

Example 1 (i). *Given*

$$M = \begin{pmatrix} 4 & -1 & 1+i \\ 0.5 & 3 & -0.5 \\ 1 & 1 & 3 \end{pmatrix},$$

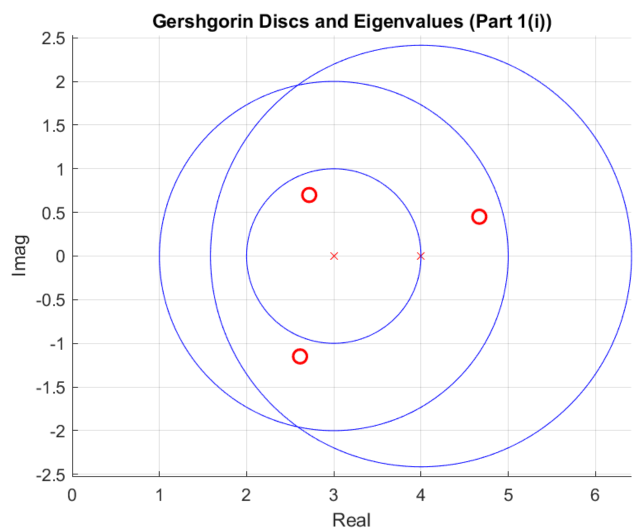
such that $|4| \geq |-1| + |1+i| = 1 + \sqrt{2} = 2.41$, $|3| \geq |0.5| + |-0.5| = 0.5 + 0.5 = 1$, $|3| \geq |1| + |1| = 2$

(i) $m_{11} = 4, m_{22} = 3, m_{33} = 3$, *so each* $m_{ii} > 0$ *for* $i = 1, 2, 3$. *Now, we have to show that* $Re(\lambda_i(M)) > 0$. *For this we make use of the Gershgorin circle theorem.*

$$\begin{aligned} &\textit{For row } 1, \quad m_{11} = 4, R_1 = 2.41. \textit{ So } D_1(M) = \{z \in \mathbb{C} : |z-4| \leq 2.41\}. \\ &\textit{For row } 2, \quad m_{22} = 3, R_2 = 1. \textit{ So } D_2(M) = \{z \in \mathbb{C} : |z-3| \leq 1\}. \\ &\textit{For row } 3, \quad m_{33} = 3, R_3 = 2. \textit{ So } D_3(M) = \{z \in \mathbb{C} : |z-3| \leq 2\}. \end{aligned}$$

Thus, these disks lie entirely in the right half of the complex plane. Hence $Re(\lambda_i(M)) > 0$, *for* $i = 1, 2, 3$.

$$\lambda_1 = 4.6688 + 0.4505i, \lambda_2 = 2.7183 + 0.6993i, \lambda_3 = 2.6128 - 1.1498i$$



Example 2 (ii). *Given*

$$M = \begin{pmatrix} 5 & 1+i & 0 \\ 1-i & 4 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

such that $|5| \geq |1+i| + |0| = \sqrt{2} + 0 = 1.41$, $|4| \geq |1-i| + |-1| = \sqrt{2} + 1 = 2.41$, $|3| \geq |0| + |-1| = 1$. *Since* $m_{12} = 1+i, m_{21} = \overline{1+i}, m_{23} = -1 = m_{32}, m_{13} = 0 = m_{31}$. *So* M *is the Hermitian matrix. Now we will show that* $\lambda_i(M) > 0$. *For this we make use of the Gershgorin circle theorem.*

$$\textit{For row } 1, \quad m_{11} = 5, R_1 = 1.41. \textit{ So } D_1(M) = (z \in \mathbb{C} : |z-5| \leq 1.41)$$

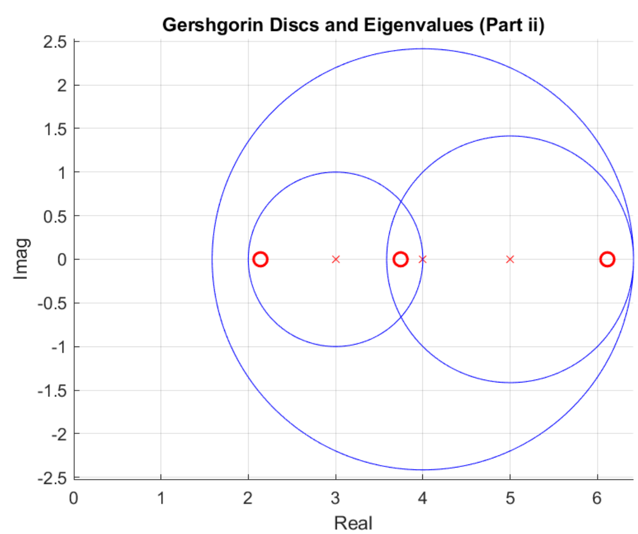
Algorithm 1 Gershgorin disks and eigenvalues - Theorem 2(i)

Input: Matrix $M \in \mathbb{C}^{n,n}$
Output: Computes the Gershgorin disks of M and verify that $Re\,(\lambda_i(M)) > 0$.
for $i = 1, \ldots, n$ **do**
    $d_i = m_{ii}, \quad R_i = \sum_{j \in \mathbb{N} \setminus \{i\}} |m_{ij}|$
end for
Compute eigenvalues $\lambda(M)$
if $Re\,(\lambda_i(M)) > 0$ **then**
    Print "Verification successful"
else
    Print "Verification failed"
end if
Plot Gershgorin disks and eigenvalues in the complex plane.
Mark centers with crosses and eigenvalues with red circles in the complex plane

$$\textit{For row}\, 2, \quad m_{22} = 4, R_2 = 2.41. \, \textit{So}\, D_2(M) = (z \in \mathbb{C} : |z-4| \le 2.41)$$
$$\textit{For row}\, 3, \quad m_{33} = 4, R_3 = 1. \, \textit{So}\, D_3(M) = (z \in \mathbb{C} : |z-3| \le 1).$$

Thus, these disks lie entirely in the right half of the complex plane. Hence $\lambda_i(M) > 0$, *for* $i = 1, 2, 3$.

$$\lambda_1 = 2.1392, \lambda_2 = 3.7459, \lambda_3 = 6.1149$$



Algorithm 2 Gershgorin disks and eigenvalues - Theorem 2(ii)

Input: Matrix $M \in \mathbb{C}^{n,n}$
Output: Computes the Gershgorin disks of matrix M, determines its eigenvalues and verify that $\lambda_i(M) > 0,\ \forall i$
for $i = 1, \ldots, n$ **do**
  $d_i = m_{ii}, \quad R_i = \sum_{j \in \mathbb{N}\setminus\{i\}} |m_{ij}|$
end for
Compute eigenvalues $\lambda(M)$
if $\lambda_i(M) > 0$ **then**
  Print "All eigenvalues are positive (verified)."
else
  Print "Verification failed: some eigenvalues are not positive."
end if
Plot Gershgorin disks using centers d_i and radii R_i.
Mark centers with crosses and eigenvalues with red circles in the complex plane

The following Theorem 4 show that if M is diagonally dominant and $|m_{i,i}| > \sum_{j\neq i} |m_{ij}|$ for at least $(n-1)$ values of $i \in \{1, 2, \cdots, n\}$, then none of its eigenvalue of is zero.

Theorem 4. *Let $M \in \mathbb{C}^{n,n}$ such that it has non-zero diagonal entries. If M is diagonally dominant and $|m_{i,i}| > \sum_{j\neq i} |m_{ij}|$ for at least $(n-1)$ values of $i \in \{1, 2, \cdots, n\}$, then $\Pi_i \lambda_i(M) \neq 0,\ \forall i$, where $\Pi_i \lambda_i(M)$ denotes the product of $\lambda_i(M),\ \forall i$.*

Proof. Given $M \in \mathbb{C}^{n,n}$ such that $m_{i,i} \neq 0$, where M is diagonally dominant and condition $|m_{i,i}| > \sum_{j\neq i} |m_{ij}|$ holds for at least $(n-1)$ value of $i \in N$. We have to show that the product of all eigenvalues of M is non-zero, that is, $\Pi_i \lambda_i(M) \neq 0,\ \forall i$. To prove this we make use of Gershgorin circle theorem that eigenvalues lie within Gershgorin disks, which are centered at $m_{i,i}$ with radius equal to absolute sum of off-diagonal entries. Since, the condition $|m_{i,i}| > \sum_{j\neq i} |m_{ij}|$ holds for at least $(n-1)$ rows, it means that eigenvalues of M are likely close to diagonal entries $m_{i,i}$ and since these entries $m_{i,i}$ are non-zero, so eigenvalues are also non-zero. Thus, we finally have $\Pi_i \lambda_i(M) \neq 0,\ \forall i$.

Alternative proof: Given $M \in \mathbb{C}^{n,n}$ such that $m_{i,i} \neq 0$, where M is diagonally dominant and condition $|m_{i,i}| > \sum_{j\neq i} |m_{ij}|$ holds for at least $(n-1)$ value of $i \in N$. We have to show that the product of all eigenvalues of M is non-zero, that is, $\Pi_i \lambda_i(M) \neq 0,\ \forall i$. In $n \times n$ matrix, $det(M) = \lambda_1 \lambda_2 \cdots \lambda_n = \Pi_i \lambda_i(M),\ \forall i \in N$. To show matrix has non-zero determinant, we have to show that matrix is invertible. Since the matrix is diagonally dominant overall but strictly diagonally dominant for at least $(n-1)$ rows. Strictly diagonal dominance ensures that the matrix is invertible. Even if the strict diagonal dominance condition does not hold for one row, the strict dominance condition for at least $(n-1)$ rows ensures that the matrix is still invertible. So, invertibility of a matrix is not destroyed by single rows that do not satisfy the strict dominance condition, because

$m_{i,i} \neq 0$ and the dominant diagonal entries in the row ensure that the matrix remains invertible overall. Thus, $det(M) \neq 0$. We finally have $\Pi_i \lambda_i(M) \neq 0, \ \forall i \in N$. □

Example 3. *Given*

$$M = \begin{pmatrix} 4 & -1 & 1+i \\ 2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

such that $m_{11} = 4, m_{22} = 2, m_{33} = 3$ *all are non-zero and*

$$|4| \geq |-1| + |1+i| = 1 + \sqrt{2} = 2.41, \quad |2| \geq |2| + |0| = 2 + 0 = 2, \quad |3| \geq |1| + |0| = 1.$$

So M is diagonally dominant and the condition of strictly diagonal dominant satisfies for 2 rows. Since 2 *rows are diagonally dominant and have non-zero centers, the disks are far from the origin. So, at least 2 eigenvalues are far from 0. The third eigenvalue may be closer to* 0*, but will not be exactly* 0*, because* $m_{22} \neq 0$*. Therefore, none of the eigenvalues is zero. Thus,* $\Pi_i \lambda_i(M) \neq 0$*, for* $i = 1, 2, 3$*. Since the determinant of M is non-zero, that is,* $det(M) = 28 - 2i \neq 0$*. And, we have*

$$\lambda_1 = 4.1671 + 0.5774i, \ \lambda_2 = 2.4226 - 1.1671, \ \lambda_3 = 2.4102 + 0.5898i,$$

and product of these eigenvalues of M is $\Pi_i \lambda_i(M) = 28 - 2i \neq 0$.

Algorithm 3 Computation of Gershgorin Disks and product of Eigenvalues - Theorem 3

Input: A matrix M of size $n \times n$

Output: Computes the Gershgorin disks, calculates the determinant and eigenvalues of matrix M, establishes the relationship $\det(M) = \prod_{i=1}^{n} \lambda_i$, and confirms the consistency between the determinant and the eigenvalue product.

Compute determinant $\det(M)$

Compute all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Compute product $P = \prod_{i=1}^{n} \lambda_i$

if $\det(M) = P$ within numerical tolerance **then**

  Print "Verification successful"

else

  Print "Verification failed"

end if

Plot Gershgorin disks using centers d_i and radii R_i.

Mark centers with crosses and eigenvalues with red circles in the complex plane.

The following Theorem 5 is to present some conditions under which the eigenvalues are on the boundary of Gershgorin disks rather than laying in the interior of such disks.

Theorem 5. *Let $M \in \mathbb{C}^{n,n}$, and let $\lambda \in \mathbb{C}$ is given. Then*

(i) λ is not in the interior of any of Gershgorin disk of M if and only if $|\lambda - m_{i,i}| \geq R_i^{'} = \sum_{j \neq i} |m_{ij}|, \ \forall i = 1 : n$.

(ii) If $\lambda \in \partial G(M)$, then it must satisfies the inequality $|\lambda - a_{i,i}| \geq R_i^{'}$.

(iii) Given $M \in \mathbb{C}^{n,n}$ is diagonally dominant if and only if $\lambda = 0$ must satisfies inequality $|\lambda - m_{i,i}| \geq R_i^{'}$.

Here ∂ denotes the boundary of $G(M) := \cup_{i=1}^{n}(\{z \in \mathbb{C} : |z - m_{i,i}| \leq R_i^{'}(M)\})$ and $R_i^{'}(M) := \sum_{j \neq i} |m_{ij}|, \ i = 1 : n$.

Proof. Given $M \in \mathbb{C}^{n,n}$ and $\lambda \in \mathbb{C}$.

(i) We know that $G_i(M) = \{z \in \mathbb{C} : |z - m_{i,i}| \leq R_i^{'}(M)\}$ is Gershgorin Disk and $(int\, G_i(M)) := \{z \in \mathbb{C} : |z - m_{i,i}| < R_i^{'}(M)\}$. Now, Lets suppose on contrary $\lambda \in (int\, G(M))$ then by definition of Gershgorin circle theorem $|\lambda - m_{i,i}| \leq \sum_{j \neq i} |m_{ij}|$, which is a contradiction. So, if λ is not in the interior of any of the Gershgorin disks of M then $|\lambda - m_{i,i}| \geq R_i^{'} = \sum_{j \neq i} |m_{ij}|, \ \forall i = 1 : n$. Conversely, if $|\lambda - m_{i,i}| \geq \sum_{j \neq i} |m_{ij}|$, then λ is not strictly inside any of the disks, it will lie on the boundary or outside the disk. So $\lambda \notin (IntG(M))$. Thus we finally have λ is not in the interior of any of Gershgorin disk of M if and only if $|\lambda - m_{i,i}| \geq R_i^{'} = \sum_{j \neq i} |m_{ij}|, \ \forall i = 1 : n$.

(ii) If $\lambda \in (\partial G(M))$ then by definition $(\partial G(M)) = \cup_{i=1}^{n}\{z \in \mathbb{C} : |z - m_{i,i}| = R_i^{'}(M)\}$, we have $|\lambda - m_{i,i}| = R_i^{'}$. The inequality will hold exactly as an equality. Thus if $\lambda \in \partial G(M)$, then we can say that it must satisfies the inequality $|\lambda - m_{i,i}| \geq R_i^{'}$.

(iii) Given M is diagonally dominant means that $|m_{i,i}| \geq \sum_{j\neq i} |m_{ij}| = R_i'$, $\forall i = 1:n$. Let $\lambda = 0$, then $|\lambda - m_{i,i}| = |0 - m_{i,i}| = |m_{i,i}|$. By definition $|m_{i,i}| \geq R_i'$. Hence $\lambda = 0$ must satisfies inequality $|\lambda - m_{i,i}| \geq R_i'$. Conversely, if λ satisfy the inequality $|\lambda - m_{i,i}| \geq R_i'$ and $\lambda = 0$ then $|m_{i,i}| \geq \sum_{j\neq i} |m_{ij}|$, $\forall i = 1:n$. So M is diagonally dominant. Thus we finally have $A \in \mathbb{C}^{n,n}$ is diagonally dominant if and only if $\lambda = 0$ must satisfies inequality $|\lambda - m_{i,i}| \geq R_i'$. □

Example 4. *Let*

$$M = \begin{pmatrix} 4 & -1 & 1+i \\ 2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

and $\lambda = 0 \in \mathbb{C}$ be given.

(i) Let's find Gershgorin disks:

For row 1, $m_{11} = 4, R_1 = 2.41$. So, $G_1(M) = (z \in \mathbb{C} : |z-4| \leq 2.41)$.

For row 2, $m_{22} = 2, R_2 = 2$. So, $G_2(M) = (z \in \mathbb{C} : |z-2| \leq 2)$.

For row 3, $m_{33} = 3, R_3 = 1$. So, $G_3(M) = (z \in \mathbb{C} : |z-3| \leq 1)$.

If $\lambda = 0$, then $|0-4| = 4 \geq 2.41, \quad 2 \geq 2, \quad 3 \geq 1$.

Hence $\lambda \notin (int\, G_i(M))$.

(ii) We know that $\lambda \in (\partial G(M))$ iff $|\lambda - m_{i,i}| = R_i'(M)$, for some i.

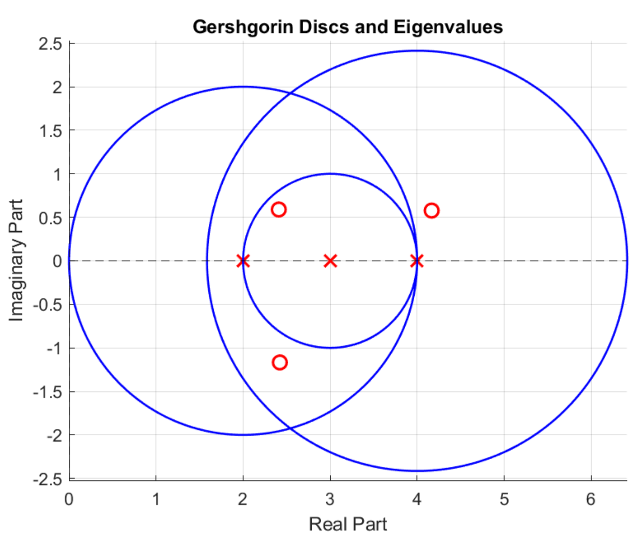
Now, if $\lambda = 0$, then $4 \neq 2.41, \quad 2 = 2, \quad 3 \neq 1$.

So, $\lambda = 0$ lies on the boundary of the disk 2. It also satisfies the inequality $|0-2| = 2 \geq 2$.

(iii) If $\lambda = 0$ satisfies the inequality $|0 - m_{ii}| \geq R_i'(M)$, we have to show that M is diagonally dominant.

So, $|0-4| = 4 \geq 2.41, \quad |0-2| = 2 \geq 2, \quad |0-3| = 3 \geq 1$.

Thus, M is the diagonally dominant matrix.



Algorithm 4 Localization of eigenvalues and diagonal dominance by using Gershgorin Disks - Theorem 4

Input: A square complex matrix M, and a complex number λ

Evaluates whether a given λ lies inside, on, or outside the Gershgorin disks of matrix M, and determines if the matrix satisfies the diagonal dominance condition.

Extract diagonal entries $d_i = m_{ii}$ and compute radii

$$R_i = \sum_{j \in \mathbb{N} \setminus \{i\}} |m_{ij}|$$

(i) Interior check:
if $|\lambda - d_i| < R_i$ for some i **then**
    Print "λ lies inside at least one Gershgorin disk"
else
    Print "λ does not lie inside any Gershgorin disk"
end if

(ii) Boundary check:
if $|\lambda - d_i| = R_i$ for some i **then**
    Print "λ lies on the boundary of disk i"
else
    Print "λ does not lie on the boundary of any disk"
end if

(iii) Diagonal dominance check:
if $|\lambda - d_i| \geq R_i$ for all i **then**
    Print "Matrix is diagonally dominant"
else
    Print "Matrix is not diagonally dominant"
end if

The following Theorem 6 is to construct conditions such that each of Gershgorin disk will pass through an eigenvalue λ of given $M \in \mathbb{C}^{n,n}$.

Theorem 6. *Let $M \in \mathbb{C}^{n,n}$, and let (λ, x) be an eigen pair of M such that λ satisfying inequality $|\lambda - m_{i,i}| \geq R_i^{'} = \sum_{j \neq i} |m_{ij}|,\ \forall i = 1, 2, \cdots n$. Let $m_{i,j} \neq 0,\ \forall i, j$, then*

(i) Each Gershgorin circle for $M \in \mathbb{C}^{n,n}$ will passes through an eigenvalue λ.

(ii) For eigenvector x, $|x_i| = \|x\|_\infty,\ \forall 1, 2, \cdots, n$.

Proof. Given $M \in \mathbb{C}^{n,n}$ and let (λ, x) be an eigen pair of M so $Mx = \lambda x$ with $x \neq 0$ such that λ satisfying inequality $|\lambda - m_{i,i}| \geq R_i^{'} = \sum_{j \neq i} |m_{ij}|,\ \forall i = 1, 2, \cdots n$ and $m_{i,j} \neq 0,\ \forall i, j,$

(i) We have to show that each Gershgorin circle for $M \in \mathbb{C}^{n,n}$ will passes through an eigenvalue λ, that is, for every i we must show that $|\lambda - m_{i,i}| = R_i'$. Since $Mx = \lambda x$, with $x \neq 0$ for each $i \in (1, 2, \cdots, n)$, we have $(\lambda - m_{i,i})x_i = \sum_{j\neq i} m_{ij}x_j$. By taking absolute and applying triangle inequality we have $|\lambda - m_{i,i}||x_i| \leq \sum_{j\neq i} |m_{ij}||x_j|$. Now, suppose on contrary that inequality $|\lambda - m_{i,i}| > \sum_{j\neq i} m_{ij}$ holds $\forall i = 1, 2, \cdots n$. When we multiplying $|x_i| \geq 0$ on both sides, we have $|\lambda - m_{i,i}||x_i| > \sum_{j\neq i} |m_{ij}||x_i|$. By comparing both equations $\sum_{j\neq i} |m_{ij}||x_i| < \sum_{j\neq i} |m_{ij}||x_j|$. This implies that there exist at least one $j \neq i$ such that $|x_j| > |x_i|$. If we assume strict inequality for each row, that is, $|x_i|$ is smaller than some other $|x_j|$, which is impossible for finite set. Hence, this is a contradiction. Thus, we finally have each Gershgorin circle for $M \in \mathbb{C}^{n,n}$ will passes through an eigenvalue λ.

(ii) Since $Mx = \lambda x$, for each $i \in \{1, 2, \cdots, n\}$, we have $(\lambda - m_{i,i})x_i = \sum_{j\neq i} m_{ij}x_j$. By taking absolute and applying triangle inequality we have $|\lambda - m_{i,i}||x_i| \leq \sum_{j\neq i} |m_{ij}||x_j|$. Given λ satisfies inequality $|\lambda - m_{i,i}| \geq \sum_{j\neq i} |m_{ij}|$, $\forall i = 1, 2, \cdots n$. When we multiplying $|x_i| \geq 0$ on both sides, we have $|\lambda - m_{i,i}||x_i| \geq \sum_{j\neq i} |m_{ij}||x_j|$. By comparing both equations $\sum_{j\neq i} |m_{ij}||x_i| \leq \sum_{j\neq i} |m_{ij}||x_j|$. So $\sum_{j\neq i}(|x_j| - |x_i|) \geq 0$. This implies that $|x_j| \geq |x_i|$, $\forall j \neq i$. But this is true for every index i, so only way it works is if $|x_1| = |x_2| = \cdots = |x_n| = |x_\infty|$. Hence $|x_i| = \|x\|_\infty$, $\forall 1, 2, \cdots, n$. □

Example 5. *Let*

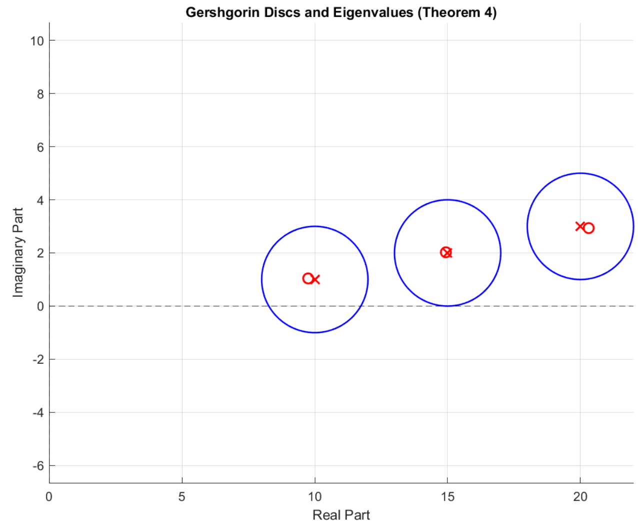
$$M = \begin{pmatrix} 10+i & 1 & 1 \\ 1 & 15+2i & 1 \\ 1 & 1 & 20+3i \end{pmatrix},$$

and $\lambda_1 = 9.7511 + 1.0410i$, $\quad \lambda_2 = 20.3140 + 2.9349i$, $\quad \lambda_3 = 14.9349 + 2.0241i$.

The corresponding eigenvectors are obtained as

$$\vec{x_1} = \begin{bmatrix} 0.9824 \\ -0.1673 + 0.0283i \\ -0.0772 + 0.0120i \end{bmatrix}; \quad \vec{x_2} = \begin{bmatrix} 0.1089 + 0.0242i \\ 0.1968 - 0.0392i \\ 0.9733 \end{bmatrix}; \quad \vec{x_3} = \begin{bmatrix} 0.1482 - 0.0217i \\ 0.9649 \\ -0.2111 + 0.0449i \end{bmatrix}$$

Let (λ_1, x_1) *be an eigen pair of M satisfying the inequality such that* $|20.3140 + 2.9349i - (10 + i)| = 10.498 \geq 2, |29.6220 + 1.0127i - (15 + 2i)| = 14.656 \geq 2, |29.6220 + 1.0127i - (20 + 3i)| = 9.822 \geq 2.$ *Since* $Mx = \lambda x$, *for each* $i = 1, 2, 3$, $(\lambda - m_{ii})x_i = \sum_{j\neq i} m_{ij}x_j$. *For* $i = 1, m_{11} = 10 + i, r_1(M) = 2$. *By taking the absolute and applying the triangle inequality, we have* $|\lambda - m_{11}||x_1| = |m_{12}x_2 + m_{13}x_3| \leq |x_2| + |x_3|$. *Suppose on contrary that* $|\lambda - m_{11}| > 2$. *So,* $2|x_1| < |x_2| + |x_3|$. *This implies that* $|x_1|$ *is less than at least one of* $|x_2|$ *or* $|x_3|$. *Similarly, for* $i = 2:\ 2|x_2| < |x_1| + |x_3|$. *Again* $|x_2|$ *is less than at least one of* $|x_1|$ *or* $|x_3|$. *Similarly, for* $i = 3:\ 2|x_3| < |x_1| + |x_2|$. *Again* $|x_3|$ *is less than at least one of* $|x_1|$ *or* $|x_2|$. *This is impossible for a finite set. So, this is a contradiction. Hence, each Gershgorin circle for* $M \in \mathbb{C}^{3,3}$ *will pass through an eigenvalue* λ.



Algorithm 5 Each Gershgorin disk will pass through and eigenvalue of M Theorem 5(i)

Input: A complex matrix M of size n, n
Compute eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors of M
Extract diagonal entries $m_{ii} = d_i$ as centers of Gershgorin disks
Compute radii

$$R_i = \sum_{j \in \mathbb{N} \setminus \{i\}} |m_{ij}|$$

for each eigenvalue λ_k **do**
    for each row $i = 1, 2, \ldots, n$ **do**
        Check inequality

$$|\lambda_k - m_{ii}| \geq R_i$$

    end for
end for
if inequality holds for all rows and eigenvalues **then**
    Print "Verified: condition satisfied"
else
    Print "Condition not satisfied for some eigenvalues"
end if
Plot Gershgorin disks using centers d_i and radii R_i.
Mark centers with crosses and eigenvalues with red circles in the complex plane

---

Algorithm 6 Each Gershgorin disk will pass through and eigenvalue of M Theorem 5(ii)

Input: A square complex matrix $M \in \mathbb{C}^{n\times n}$
Output: Eigenpairs satisfying $|\lambda - m_{ii}| \geq R_i$
Compute radii for each row: $R_i = \sum_{j\in\mathbb{N}\setminus\{i\}} |m_{ij}|$, for $i = 1, \dots, n$
Compute all eigenpairs (λ_k, x_k) of M
Set numerical tolerance ε (e.g. $\varepsilon = 10^{-10}$)
for each eigenpair (λ_k, x_k), $k = 1, \dots, n$ **do**
    if for all $i = 1, \dots, n$ we have $|\lambda_k - m_{ii}| \geq R_i$ and $m_{ij} \neq 0$ for all i, j **then**
        (i) Circles-through-eigenvalue check:
        for each $i = 1, \dots, n$ **do**
            if $\big||\lambda_k - m_{ii}| - R_i\big| \leq \varepsilon$ **then**
                Record: "Gershgorin circle i passes through λ_k (within tol)"
            else
                Record: "Gershgorin circle i does not pass through λ_k (within tol)"
            end if
        end for
        (ii) Eigenvector infinity-norm check:
        Compute $s = \max_{1\leq t\leq n} |(x_k)_t|$
        if for all $t = 1, \dots, n$ we have $||(x_k)_t| - s| \leq \varepsilon$ **then**
            Record: "All components satisfy $|(x_k)_t| = \|x_k\|_\infty$ (within tol)"
        else
            Record: "Not all components equal $\|x_k\|_\infty$"
        end if
    else
        Record: "Eigenpair (λ_k, x_k) does not satisfy precondition; skip checks"
    end if
end for
Print all recorded results

Corollary 2. *Let $M \in \mathbb{C}^{n,n}$. Let $m_{i,j} \neq 0$. Consider that M has a diagonal dominant structure and if there exist $k \in (1, 2, \cdots, n)$ such that $|m_{k,k}| > R'_k$, then M must not even a single eigenvalue from its spectrum to be exactly equal to zero.*

Proof. By Gershgorin circle theorem, $D_i(M) = \{z \in \mathbb{C} : |z - m_{i,i}| \leq \sum_{j\neq i} |m_{i,j}|\}$. Since $|m_{i,i}| \geq R'_i$, But for some specific index k, $|m_{k,k}| > R'_k$. So, the disk doesn't contain the origin because $|0 - m_{k,k}| = |m_{k,k}| > R'_k$. It implies that $0 \notin D_k(M)$. Hence 0 is not an eigenvalue of M. Thus, M must not have even a single eigenvalue from its spectrum to be exactly equal to zero. $\square$

3.2. **Gershgorin type eigenvalue results.** In this section, we present some new Gershgorin type inclusion results on the inclusion of the eigenvalues in Gershgorin disks.

Theorem 7. *For any $A \in \mathbb{C}^{n,n}, n \geq 2$, and any $\lambda \in \sigma(A)$, there is a pair of distinct integers i and j in $\mathbb{N}$ such that $\lambda \in K_{i,j} := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq R_i(A) \cdot R_j(A)\}$. As this is true for each λ in $\sigma(A)$, then $\sigma(A) \subseteq \kappa(A) := \bigcup_{i \neq j, i, j \in N} K_{i,j}(A)$.*

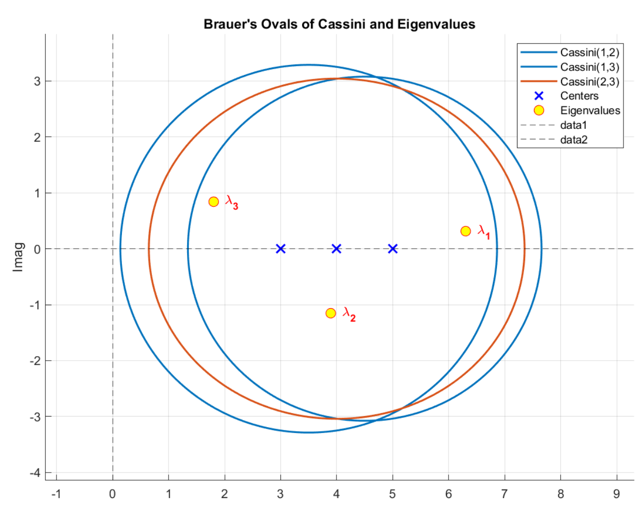
Proof. We have to show that $|\lambda - a_{i,i}| \cdot |\lambda - a_{j,j}| \leq R_i(A) \cdot R_j(A)$. To prove this, consider $\lambda \in \mathbb{C}$ be an eigenvalue of A with a non-zero eigenvector $\mathbf{x} = (x_1, \cdots x_n)^T$, i.e. $A\mathbf{x} = \lambda \mathbf{x}$, that is, $\sum_{j \in N} a_{i,j} x_j = \lambda x_i$. So, we have $\sum_{j \neq i} a_{i,j} x_j = (\lambda - a_{i,i}) x_i$. Let $s \in N$ such that $|x_s| = max|x_k| > 0$. For $i = s$, by taking absolute and apply triangle inequality, we have $|\lambda - a_{s,s}||x_s| \leq \sum_{j \neq s} |a_{s,j}||x_j| \leq r_s(A) \cdot |x_s|$. Similarly, for $i = t$, by taking absolute an apply triangle inequality, we have $|\lambda - a_{t,t}||x_t| \leq \sum_{j \neq t} |a_{t,j}||x_j| \leq r_t(A)|x_t|$. Thus, after multiplying these two equations, we have $|\lambda - a_{t,t}| \cdot |\lambda - a_{s,s}| \leq r_t(A) \cdot r_s(A)$. Then $\lambda \in K_{i,j}$ and hence $\lambda \in \kappa(A)$. As this is valid for each $\lambda \in \sigma(A)$, then $\sigma(A) \subseteq \kappa(A)$. □

Example 6. *Let*

$$A = \begin{pmatrix} 4 & 2+i & -1 \\ 1-i & 3 & 2 \\ 2 & 1 & 5 \end{pmatrix};$$

$\lambda_1 = 6.3021 + 0.3134i, \quad \lambda_2 = 3.8938 - 1.1530i, \quad \lambda_3 = 1.8041 + 0.8397i.$

We choose $\lambda_1 = 6.3021 + 0.3134i$, *and test the theorem on it. Here,* $r_1 = |2 + i| + |-1| = \sqrt{5} + 1 = 3.2361$, $r_2 = |1 - i| + |2| = \sqrt{2} + 2 = 3.4142$, $r_3 = |2| + |1| = 2 + 1 = 3$. *Now pick two distinct integers* $i = 1, j = 3$. *So* $|\lambda - a_{1,1}| = |6.3021 + 0.3134i - 4| = |2.3021 - 0.3134i| \approx 2.3233$, $|\lambda - a_{3,3}| = |6.3021 + 0.3134i - 5| = |1.3021 - 0.3134i| \approx 1.3383$. *Thus after multiplying these two we have,* $|\lambda - a_{1,1}|.|\lambda - a_{3,3}| \approx 2.3233 \times 1.3383 \approx 3.110$ *and* $r_1 \times r_3 \approx 3.2361 \times 3 \approx 9.7083$. *Hence, we finally have* $|\lambda - a_{1,1}|.|\lambda - a_{3,3}| \leq r_1.r_3$.



Algorithm 7 Computation of Brauer's Ovals of Cassini and Eigenvalues

1: **Input:** A square complex matrix A of size $n \times n$

2: **Output** Constructs Brauer's Cassini ovals for matrix A, computes its eigenvalues, and visualizes the eigenvalue spectrum with respect to the Cassini regions in the complex plane.

3: Extract diagonal entries a_{ii} as centers

4: Compute radii

$$R_i = \sum_{j \in \mathbb{N} \setminus \{i\}} |m_{ij}|$$

5: Compute eigenvalues $\lambda_1, \ldots, \lambda_n$ of A

6: **Construct Brauer's Ovals of Cassini:**

7: **for** each pair of indices (i, j) with $i < j$ **do**

8: Define the set of points $z \in \mathbb{C}$ such that

$$|z - a_{ii}| \cdot |z - a_{jj}| \leq R_i R_j$$

9: **end for**

10: Plot Cassini ovals for all pairs (i, j)

11: Mark centers a_{ii} as crosses

12: Mark eigenvalues λ_k as points with labels

4. Conclusion

In this article we studied the localization of the spectrum for n-dimensional complex matrices. Building on classical ideas from matrix theory and numerical linear algebra, we derived new Gershgorin-type bounds and sharpened existing eigenvalue-enclosure results. To demonstrate the practicality of these theoretical advances, we designed corresponding algorithms and implemented them in MATLAB, validating their effectiveness on representative test cases. Our results offer tighter, computationally tractable spectral enclosures that can improve diagnostics, preconditioning and shift selection in eigenvalue solvers. Future work will focus on scaling these methods to very large structured problems and integrating adaptive selection strategies that respond to non-normality and conditioning.

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