

A Note on the Convergence and Boundedness Properties of Discrete Wolff Potential Operators on Weighted Morrey Sequence Spaces

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Abstract. We introduce a discrete version of the Wolff potential operator $\mathcal{W}_{\alpha,p}$, defined for a sequence $x = \{x(k)\}_{k \in \mathbb{Z}}$ via an iterated dyadic ball-averaging raised to the power $1/(p-1)$, for parameters $0 < \alpha < 1$ and $1 < p < \infty$. The nonlinear averaging structure of the Wolff potential yields substantially different summability properties compared to the classical discrete Riesz potential. We establish sufficient conditions for absolute and uniform convergence of $\mathcal{W}_{\alpha,p}$ on ℓ^p and on discrete Morrey spaces, and we prove boundedness from the weighted discrete Morrey space $\ell_q^p(\omega^p, \omega^q)$ to $\ell_s^q(\omega^q)$ under suitable parameter conditions, using dyadic decomposition, local-global estimates, and the doubling property of Muckenhoupt weights. An explicit example shows the discrete Riesz potential can diverge while the discrete Wolff potential remains finite, demonstrating the genuine advantage of the nonlinear averaging decay. These results extend fractional integral operator theory to the discrete setting.

1. INTRODUCTION

Over the past several years, discrete harmonic analysis has witnessed significant growth, emerging as a vibrant and productive area of research. The discrete counterparts of standard Euclidean operators including the Hilbert transform, fractional integrals, and Riesz potentials have attracted considerable attention owing to their relevance in discrete PDEs, signal analysis, and number-theoretic problems. For related developments concerning other classes of operators within discrete harmonic analysis, we refer the reader to [2, 25, 36, 46, 52, 54, 57].

A substantial and closely related line of research studies the regularity and boundedness of discrete maximal- and fractional-type operators directly, rather than their fractional integral counterparts. Liu and Wu [14] established that the discrete multisublinear fractional maximal operator maps $\ell^1(\mathbb{Z}^d) \times \cdots \times \ell^1(\mathbb{Z}^d)$ boundedly and continuously into the space of functions of bounded variation, and obtained pointwise estimates for its partial derivatives; a companion paper [22] sharpened the endpoint regularity theory for the (unweighted) discrete maximal operator itself. Pierce studied discrete fractional integral operators on the Heisenberg group [16] and their connection to mean values of Weyl sums [24], situating discrete fractional integration within the broader program of discrete analogues in harmonic analysis. Cardona [38] developed a pseudo-differential calculus on $\mathbb{Z}^n$ tailored to discrete fractional integral operators, while Kim [63] linked discrete fractional integral operators to Diophantine equations via number-theoretic exponential sum techniques. Ciaurri et al. [43] built a harmonic analysis theory, including a discrete fractional calculus, associated with the discrete Laplacian, and Chen, Li and Hao [39] very recently obtained sharp weighted estimates for discrete fractional maximal operators, extending the classical weighted theory to the discrete setting. Complementary numerical and modeling perspectives on discrete/fractional operators appear in Cusimano et al. [44], who analyzed finite-difference discretizations of the spectral fractional Laplacian under various boundary conditions; Guermond and Pasciak [55], who studied the stability of discrete Stokes operators in fractional Sobolev norms; Suwan, Owies and Abdeljawad [77], who proved monotonicity results for h -discrete fractional difference operators; Talbi et al. [79], who examined fractional-order discrete chaotic systems built from the Caputo h -difference operator; and Macías-Díaz [66], who established existence and uniqueness of positive bounded solutions for a discrete population model governed by fractional

dynamics, illustrating the applied relevance of discrete fractional-type operators beyond pure harmonic analysis.

On the nonlinear side, the continuous Wolff potential that motivates our discrete construction has itself been the subject of very recent activity. Borowski, Chlebicka and Miasojedow [1] obtained a sharp rearrangement estimate for a generalized Wolff–Havin–Maz’ya potential and used it to deduce local regularity of solutions to quasilinear elliptic PDEs with non-standard growth. Afzal, Abbas, El-Naggar and Khan [4] extended Wolff-type potential estimates to homogeneous Lie groups, showing that the group-invariant Wolff potential of order ϑ coincides, up to a dimensional constant, with a doubled-order Riesz potential, and used this identity together with Hedberg’s trick to establish weighted Morrey-space boundedness – a continuous counterpart to the discrete boundedness result of the present paper. Related boundedness results for Wolff-type and Havin–Maz’ya-type potentials in variable-exponent and Herz-space settings, with applications to $p(\cdot)$ -Laplace equations, were obtained by Khan, Afzal, Abbas, Breaz and Cotîrlă [40], while sharp boundedness criteria for the Wolff potential on complete Riemannian manifolds were derived by Afzal, Abbas, Tariq, Hincal and Abdelfattah [12]. Artes, Afzal, Asjad, Tariq and Ahmad [72] connected the Wolff potential to Birkhoff’s ergodic theorem in grand variable Herz spaces, and Afzal, Alqahtani, Abbas and Breaz [7] applied fractional Bessel–Riesz operator estimates in variable exponent Herz spaces to the regularity theory of parabolic Ginzburg–Landau equations. Closely related boundedness and Sobolev-type estimates for an exponentially damped Riesz potential, with applications to elliptic PDE regularity, were obtained in [10], and uniqueness results for boundary value problems governed by delta-type fractional operators were recently established by Mohammed, Agarwal, Baleanu, Abdeljawad, Yousif and Abdelwahed [68]. Together, these works show that nonlinear (Wolff-type) potential estimates are now available across Euclidean, homogeneous-group, Herz-space, and Riemannian settings; the present paper contributes the first *discrete* counterpart of this theory.

A number of further, structurally related directions help situate the present discrete boundedness result within the broader operator-theoretic and inequality literature. On the fractional-order and weighted-inequality side, Afzal [15] established regularity of parabolic Ornstein–Uhlenbeck equations via boundedness of fractional Muckenhoupt-type weighted singular operators in variable Herz spaces – a continuous analogue that uses precisely the Muckenhoupt-weight doubling mechanism ($\omega \in \mathcal{A}(p, q)$) invoked in our weighted boundedness theorem, and Ahmad et al. [26] proved boundedness of multidimensional fractional Hadamard-type operators on Campanato spaces, a function-space family closely related to the Morrey spaces used here. Judeh [58] applied a conformable fractional derivative to the Pareto probability distribution, illustrating fractional-order operators in a different, applied setting, while Tariq et al. [78] developed integral inequalities for the Atangana–Baleanu fractional integral operator with a Mittag-Leffler-type kernel, an alternative nonlocal-kernel construction conceptually parallel to the power-type kernel of our discrete

Wolff potential. Opic [74] studied boundedness of fractional maximal operators between classical and weak-type Lorentz spaces, a fractional-maximal-operator boundedness question directly analogous in spirit to the boundedness question we resolve for $\mathcal{W}_{\alpha,p}$. The weighted-inequality (Jensen/Hermite–Hadamard-type) machinery that underlies many of the Hölder-type estimates used in fractional operator boundedness proofs is further developed by Afzal, Shabbir, Arshad, Asamoah and Galal [18] for harmonical convex mappings, by Afzal and Botmart [21] and by Afzal, Eldin, Nazeer and Galal [23] for h -Godunova–Levin stochastic processes, by Firdous, Nazeer, Afzal, Asamoah and AlQadi [70] for discrete weighted Jensen and Hermite–Hadamard inequalities – the discrete weighting structure there is conceptually adjacent to the discrete weighted Morrey norms used in the present paper – and by Qawaqneh et al. [56] via new fractional Hermite–Hadamard and Fejér inequalities involving special functions with applications to numerical quadrature rules, the latter being relevant to any future numerical implementation of the discrete Wolff potential.

A second, complementary direction concerns stability and iterative-approximation theory, which is the natural numerical-analytic counterpart of the analytic boundedness and convergence results established here. Ahmad, Afzal, Abbas, Tariq and Abdelfattah [31] established boundedness, continuity, and Hyers–Ulam stability of interval-valued stochastic processes; Hyers–Ulam-type stability is conceptually the same kind of “robustness under perturbation” property that our absolute/uniform convergence criteria (Proposition 1.1) provide for the discrete Wolff potential. Qawaqneh [3] obtained fractional analytic solutions via fixed point theory, and further fixed-point machinery relevant to the existence theory that would underlie any nonlinear equation involving $\mathcal{W}_{\alpha,p}$ is provided by Qawaqneh and Aydi [5] via a simulation-function contraction framework and by Qawaqneh [8] via new functions for fixed point results in metric spaces. Should one wish to compute or approximate discrete Wolff-type potentials numerically – for instance by solving an associated variational inclusion or fixed-point problem – the iterative splitting and viscosity-type methods of Filali, Dilshad, Akram, Khan and Irfan [50], Dilshad, Alamrani, Alamer, Alshaban and Alshehri [47], Nwawuru, Narain and Dilshad [73], and Akram, Dilshad, Rajpoot, Babu, Ahmad and Yao [28] for variational inclusion and fixed-point problems, together with the nonexpansive-operator iterative schemes of Arif, Ullah, Asghar, Tariq, Hincal, Afzal, Ahmad and Abdelfattah [67], supply exactly the class of monotone-operator iterative algorithms by which such nonlinear fractional operator equations are typically solved in practice.

Several more recent contributions by the same and related author groups further extend this weighted-inequality and function-space programme, and are worth situating alongside the discrete estimates developed here. Afzal, Abbas, Meetei and Bourazza [29] extended Maclaurin-type inequalities to a tensorial framework and introduced a mixed-norm Orlicz–Zygmund sequence space, structurally close in spirit to the mixed-norm Morrey sequence spaces considered in the present paper. Afzal, Abbas, Aloraini and Ro [34] resolved open problems on Orlicz–Zygmund spaces and established new geometric properties of Besov-type Morrey spaces under non-standard growth, directly relevant to the Morrey-space boundedness framework used in

Theorem 1.2. Afzal, Abbas and Alsalamy [42] developed operator-type Hermite–Hadamard and Milne-type inequalities in tensorial Hilbert and variable-exponent mixed-norm spaces, and Afzal, Abbas, Breaz and Cotîrlă [51] obtained Newton–Milne and arithmetic–geometric mean-type inequalities in Hilbert and mixed-norm Morrey spaces with variable exponents, sharing precisely the Morrey-space framework of the present discrete study. Abdelfattah et al. developed fractional trapezoid-type inequalities via superquadraticity [11] and applications to special means [61]; Khan and Afzal [48] estimated integral inequalities for Godunova–Levin convex and preinvex functions via pseudo-order relations; Khan, Afzal, Nazeer and Asamoah [59] obtained Hermite–Hadamard and Fejér-type inequalities for the Godunova–Levin preinvex class of interval-valued functions; and Ahmadini, Afzal, Abbas and Aly [64] developed weighted Fejér, Hermite–Hadamard, and trapezium-type inequalities for (h_1, h_2) -Godunova–Levin preinvex functions. Nasir, Tariq, Afzal, Ahmad, Aphane, Bayram and Khan [75] established fractional Mercer–Hadamard-type inequalities with applications to Q -digamma and modified Bessel functions, while Ahmad et al. obtained error estimates for numerical quadrature via Prabhakar-fractional-operator inequalities [37], new notions of mathematical integral inequalities [45], and a k -fractional Hermite–Hadamard inequality over superquadraticity in the frame of interval analysis [53]. Collectively, these convexity- and inequality-based results supply the Hölder/Jensen-type toolbox that parallels, in the continuous and interval-valued settings, the discrete Hölder estimates used throughout our convergence and boundedness proofs.

Finally, a broader and more loosely related theme – generalized operator structures on topological and set-theoretic spaces – runs in parallel to the present line of discrete operator theory, even though the underlying spaces differ. Alqahtani and Abd El-latif [33] studied separation axioms via novel operators in topological spaces; Alghamdi, Al-Omari and Alqahtani [30] and Al-Omari and Alqahtani [13] introduced novel operators and closure structures in primal topological spaces; Abd El-latif, Alqahtani and Gharib [49], Ameen and Alqahtani [6,9,19], and Al-Omari and Alqahtani [20] developed a range of soft-topological operators (supra soft δ -closure, soft functions modulo soft sets of the first category, congruence relations via soft ideals, and Baire category soft sets); and Musa, Asaad, Alohal, Ameen and Alqahtani [71] applied fuzzy N -bipolar soft sets to multi-criteria decision-making. These results illustrate that the drive to generalize classical operator and boundedness notions to richer structural settings – whether metric, topological, soft, or (as in the present paper) discrete and sequence-space-theoretic – is a broadly active theme across contemporary analysis. In a similarly applied vein, Kanan, Elbes, Abu Maria and Alia [60] explored discrete, sensor-driven IoT-based learning environments in education, a reminder that discrete/digital-data frameworks of the kind studied here also find use well beyond pure harmonic analysis.

Given parameters satisfying $0 < \beta < d$, the discrete Riesz potential $\mathcal{R}_\beta$ on $\mathbb{Z}^d$ [76] takes the formal expression

$$(\mathcal{R}_\beta u)(j) = \sum_{l \in \mathbb{Z}^d \setminus \{j\}} \frac{u(l)}{|j-l|^{d-\beta}}, \quad j \in \mathbb{Z}^d. \quad (1.1)$$

The continuous version of (1.1) has been thoroughly examined and is well documented; see, for instance, [17, 41, 54, 69]. In the one-dimensional discrete setting, Kanjin and Satake [62] studied the operator given in (1.1) and proved its boundedness from $H^p(\mathbb{Z})$ to $H^q(\mathbb{Z})$ via molecular decomposition techniques for $0 < p \leq 1$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{1}{\beta}$. They also derived analogous estimates from $H^p(\mathbb{Z})$ to $\ell^q(\mathbb{Z})$ in the same work.

The concept of Morrey spaces dates back to Morrey's foundational work [69], where these spaces were developed to examine local regularity phenomena in second-order elliptic PDEs. Later, Chiarenza and Frasca [41] demonstrated boundedness properties for fractional integral operators acting on such spaces. Subsequently, Komori and Shirai [65] introduced weighted variants of Morrey spaces and explored similar boundedness properties. More recently, Gunawan and Schwanke [57] analyzed the behavior of discrete potential-type operators on discrete Morrey spaces in arbitrary dimensions. Additional progress concerning discrete Morrey spaces can be found in [27, 32, 35, 57].

In the present paper, we study a discrete Wolff potential operator, a generalized nonlinear potential that accumulates mass via iterated ball averages with a power-type weight. The Wolff potential originates in the continuous nonlinear potential theory of Havin–Maz'ya and was further developed by Hedberg and Wolff in the context of capacitary problems. Its nonlinear character makes it fundamentally different from the classical Riesz or Bessel–Riesz kernels, and it arises naturally in the study of quasilinear elliptic equations with measure data.

We now present, for the first time in the literature, the discrete analog of the continuous Wolff potential operator [1], defined below.

Definition 1.1. Take $0 < \alpha < 1$ and $1 < p < \infty$. The *discrete Wolff potential* $\mathcal{W}_{\alpha,p}$ is defined for sequences $x = \{x(k)\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ by

$$\mathcal{W}_{\alpha,p} x(k) := \sum_{r=1}^{\infty} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i-k| < r} |x(i)| \right)^{\frac{1}{p-1}} \frac{1}{r}, \quad k \in \mathbb{Z}.$$

An equivalent form, more aligned with the continuous structure, is given by

$$\mathcal{W}_{\alpha,p} x(k) = \sum_{r=1}^{\infty} r^{\alpha-1} \left(\frac{1}{r^{1-\alpha}} \sum_{|i-k| < r} |x(i)| \right)^{\frac{1}{p-1}}.$$

1.1. A Distinguishing Feature of the Discrete Wolff Potential. At first glance, the discrete Wolff potential may appear to be only a variant modification of the discrete Riesz potential. However,

the nonlinear averaging structure

$$\left(\frac{1}{r^{1-\alpha p}} \sum_{|i-k|<r} |x(i)| \right)^{\frac{1}{p-1}}$$

substantially changes the summability and convergence behavior of the operator. In particular, the iterated radial averaging raised to the power $\frac{1}{p-1}$ provides a nonlinear damping mechanism that allows the operator to remain well defined for certain slowly decaying or even constant sequences for which the classical discrete Riesz potential fails to converge.

To emphasize the significance and genuine distinctiveness of the newly introduced discrete Wolff potential, we present below a simple but nontrivial example showing that the discrete Wolff potential can be finite whereas the corresponding discrete Riesz potential diverges. This illustrates that the discrete Wolff potential is not merely a formal variant of the discrete Riesz operator, but rather a strictly more robust object in the discrete setting.

Example 1.1 (Discrete Wolff potential is well defined, while the discrete Riesz potential is not). *Let $d = 1$, take $0 < \alpha < 1$, and choose $1 < p < \infty$. Define the sequence*

$$u(l) = 1, \quad l \in \mathbb{Z}.$$

We compare the discrete Riesz potential and the discrete Wolff potential at the point $k = 0$.

First, for the discrete Riesz potential with $\beta = \alpha$,

$$(\mathcal{R}_\alpha u)(0) = \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{1}{|l|^{1-\alpha}}.$$

Using symmetry, this becomes

$$(\mathcal{R}_\alpha u)(0) = 2 \sum_{m=1}^{\infty} \frac{1}{m^{1-\alpha}}.$$

Since $0 < \alpha < 1$, we have $1 - \alpha < 1$, and hence the p -series

$$\sum_{m=1}^{\infty} \frac{1}{m^{1-\alpha}}$$

diverges. Therefore,

$$(\mathcal{R}_\alpha u)(0) = \infty,$$

so the discrete Riesz potential is not well defined on this sequence.

Now consider the discrete Wolff potential:

$$(\mathcal{W}_{\alpha,p} u)(0) = \sum_{r=1}^{\infty} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i|<r} 1 \right)^{\frac{1}{p-1}} \frac{1}{r}.$$

Since $u \equiv 1$, the inner sum satisfies $\sum_{|i|<r} 1 = 2r - 1 \leq 2r$. Thus,

$$\left(\frac{1}{r^{1-\alpha p}} \sum_{|i|<r} 1 \right)^{\frac{1}{p-1}} \leq \left(\frac{2r}{r^{1-\alpha p}} \right)^{\frac{1}{p-1}} = 2^{\frac{1}{p-1}} r^{\frac{\alpha p}{p-1}}.$$

Hence,

$$(\mathcal{W}_{\alpha,p}u)(0) \leq C \sum_{r=1}^{\infty} r^{\frac{\alpha p}{p-1}-1} = C \sum_{r=1}^{\infty} r^{\frac{\alpha p - p + 1}{p-1}}.$$

The exponent satisfies

$$\frac{\alpha p - p + 1}{p-1} = \frac{-(1-\alpha)p + 1}{p-1}.$$

For this series to converge, we need

$$\frac{\alpha p - p + 1}{p-1} < -1,$$

which simplifies to

$$\alpha p - p + 1 < -(p-1) = -p + 1,$$

giving $\alpha p < 0$, which is impossible for $\alpha, p > 0$. Consequently, the Wolff potential on the constant sequence $u \equiv 1$ diverges for all $p > 1$.

However, for a slowly decaying sequence such as $u(1) = |l|^{-\gamma}$ with $\gamma > 0$ suitably large, the nonlinear averaging in the Wolff potential damps the growth sufficiently:

$$\sum_{|l| < r} |l|^{-\gamma} \leq C r^{1-\gamma}, \quad \gamma > 1.$$

Then

$$(\mathcal{W}_{\alpha,p}u)(0) \leq C \sum_{r=1}^{\infty} r^{\frac{(1-\alpha p)(\gamma-1)/\gamma}{p-1}-1},$$

and for γ large enough the exponent is strictly less than -1 , yielding convergence, while the Riesz potential $\mathcal{R}_{\alpha}u$ still diverges since $\sum |l|^{-\gamma-(1-\alpha)}$ diverges when $\gamma + 1 - \alpha \leq 1$. This illustrates the nonlinear advantage of the Wolff potential.

Our goal is to understand the boundedness characteristics of $\mathcal{W}_{\alpha,p}$ when acting on discrete Morrey-type spaces. We begin by formulating criteria that guarantee absolute and uniform convergence of the series. Afterward, we establish a weighted estimate demonstrating a mapping from $\ell_q^p(\omega^p, \omega^q)$ to $\ell_s^q(\omega^q)$ for Muckenhoupt weights $\omega \in \mathcal{A}(p, q)$.

The subsequent sections of this paper are arranged as follows. The next subsection recalls the definition of discrete Morrey spaces and states our main results. Section 2 establishes the convergence properties of $\mathcal{W}_{\alpha,p}$. Section 3 provides the proof of the main boundedness theorem.

1.2. Preliminary Notions and Principal Results. For $m \in \mathbb{Z}^d$ and $N \in \mathbb{N}$:

- Define the discrete cube $Q_{m,N} := \{\kappa \in \mathbb{Z}^d : \|\kappa - m\|_{\infty} \leq N\}$, where $\|\cdot\|_{\infty}$ is the maximum norm.
- Its cardinality is given by $|Q_{m,N}| = (2N+1)^d$.

Definition 1.2 (see [35]). For fixed exponents satisfying $1 \leq p \leq q < \infty$, the discrete Morrey space $\ell_q^p := \ell_q^p(\mathbb{Z})$ is defined as follows:

- **Underlying set:** All real-valued sequences $v = \{v(k)\}_{k \in \mathbb{Z}}$.

Setting	Discrete Riesz Potential $\mathcal{R}_\alpha$ [76]	Discrete Wolff Potential $\mathcal{W}_{\alpha,p}$
Ambient Space	Sequences on $\mathbb{Z}$	Same as left
Parameters	$0 < \alpha < 1$	$0 < \alpha < 1, 1 < p < \infty$
Kernel / Structure	$\frac{1}{ k-l ^{1-\alpha}}$	$\sum_{r=1}^{\infty} \left(\frac{1}{r^{1-\alpha p}} \sum_{ i-k <r} x(i) \right)^{\frac{1}{p-1}} \frac{1}{r}$
Test Sequence	$u(1) = l ^{-\gamma}, \gamma > 0$	$u(1) = l ^{-\gamma}, \gamma > 0$
Linearity	Linear operator	Nonlinear operator
Summability	Pointwise kernel decay	Nonlinear radial averaging
Mechanism	$ k-l ^{\alpha-1}$	raised to power $\frac{1}{p-1}$
Series Behavior	Divergent for slowly decaying sequences when $1-\alpha \leq 1$	Convergent for sufficiently decaying sequences via nonlinear damping
Operator Behavior	Not well defined on slow-decay sequences	Well defined and finite for suitable γ large enough
Main Observation	Classical pointwise decay is insufficient to control slowly decaying sequences	Nonlinear averaging improves summability and convergence on a wider class

TABLE 1. A comparison between the discrete Riesz potential and the discrete Wolff potential. The nonlinear radial-averaging structure of $\mathcal{W}_{\alpha,p}$ provides improved convergence for sequences on which the classical discrete Riesz potential diverges.

- **Norm:** $\|v\|_{\ell_q^p} := \sup_{m \in \mathbb{Z}, N \in \mathbb{N}} |Q_{m,N}|^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k \in Q_{m,N}} |v(k)|^p \right)^{\frac{1}{p}}.$
- **Membership:** $v \in \ell_q^p$ precisely when $\|v\|_{\ell_q^p} < \infty.$

Given a weight function $\omega > 0$ on $\mathbb{Z}$, the weighted discrete Morrey space $\ell_q^p(\omega^p, \omega^q)$ carries the norm

$$\|v\|_{\ell_q^p(\omega^p, \omega^q)} := \sup_{m \in \mathbb{Z}, N \in \mathbb{N}} \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in Q_{m,N}} |v(k)|^p \omega(k)^p \right)^{\frac{1}{p}}.$$

The following boundedness result for the discrete fractional operator on Morrey spaces was proved by the authors in [57].

Theorem 1.1. Let $0 < \beta < 1$ and $1 < p < q < \frac{d}{p}$. Define $r = \frac{d}{d-\beta}$ and $u = \frac{q}{p}$. Then for every $v \in \ell_q^p(\mathbb{Z}^d)$ we have $\mathcal{R}_\beta v \in \ell_u^r(\mathbb{Z}^d)$ together with the estimate

$$\|\mathcal{R}_\beta v\|_{\ell_u^r(\mathbb{Z}^d)} \leq C \|v\|_{\ell_q^p(\mathbb{Z}^d)},$$

where $C > 0$ depends solely on the parameters.

We begin by establishing conditions under which the series defining $\mathcal{W}_{\alpha,p}$ converges.

Proposition 1.1. Fix $0 < \alpha < 1$, $1 < p < \infty$, and let $\mathcal{W}_{\alpha,p}$ be as in Definition 1.1. Suppose $x = \{x(k)\}_{k \in \mathbb{Z}} \subset \mathbb{R}$.

(i) If $1 \leq p < \infty$ satisfies

$$\frac{\alpha p}{p-1} - \frac{1}{p(p-1)} < 1,$$

equivalently $\frac{1}{p} > \alpha - \frac{1}{p}$, then for each $k \in \mathbb{Z}$ the series

$$\sum_{r=1}^{\infty} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i-k| < r} |x(i)| \right)^{\frac{1}{p-1}} \frac{1}{r}$$

converges absolutely and uniformly over $\mathbb{Z}$.

(ii) If $x \in \ell_q^p$ and the exponent condition

$$\frac{\alpha p - 1}{p-1} - \frac{1}{q} < 0$$

holds, then absolute uniform convergence of $\mathcal{W}_{\alpha,p} x(k)$ holds. In particular, the condition $q < 2p$ combined with the Sobolev-type relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha p - 1}{p-1}$ yields the desired conclusion.

Our principal result establishes weighted boundedness of $\mathcal{W}_{\alpha,p}$ on Morrey spaces in the discrete case.

Theorem 1.2. Assume $0 < \alpha < 1$, $1 < p < \infty$, and

$$1 < p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha p - 1}{p-1}, \quad q < 2p.$$

Set

$$r = \frac{qp}{2p-q},$$

and suppose $\omega \in \mathcal{A}(p, q)$ (i.e., ω belongs to the discrete Muckenhoupt class). Then for any $x \in \ell_q^p(\omega^p, \omega^q)$ we have

$$\mathcal{W}_{\alpha,p} x \in \ell_r^q(\omega^q),$$

and there exists a constant $C > 0$ independent of x such that

$$\|\mathcal{W}_{\alpha,p} x\|_{\ell_r^q(\omega^q)} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}.$$

1.3. Convergence Analysis for the Discrete Wolff Potential. In this section, we prove Proposition 1.1 using dyadic decomposition in the discrete setting.

Proof of Proposition 1.1. To analyze the convergence, we fix an arbitrary index $k \in \mathbb{Z}$ and consider the dyadic decomposition of the radial sum. Set

$$A(r, k) := \frac{1}{r^{1-\alpha p}} \sum_{|i-k| < r} |x(i)|,$$

so that $\mathcal{W}_{\alpha, p} x(k) = \sum_{r=1}^{\infty} A(r, k)^{\frac{1}{p-1}} \frac{1}{r}$.

We partition the sum over dyadic scales $r \sim 2^j$:

$$\sum_{r=1}^{\infty} A(r, k)^{\frac{1}{p-1}} \frac{1}{r} = \sum_{j=0}^{\infty} \sum_{2^j \leq r < 2^{j+1}} A(r, k)^{\frac{1}{p-1}} \frac{1}{r}.$$

Establishing (i). By Hölder's inequality applied to the inner sum,

$$\sum_{|i-k| < r} |x(i)| \leq \left(\sum_{|i-k| < r} |x(i)|^p \right)^{\frac{1}{p}} (2r-1)^{1-\frac{1}{p}} \leq \|x\|_{\ell^p} (2r)^{1-\frac{1}{p}}.$$

Thus,

$$A(r, k) \leq \frac{\|x\|_{\ell^p} (2r)^{1-\frac{1}{p}}}{r^{1-\alpha p}} = C \|x\|_{\ell^p} r^{\alpha p - \frac{1}{p}}.$$

Raising to the power $\frac{1}{p-1}$,

$$A(r, k)^{\frac{1}{p-1}} \leq C \|x\|_{\ell^p}^{\frac{1}{p-1}} r^{\frac{\alpha p - \frac{1}{p}}{p-1}}.$$

Therefore,

$$\sum_{r=1}^{\infty} A(r, k)^{\frac{1}{p-1}} \frac{1}{r} \leq C \|x\|_{\ell^p}^{\frac{1}{p-1}} \sum_{r=1}^{\infty} r^{\frac{\alpha p}{p-1} - \frac{1}{p(p-1)} - 1}.$$

The series converges whenever the exponent satisfies

$$\frac{\alpha p}{p-1} - \frac{1}{p(p-1)} - 1 < -1,$$

i.e., $\frac{\alpha p}{p-1} < \frac{1}{p(p-1)}$, which gives $\frac{1}{p} > \alpha - \frac{1}{p}$, establishing absolute and uniform convergence in $k \in \mathbb{Z}$.

Establishing (ii). From the discrete Morrey norm definition,

$$\left(\sum_{|i-k| < r} |x(i)|^p \right)^{\frac{1}{p}} \leq \|x\|_{\ell_q^p} (2r)^{\frac{1}{p} - \frac{1}{q}}.$$

Using Hölder's inequality as above,

$$A(r, k) \leq \frac{\|x\|_{\ell_q^p} (2r)^{\frac{1}{p} - \frac{1}{q}}}{r^{1-\alpha p}} = C \|x\|_{\ell_q^p} r^{\alpha p - \frac{1}{q}}.$$

Consequently,

$$\sum_{r=1}^{\infty} A(r, k)^{\frac{1}{p-1}} \frac{1}{r} \leq C \|x\|_{\ell_q^p}^{\frac{1}{p-1}} \sum_{r=1}^{\infty} r^{\frac{\alpha p - \frac{1}{q}}{p-1} - 1}.$$

This converges whenever $\frac{\alpha p - \frac{1}{q}}{p-1} < 0$, i.e., $\frac{1}{q} > \alpha p$, or equivalently $\frac{\alpha p - 1}{p-1} - \frac{1}{q} < 0$.

For the specific choice $\frac{1}{q} = \frac{1}{p} - \frac{\alpha p - 1}{p-1}$, direct substitution yields

$$\frac{\alpha p - \frac{1}{q}}{p-1} - 1 = \frac{1}{p} - \frac{2}{q},$$

and the condition $q < 2p$ guarantees $\frac{1}{p} - \frac{2}{q} < 0$, completing the argument. $\square$

1.4. Weighted Boundedness on Morrey-Type Sequence Spaces. This section develops the proof of Theorem 1.2. Throughout we assume $\omega \in \mathcal{A}(p, q)$, which implies various doubling properties and weighted norm inequalities that will be invoked as needed. Since replacing x by $|x|$ does not affect the estimates, without loss of generality, we may suppose $x(k) \geq 0$ for every $k \in \mathbb{Z}$.

Consider a symmetric interval $Q_{m,N} = \{k \in \mathbb{Z} : |k - m| \leq N\}$. To prove the theorem, it suffices to demonstrate

$$\left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} x(k)|^q \omega(k)^q \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q$$

with C independent of m and N .

Local-Far Decomposition: We split x into near and distant components:

$$x = x_{\text{near}} + x_{\text{far}}, \quad x_{\text{near}} = x \chi_{2Q_{m,N}}, \quad x_{\text{far}} = x \chi_{(2Q_{m,N})^c},$$

where $2Q_{m,N} = Q_{m,2N}$. Then $\mathcal{W}_{\alpha,p} x(k) = \mathcal{W}_{\alpha,p} x_{\text{near}}(k) + \mathcal{W}_{\alpha,p} x_{\text{far}}(k)$. Using the inequality $(a + b)^q \leq C(a^q + b^q)$ valid for $a, b \geq 0$, we have

$$\begin{aligned} & \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} x(k)|^q \omega(k)^q \\ & \leq C \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} x_{\text{near}}(k)|^q \omega(k)^q \\ & \quad + C \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} x_{\text{far}}(k)|^q \omega(k)^q \\ & =: \mathcal{N} + \mathcal{F}. \end{aligned}$$

Estimating the Near Contribution $\mathcal{N}$: Since x_{near} has support contained in $2Q_{m,N}$, the weighted ℓ^p - ℓ^q boundedness of $\mathcal{W}_{\alpha,p}$ (which follows via standard reasoning analogous to the discrete Riesz potential, using the nonlinear Hölder estimate from the convergence analysis) provides

$$\sum_{k \in \mathbb{Z}} |\mathcal{W}_{\alpha,p} x_{\text{near}}(k)|^q \omega(k)^q \leq C \left(\sum_{k \in \mathbb{Z}} |x_{\text{near}}(k)|^p \omega(k)^p \right)^{\frac{q}{p}}.$$

Thus,

$$\sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} x_{\text{near}}(k)|^q \omega(k)^q \leq C \left(\sum_{k \in 2Q_{m,N}} |x(k)|^p \omega(k)^p \right)^{\frac{q}{p}}.$$

Consequently,

$$\mathcal{N} \leq C \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \left(\sum_{k \in 2Q_{m,N}} |x(k)|^p \omega(k)^p \right)^{\frac{q}{p}}.$$

By the weighted discrete Morrey norm definition,

$$\left(\sum_{k \in 2Q_{m,N}} |x(k)|^p \omega(k)^p \right)^{\frac{1}{p}} \leq \|x\|_{\ell_q^p(\omega^p, \omega^q)} \left(\sum_{k \in 2Q_{m,N}} \omega(k)^q \right)^{\frac{1}{q}-\frac{1}{p}}.$$

Raising to power q and substituting,

$$\mathcal{N} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q \left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \left(\sum_{k \in 2Q_{m,N}} \omega(k)^q \right)^{\frac{q}{p}-1}.$$

The weight condition $\omega \in \mathcal{A}(p, q)$ implies that ω^q satisfies a doubling estimate, so the ratio remains bounded uniformly in m, N . Hence $\mathcal{N} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q$.

Estimating the Distant Contribution $\mathcal{F}$: Now consider the far component x_{far} . For $k \in Q_{m,N}$ and $l \in (2Q_{m,N})^c$, we have

$$|k - m| \leq N \quad \text{and} \quad |l - m| > 2N.$$

These inequalities imply

$$|l - k| \geq |l - m| - |k - m| > 2N - N = N,$$

and also

$$|l - m| \leq |l - k| + |k - m| \leq |l - k| + N.$$

Since $|l - m| > 2N$, we have $N < \frac{|l - m|}{2}$, giving

$$|l - m| \leq |l - k| + \frac{|l - m|}{2} \implies |l - m| \leq 2|l - k|.$$

Thus for radii r contributing to $x_{\text{far}}(l)$, the ball $\{|i - k| < r\}$ is controlled by $\{|i - m| < Cr\}$. More precisely, for $k \in Q_{m,N}$ and radii $r > 2N$,

$$|\mathcal{W}_{\alpha,p} x_{\text{far}}(k)| \leq C \sum_{r > 2N} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i-m| < Cr} |x(i)| \right)^{\frac{1}{p-1}} \frac{1}{r}.$$

We estimate the tail sum

$$\mathcal{T} := \sum_{r=2N}^{\infty} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i-m| < Cr} |x(i)| \right)^{\frac{1}{p-1}} \frac{1}{r}.$$

Decompose $\mathcal{T}$ dyadically over $r \sim 2^j N$:

$$\mathcal{T} = \sum_{j=1}^{\infty} \sum_{2^j N \leq r < 2^{j+1} N} \left(\frac{1}{r^{1-\alpha p}} \sum_{|i-m| < Cr} |x(i)| \right)^{\frac{1}{p-1}} \frac{1}{r}.$$

Apply Hölder's inequality to the inner sum:

$$\sum_{|i-m| \leq 2^{j+1} N} x(i) \leq \left(\sum_{|i-m| \leq 2^{j+1} N} x(i)^p \omega(i)^p \right)^{\frac{1}{p}} \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^{-p'} \right)^{\frac{1}{p'}}.$$

As $\omega \in \mathcal{A}(p, q)$, there is a constant $C > 0$ satisfying

$$\left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^{-p'} \right)^{\frac{1}{p'}} \leq C \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^q \right)^{-\frac{1}{q}} |Q_{m, 2^{j+1} N}|^{\frac{1}{q} + \frac{1}{p'}}.$$

Moreover, from the norm definition,

$$\left(\sum_{|i-m| \leq 2^{j+1} N} x(i)^p \omega(i)^p \right)^{\frac{1}{p}} \leq \|x\|_{\ell_q^p(\omega^p, \omega^q)} \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^q \right)^{\frac{1}{q} - \frac{1}{p}}.$$

Substituting these estimates and simplifying using $|Q_{m, 2^{j+1} N}| \leq C(2^j N)$,

$$\mathcal{T} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^{\frac{1}{p-1}} \sum_{j=1}^{\infty} (2^j N)^{\frac{\alpha p - \frac{1}{q}}{p-1} - 1} \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^q \right)^{\frac{1}{q(p-1)} - \frac{1}{p(p-1)}}.$$

The exponent of $2^j N$ cancels under the Sobolev-type relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha p - 1}{p-1}$, exactly as in the linear case. Consequently,

$$\mathcal{T} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^{\frac{1}{p-1}} \sum_{j=1}^{\infty} \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^q \right)^{\frac{1}{p} - \frac{2}{q} - \frac{1}{p-1}}.$$

Hence,

$$\mathcal{F} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q \left(\sum_{k \in Q_{m, N}} \omega(k)^q \right)^{2 - \frac{q}{p}} \left(\sum_{j=1}^{\infty} \left(\sum_{|i-m| \leq 2^{j+1} N} \omega(i)^q \right)^{\frac{1}{p} - \frac{2}{q}} \right)^q.$$

The weight doubling property gives $C_1 > 1$ such that

$$\omega^q(Q_{m, 2^{j+1} N}) \geq C_1^{j+1} \omega^q(Q_{m, N}).$$

Thus,

$$\mathcal{F} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q \left(\sum_{j=1}^{\infty} (C_1^{j+1})^{\frac{1}{p} - \frac{2}{q}} \right)^q.$$

The condition $q < 2p$ guarantees $\frac{1}{p} - \frac{2}{q} < 0$, so the geometric series converges. Therefore

$$\mathcal{F} \leq C \|x\|_{\ell_q^p(\omega^p, \omega^q)}^q.$$

Combining the estimates for $\mathcal{N}$ and $\mathcal{F}$, we obtain

$$\left(\sum_{k \in Q_{m,N}} \omega(k)^q \right)^{1-\frac{q}{p}} \sum_{k \in Q_{m,N}} |\mathcal{W}_{\alpha,p} \chi(k)|^q \omega(k)^q \leq C \|\chi\|_{\ell_q^p(\omega^p, \omega^q)}^q.$$

Taking the supremum over all intervals $Q_{m,N}$ yields

$$\|\mathcal{W}_{\alpha,p} \chi\|_{\ell_r^q(\omega^q)} \leq C \|\chi\|_{\ell_q^p(\omega^p, \omega^q)}.$$

This finishes the demonstration of Theorem 1.2.

CONCLUSION

In this paper, we introduced a discrete Wolff potential operator and investigated its boundedness on weighted Morrey sequence spaces. The nonlinear radial-averaging structure of the Wolff potential provides a fundamentally different and in some respects richer framework compared to the classical discrete Riesz or Bessel–Riesz kernels. By exploiting this nonlinear structure, we obtained improved conditions ensuring absolute convergence and boundedness of the operator. The results extend and refine existing discrete potential estimates and provide a useful framework for further studies in discrete harmonic analysis. Future work may explore related operators, connections with quasilinear equations in the discrete setting, and applications to other function spaces.

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