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Fractal Attractor Associated with Banach and Reich Contractions on Controlled Strong $b-$Metric Space

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Abstract. This article aims to derive the controlled strong $b-$Banach fractal (CSbB-Fractal) by Banach contraction and the controlled strong $b-$Reich fractal (CSbR-Fractal) by Reich contraction on the controlled strong $b-$metric space (CSbMS). In this context, a new class of Iterated function system (IFS) and its fractal attractors are constructed as a general case of certain existing systems by the Hutchinson-Barnsley (HB) theory on CSbMS and illustrated with examples. Also, the Reich fixed point theorem is discussed with examples and applications. Furthermore, the findings of the study in generalized spaces can offer a new approach to generate a novel type of fractal attractor.

1. Introduction

The classical geometry has been utilized to examine natural and human-made objects, commonly considered the theory of topological and Euclidean dimensions, regardless of the object's complexity or smoothness. Mandelbrot established the fractal notion to resolve this difficulty by analyzing it through the self-similar property. Fractal originated via the Latin term fractus, which means shattered or fragmented. It was used to characterize things that are too irregular to fit within a standard geometrical framework. The fundamental characteristic of a fractal object exhibits self-similarity at various scales, which means the same pattern across multiple scales of measure. Mandelbrot mathematically defined 'a fractal as a set whose Hausdorff dimension is strictly greater than its topological dimension'. Fractals are recurring patterns seen in many natural phenomena, such as mountains, clouds, coastlines, and blood vessels ([1]– [9]).

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Researchers often employ fractals on numerous scientific domains, namely formulation of graph polynomial, topological structures, quantum theory, neural network, fractal time derivative operators and other physical problems ([10]– [12]). Hutchinson established the notion of IFS, it was popularized by Barnsley to construct fractals. IFS provides an effective technique for modelling complex structures and generating fractals. In the application of discrete dynamical systems, IFS serves as a vital method for constructing fractal attractors. Many researchers have developed various fractal sets through IFS and such as Orbital, Kannan, Fisher, Relational, Coupled, Tiling, fuzzy, Cyclic IFSs ([13]– [23]).

The fixed point theory is regarded as a key instrument of mathematical and nonlinear analysis, it has been widely used across diverse scientific fields such as physics, chemistry, computer science, and biology. Furthermore, the generalization of fixed points in various metric spaces has been an interesting study area due to its applications in multiple disciplines ([24]– [28]).

Kirik and Shahzad [25] proposed the theory of strong $b-$metric space (SbMS) as an extension of $b-$metric space (bMS), while the distance function in SbMS is continuous but in $b-$metric, it need not be continuous. Recently, the idea of CSbMS was presented by Santina et al. [26] as an expansion of SbMS by replacing the constant with a function in the triangle inequality. The concept of CSbMS is understood as a broader and more generalized framework that encompasses both SbMS and the more familiar Metric Spaces (MS) as particular cases. Specifically, a CSbMS introduces additional flexibility in its definition through the inclusion of a control function and a control variable factor within the triangular inequality. These modifying parameters allow for a wider range of distance-like structures that can capture more complex geometric behaviors. When the control function used in the triangular inequality of a CSbMS is taken to be a constant value, the additional generality it provides disappears. In that case, the CSbMS simplifies directly into SbMS. On the other hand, if the control variable factor appearing in the triangular inequality of the CSbMS is specifically set to one, then the scaling effect in the inequality vanishes, and the system further reduces to an ordinary metric space. However, the converse statements of these cases do not necessarily hold true. Recognizing these distinctions is important because it emphasizes the hierarchical nature of these spaces: CSbMS generalizes SbMS, and SbMS generalizes MS. This generalization provides a richer mathematical framework, which becomes particularly useful when exploring fractal attractors and Iterated Function Systems. In such studies, the flexibility offered by CSbMS allows researchers to model more complicated geometric and dynamical structures that cannot be adequately described within the more restrictive settings of SbMS or MS. Hence, the development of fractal attractors and IFS within the generalized CSbMS framework is a natural and motivated extension of previous work, aimed at capturing a broader spectrum of fractal behaviors.

The Hausdorff CSbMS is complete and the construction of Kannan fractal attractor on CSbMS was established in [29]. So far, the research study on the Reich fixed point theorem on CSbMS and its fractal structures by Banach and Reich contractions has not been discussed. This motivated us to introduce the Reich fixed point theorem and a new type of fractal attractor on CSbMS by Banach and Reich contractions.

The paper is structured as follows: Section 2 describes the fundamental concepts required for this work. Section 3 demonstrates the Reich fixed point theorem on CSbMS with examples. In Section 4, the construction of Banach fractal on CSbMS is presented. In addition, the Reich fractal on CSbMS is constructed. Section 5 the application of CSbMS is discussed. Finally, Section 6 summarizes the results.

2. Preliminaries

The fundamental concepts required for the proposed theory of constructing generalized fractals are discussed in this section.

Definition 2.1 ([25]). *Consider $Z \neq \emptyset$, $t \geq 1$ also $d : Z \times Z \to [0, \infty)$ fulfills the following conditions*

(a). $d(a, b) = 0$ *iff* $a = b$,
(b). $d(a, b) = d(b, a)$,
(c). $d(a, b) \leq d(a, s) + td(s, b), \forall\, a, b, s \in Z$.

Then d is known as strong $b-$metric on Z. The pair (Z, d) is called a SbMS.

Definition 2.2 ([26]). *Consider $Z \neq \emptyset$, $\alpha : Z \times Z \to [1, \infty)$ also $d : Z \times Z \to [0, \infty)$ fulfills the following conditions*

(a). $d(a, b) = 0$ *iff* $a = b$,
(b). $d(a, b) = d(b, a)$,
(c). $d(a, b) \leq d(a, s) + \alpha(s, b)d(s, b), \forall\, a, b, s \in Z$.

Then d is known as controlled strong $b-$metric on Z. The pair (Z, d) is called a CSbMS.

Definition 2.3 ([26]). *Let (Z, d) be a CSbMS and $\{a_r\}_{r\geq 0}$ be a sequence in Z.*

(i). *$\{a_r\}$ converges to some $a \in Z$, if $\forall\, \varepsilon \geq 0$, $\exists\, K \in \mathbb{N} \ni d(a_r, a) \leq \varepsilon$, $\forall\, r \geq K$. i.e $\lim_{r\to\infty} a_r = a$.*
(ii). *$\{a_r\}$ is Cauchy, if $\forall\, \varepsilon \geq 0$, $\exists\, K \in \mathbb{N} \ni d(a_m, a_n) \leq \varepsilon$, $\forall\, m, n \geq K$.*
(iii). *If every Cauchy sequence in (Z, d) is convergent in Z, then (Z, d) is complete controlled strong $b-$metric space (CCSbMS).*

Theorem 2.1 ([26]). *Given CCSbMS (Z, d), suppose $g : Z \to Z$ is a mapping where $d(g(a), g(b)) \leq \phi d(a, b)$, $\forall\, a, b \in Z$, where $\phi \in (0, 1)$ and $\alpha : Z \times Z \to [1, \infty)$. For $a_0 \in Z$, choose $a_r = g^r(a_0)$. Assume that $\sup_{k\geq 1} \lim_{i\to\infty} \alpha(a_{i+1}, a_k) < \frac{1}{\phi}$, and also consider that $\lim_{r\to\infty} \alpha(a, a_r)$ & $\lim_{r\to\infty} \alpha(a_r, a)$ exist and finite, for each $a \in Z$. Then, g possesses a unique fixed point.*

Definition 2.4 ([29]). *Given CCSbMS (Z, d), and let $K_0(Z)$ denote the set of all non empty compact subsets of Z. For $a \in Z$ and $E, F \in K_0(Z)$, define*

$$d(a, F) = \inf \{d(a, b) : b \in F\}$$

and

$$d(E,F) = \sup\{d(a,F) : a \in E\}.$$

Now, define $H_d : K_0(Z) \times K_0(Z) \to [0,\infty)$ *as*

$$H_d(E,F) = \max\left\{\sup_{a\in E} d(a,F), \sup_{b\in F} d(b,E)\right\}, \text{ where } E,F \in K_0(Z).$$

Then H_d *is known as Hausdorff controlled strong* $b-$*metric on* $K_0(Z)$. *The pair* $(K_0(Z), H_d)$ *is called a Hausdorff controlled strong* $b-$*metric space (HCSbMS).*

Theorem 2.2 ([29]). *If* (Z,d) *is a complete CSbMS with* $\lim_{r,k\to\infty} \alpha(a_r,a_k)\phi < 1$, $\forall a_r, a_k \in Z$, *where* $\phi \geq 1$, *then* $(K_0(Z), H_d)$ *is also a complete HCSbMS.*

Theorem 2.3 (Reich Fixed Point Theorem [30]). *Let* (Z,d) *be a complete metric space and* $g : Z \to Z$. *If* ϕ, ω, δ *are non-negative such that*

$$d(g(a), g(b)) \leq \phi d(a, g(a)) + \omega d(b, g(b)) + \delta d(a,b), \text{ where } \phi + \omega + \delta < 1 \text{ and } a,b \in Z,$$

then g *is known as Reich contraction on* Z. *Then,* g *possesses a unique fixed point in* Z.

Definition 2.5 ([5]). *The similarity dimension* D_s *is defined as*

$$D_s = \frac{log(N)}{log(\frac{1}{\epsilon})}$$

where N *is self-similar segments and* ϵ *denotes the scaling ratio.*

3. Controlled Strong $b-$Reich Fixed Point Theorem

Theorem 3.1. *Given CCSbMS* (Z,d), *suppose* $g : Z \to Z$ *is a Reich mapping with contraction ratios* ϕ, ω, δ. *For* $a_0 \in Z$, *choose* $a_r = g^r(a_0)$. *Assume that*

$$\sup_{k\geq 1} \lim_{i\to\infty} \alpha(a_{i+1}, a_k) < \left(\frac{1-\omega}{\phi+\delta}\right), \tag{3.1}$$

and also consider that

$$\lim_{r\to\infty} \alpha(a,a_r) \;\&\; \lim_{r\to\infty} \alpha(a_r,a) \tag{3.2}$$

exist and finite, for each $a \in Z$. *Then,* g *possesses a unique fixed point.*

Proof. Let $a_r = g^r(a_0)$, for $a \in Z$.

$$\begin{aligned} d(a_r, a_{r+1}) &= d(g(a_{r-1}), g(a_r)) \\ &\leq \phi d(a_{r-1}, g(a_{r-1})) + \omega d(a_r, g(a_r)) + \delta d(a_{r-1}, a_r) \\ &\leq \phi d(a_{r-1}, a_r) + \omega d(a_r, a_{r+1}) + \delta d(a_{r-1}, a_r) \end{aligned}$$

Then,

$$d(a_r, a_{r+1}) \leq \left(\frac{\phi+\delta}{1-\omega}\right) d(a_{r-1}, a_r)$$

$$\leq \left(\frac{\phi+\delta}{1-\omega}\right)^2 d(a_{r-2},a_{r-1})$$

$$\vdots$$

$$d(a_r,a_{r+1}) \leq \left(\frac{\phi+\delta}{1-\omega}\right)^r d(a_0,a_1) = F^r d(a_0,a_1), \text{ where } F = \left(\frac{\phi+\delta}{1-\omega}\right).$$

For all $r,k \in \mathbb{N}$ $(r<k)$, then

$$\begin{aligned}
d(a_r,a_k) &\leq d(a_r,a_{r+1}) + \alpha(a_{r+1},a_k)d(a_{r+1},a_k)\\
&\leq d(a_r,a_{r+1}) + \alpha(a_{r+1},a_k)d(a_{r+1},a_{r+2}) + \alpha(a_{r+1},a_k)\alpha(a_{r+2},a_k)d(a_{r+2},a_k)\\
&\leq d(a_r,a_{r+1}) + \alpha(a_{r+1},a_k)d(a_{r+1},a_{r+2}) + \alpha(a_{r+1},a_k)\alpha(a_{r+2},a_k)d(a_{r+2},a_{r+3})\\
&+ \alpha(a_{r+1},a_k)\alpha(a_{r+2},a_k)\alpha(a_{r+3},a_k)d(a_{r+3},a_k)\\
&\leq \dots\\
&\leq d(a_r,a_{r+1}) + \sum_{i=r+1}^{k-2}\left(\prod_{j=r+1}^{i}\alpha(a_j,a_k)\right)d(a_i,a_{i+1}) + \prod_{j=r+1}^{k-1}\alpha(a_j,a_k)d(a_{k-1},a_k)\\
&\leq F^r d(a_0,a_1) + \sum_{i=r+1}^{k-2}\left(\prod_{j=r+1}^{i}\alpha(a_j,a_k)\right)F^i d(a_0,a_1) + \prod_{j=r+1}^{k-1}\alpha(a_j,a_k)F^{k-1}d(a_0,a_1)\\
&\leq F^r d(a_0,a_1) + \sum_{i=r+1}^{k-1}\left(\prod_{j=r+1}^{i}\alpha(a_j,a_k)\right)F^i d(a_0,a_1)\\
&= F^r d(a_0,a_1) + \sum_{i=0}^{k-1}\left(\prod_{j=0}^{i}\alpha(a_j,a_k)\right)F^i d(a_0,a_1) - \sum_{i=0}^{r}\left(\prod_{j=0}^{i}\alpha(a_j,a_k)\right)F^i d(a_0,a_1)
\end{aligned} \tag{3.3}$$

$$\text{Let, } S_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\alpha(a_j,a_k)\right)F^i$$

From Eqn. 3.3, we have

$$d(a_r,a_k) \leq d(a_0,a_1)\left[F^r + (S_{k-1}-S_r)\right]. \tag{3.4}$$

Using the ratio test on the term,

$$a_i = \left(\prod_{j=0}^{i}\alpha(a_j,a_k)\right)F^i$$

$$\frac{a_{i+1}}{a_i} = \frac{\prod_{j=0}^{i+1}\alpha(a_j,a_k)F^{i+1}}{\prod_{j=0}^{i}\alpha(a_j,a_k)F^i}$$

$$\text{But } \frac{\prod_{j=0}^{i+1}\alpha(a_j,a_k)}{\prod_{j=0}^{i}\alpha(a_j,a_k)} = \alpha(a_{i+1},a_k)$$

$$\text{then } \frac{a_{i+1}}{a_i} = \alpha(a_{i+1}, a_k)F$$

Applying Eqn. (3.1), we have $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < \frac{1}{F} * F = 1$.
Therefore, $\lim_{r\to\infty} S_r$ exists and $\{S_r\}$ is Cauchy.
As limit $r, k \to \infty$ in Eqn. (3.4), then

$$\lim_{r,k\to\infty} d(a_r, a_k) = 0. \tag{3.5}$$

Thus, $\{a_r\}$ is Cauchy in CCSbMS (Z, d). So $\exists\, a \in Z$ such that,

$$\lim_{r\to\infty} d(a_r, a) = 0. \tag{3.6}$$

To show a is a fixed point of g. By CSbMS triangular inequality,

$$d(a, a_{r+1}) \leq d(a, a_r) + \alpha(a_r, a_{r+1})d(a_r, a_{r+1}).$$

Using Eqns. (3.1), (3.2), (3.5), we have

$$\lim_{r\to\infty} d(a_r, a_{r+1}) = 0. \tag{3.7}$$

Using CSbMS triangular inequality, where

$$\begin{aligned} d(a, g(a)) &\leq d(a, a_{r+1}) + \alpha(a_{r+1}, g(a))d(a_{r+1}, g(a)) \\ &\leq d(a, a_{r+1}) + \alpha(a_{r+1}, g(a))d(g(a_r), g(a)) \\ &\leq d(a, a_{r+1}) + \alpha(a_{r+1}, g(a))\left[\left(\frac{\phi + \delta}{1 - \omega}\right)d(a_r, a)\right] \end{aligned}$$

As limit $r \to \infty$ and from Eqns. (3.2), (3.7), we have $d(a, g(a)) = 0$, such that $a = g(a)$.
Suppose g has two fixed points, say a, b. Then

$$\begin{aligned} d(a, b) &= d(g(a), g(b)) \\ &\leq \phi d(a, g(a)) + \omega d(b, g(b)) + \delta d(a, b) \\ d(a, b) &= 0, \; so\; a = b. \end{aligned}$$

Hence, g possesses a unique fixed point. □

If $\phi = \omega = 0$, then the Reich map is reduced to a Banach contraction map. If $\delta = 0$ and $\phi = \omega$, then it is reduced to a Kannan contraction map. If $\phi = \omega$, then it is reduced to a Fisher-type contraction map.

Example 3.1. *Consider $Z = \{0, 1, 2\}$ and take $d : Z \times Z \to [0, \infty)$ as $d(a, b) = d(b, a)$ also $d(a, a) = 0$, where $a, b \in Z$ and*

$$d(1, 2) = 3,\; d(2, 0) = 5,\; d(0, 1) = 7.$$

Consider $\alpha : Z \times Z \to [1, \infty)$, where

$$\alpha(a, a) = 1,\; \alpha(a, b) = \alpha(b, a),\; \alpha(1, 2) = \frac{7}{3},\; \alpha(2, 0) = \frac{8}{5},\; \alpha(0, 1) = \frac{5}{3}.$$

It can be easily prove that (Z, d) is a CSbMS. Now, define $g : Z \to Z$ by

$$g(a) = \begin{cases} 1; & \text{if} \quad a = 0 \\ 2; & \text{if} \quad a \in \{1, 2\} \end{cases}$$

Take $\phi = 0.3$, $\omega = 0.2$, $\delta = 0.4$, where $\phi + \omega + \delta < 1$. Then, g fulfills all assumptions of Theorem 3.1, consequently g is Reich contraction map also g has a fixed point (unique), which is $a = 2$.

4. Generalized Fractal Attractor on Controlled Strong $b-$Metric Space

In this section, the generalized cases of fractals are constructed in Controlled Strong $b-$Metric Space with generalized contractions.

4.1. Controlled Strong $b-$Banach Fractal.

Theorem 4.1. *Given CCSbMS (Z, d), suppose $g : Z \to Z$ is a continuous Banach mapping with the contraction ratio ϕ. Then $g : K_0(Z) \to K_0(Z)$ is defined by $g(E) = \{g(a) : a \in E\}$, $\forall E \in K_0(Z)$, is a Banach contraction on $(K_0(Z), H_d)$ with the contraction ratio $\phi \in (0, 1)$.*

Proof. Let $E, F \in K_0(Z)$. Then

$$\begin{aligned} H_d(g(E), g(F)) &= \max\{d(g(E), g(F)), d(g(F), g(E))\} \\ &\leq \max\{\phi d(E, F), \phi d(F, E)\} \\ &\leq \phi \left[\max\{d(E, F), d(F, E)\}\right] \\ H_d(g(E), g(F)) &\leq \phi H_d(E, F). \end{aligned}$$

Hence g is a Banach contraction on $(K_0(Z), H_d)$. □

Definition 4.1. *(Banach IFS on CSbMS). Consider (Z, d) is CCSbMS and a family of Banach contractions g_r $(r = 1, 2, \ldots, K;\ K \in \mathbb{N})$ on Z having the corresponding contractivity ratios ϕ_r. Then, $\{Z; g_r\}_{r=1}^{K}$ is said to be a controlled strong $b-$Banach iterated function system (CSbB-IFS), where the contractivity factor of this system is $\phi = \max\limits_{r=1}^{K}\{\phi_r\}$.*

Definition 4.2. *(HB Operator of Banach Contractions on CSbMS). Given CCSbMS (Z, d), suppose $\{Z; g_r,\ r = 1, 2, \ldots, K\}$ is an CSbB-IFS with the contraction ratio $\phi = \max\limits_{r=1}^{K}\{\phi_r\}$. Suppose $G : K_0(Z) \to K_0(Z)$ as $G(E) = \bigcup\limits_{r=1}^{K} g_r(E), \forall E \in K_0(Z)$. Then G is called as HB operator for CSbB-IFS on CSbMS.*

Theorem 4.2. *(Banach Contractivity of HB Operator). Given CCSbMS (Z, d), suppose $\{Z; g_r,\ r = 1, 2, \ldots, K\}$ is an CSbB-IFS. Then the HB operator G is a Banach map in $(K_0(Z), H_d)$ with the contraction ratio $\phi = \max\limits_{r=1}^{K}\{\phi_r\}$.*

Proof.

$$H_d(G(E), G(F)) = H_d\bigg(g_1(E) \cup g_2(E) \cup \ldots \cup g_K(E), g_1(F) \cup g_2(F) \cup \ldots \cup g_K(F)\bigg)$$

$$
\begin{aligned}
&\le \max\Big\{H_d(g_1(E), g_1(F)), H_d(g_2(E), g_2(F)), \ldots, H_d(g_K(E), g_K(F))\} \\
&\le \max\Big\{\phi_1 H_d(E,F), \phi_2 H_d(E,F), \ldots, \phi_K H_d(E,F)\Big\} \\
&\le \max_{1\le r\le K}\Big\{\phi_r\Big\}\Big[\max\Big\{H_d(E,F), H_d(E,F), \ldots, H_d(E,F)\Big\}\Big] \\
&\le \phi H_d(E,F).
\end{aligned}
$$

Thus, G is a Banach map, where $\phi = \max\limits_{1\le r\le K}\Big\{\phi_r\Big\}$. □

Theorem 4.3. *(HB Theorem for CSbB-IFS on CSbMS). Given CCSbMS (Z,d), suppose $\{Z; g_r,\ r = 1,2,\ldots,K\}$ is an CSbB-IFS with $\phi = \max\Big\{\phi_i\Big\}_{i=1}^{K}$, where ϕ_i is the contraction ratio of g_i and G be the HB operator of the given IFS. For $a_0 \in Z$, choose $a_r = G^r(a_0)$. Assume that $\sup\limits_{k\ge1}\lim\limits_{i\to\infty}\alpha(a_{i+1}, a_k) < \frac{1}{\phi}$, and also consider that $\lim\limits_{r\to\infty}\alpha(a, a_r)$ & $\lim\limits_{r\to\infty}\alpha(a_r, a)$ exist and finite, for each $a_0 \in Z$. Then, the HB transformation $G : K_0(Z) \to K_0(Z)$, is a continuous Banach contraction map in CCSbMS $(K_0(Z), H_d)$ with the contraction ratio ϕ. Moreover, G admits a fixed point (unique) $E \in K_0(Z)$, such that*

$$G(E) = \bigcup_{i=1}^{K} g_i(E) = E,$$

given by $E = \lim\limits_{r\to\infty} G^r(F)$, $\forall\, F \in K_0(Z)$.

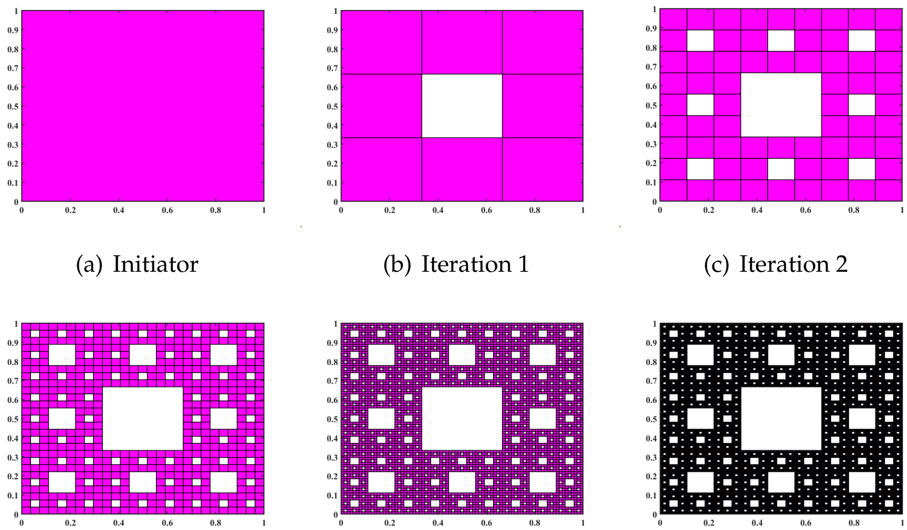
Proof. The HCSbMS $(K_0(Z), H_d)$ is Complete by Theorem 2.2, because (Z,d) is CCSbMS. From Theorem 4.2, the HB operator, G is a Banach contraction map in HCSbMS. Since $G : K_0(Z) \to K_0(Z)$, given by $G(E) = \bigcup\limits_{i=1}^{K} g_i(E), \forall E \in K_0(Z)$ is satisfied all assumptions of Theorem 2.1. Hence, we can conclude that G possesses a unique fixed point, say E. Therefore $E = \lim\limits_{r\to\infty} G^r(F)$, $\forall\, F \in K_0(Z)$. Hence proved. □

Based on the fixed point generated in Theorem 4.3, we shall define the fractal set in HCSbMS through CSbB-IFS.

Definition 4.3 (Controlled Strong $b-$Banach Fractals)**.** The set $E \in K_0(Z)$, obtained in Theorem 4.3 as a fixed point of G (HB Operator) on $K_0(Z)$ for CSbB-IFS, is called as controlled strong $b-$Banach fractal on CSbMS. Then $E \in K_0(Z)$ is called as a fractal generated by a CSbB-IFS in CSbMS.

Example 4.1. *Let Z be a unit square in $\mathbb{R}^2$ where, $A = (0,0)\,, B = (1,0)\,, C = (0,1)\,, D = (1,1)$ be the vertices, and consider the CSbB-IFS in Z has the following contractions*
$g_1(a,b) = \left(\frac{1}{3}a, \frac{1}{3}b\right)$; $g_2(a,b) = \left(\frac{1}{3}a, \frac{1}{3}b + \frac{1}{3}\right)$; $g_3(a,b) = \left(\frac{1}{3}a, \frac{1}{3}b + \frac{2}{3}\right)$; $g_4(a,b) = \left(\frac{1}{3}a + \frac{1}{3}, \frac{1}{3}b\right)$;
$g_5(a,b) = \left(\frac{1}{3}a + \frac{1}{3}, \frac{1}{3}b + \frac{2}{3}\right)$; $g_6(a,b) = \left(\frac{1}{3}a + \frac{2}{3}, \frac{1}{3}b\right)$; $g_7(a,b) = \left(\frac{1}{3}a + \frac{2}{3}, \frac{1}{3}b + \frac{1}{3}\right)$;
$g_8(a,b) = \left(\frac{1}{3}a + \frac{2}{3}, \frac{1}{3}b + \frac{2}{3}\right)$.

The first five iterations of the Sierpinski carpet are constructed in CSbMS by using CSbB-IFS, when the control factor in the triangular inequality $\alpha(a,b) = 1$. All the above contractions have contractivity factor $\frac{1}{3}$. By Theorem 4.3 the resulting fractal is called Sierpinski carpet as given in Fig. 1



(a) Initiator (b) Iteration 1 (c) Iteration 2

(d) Iteration 3 (e) Iteration 4 (f) Iteration 5

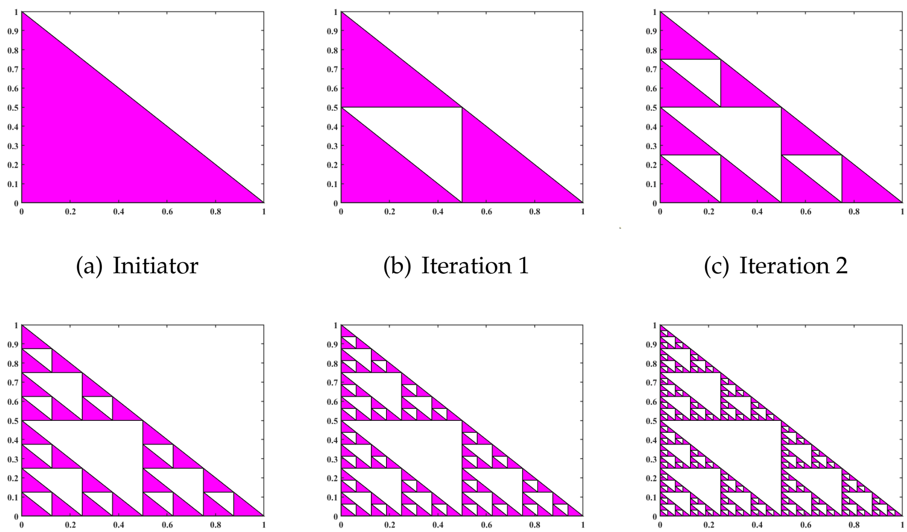
FIGURE 1. Construction of Sierpinski Carpet in CSbMS with Control Factor $\alpha(a,b) = 1$

Iteration	**Scaling ratio**	**Similarity dimension**
$N_1 = 8$	$\epsilon_1 = \frac{1}{3}$	$D_1 = \frac{\log(8)}{\log(3)} = 1.8928$
$N_2 = 64$	$\epsilon_2 = \frac{1}{9}$	$D_2 = \frac{\log(64)}{\log(9)} = 1.8928$
$N_3 = 512$	$\epsilon_3 = \frac{1}{27}$	$D_3 = \frac{\log(512)}{\log(27)} = 1.8928$
$N_4 = 4096$	$\epsilon_4 = \frac{1}{81}$	$D_4 = \frac{\log(4096)}{\log(81)} = 1.8928$
$N_5 = 32768$	$\epsilon_5 = \frac{1}{243}$	$D_5 = \frac{\log(32768)}{\log(243)} = 1.8928$

Using this similarity dimension theory, we can consider the Sierpinski carpet as a fractal in CSbMS for the case of $\alpha(a,b) = 1$.

Example 4.2. *Let Z be an right angle triangle $ABC \subseteq [0,1]^2$ where, $A = (0,0)\,, B = (1,0)\,, C = (0,1)$ be the vertices, and consider the CSbB-IFS in Z has the following contractions*
$g_1(a,b) = \left(\frac{1}{2}a, \frac{1}{2}b\right)$; $g_2(a,b) = \left(\frac{1}{2}a + \frac{1}{2}, \frac{1}{2}b\right)$; $g_3(a,b) = \left(\frac{1}{2}a, \frac{1}{2}b + \frac{1}{2}\right)$.

The first five iterations of the Sierpinski-type triangle are constructed in CSbMS using CSbB-IFS, when the control factor in the triangular inequality $\alpha(a,b) = 1$. All the above contractions have contractivity factor $\frac{1}{2}$. By Theorem 4.3 the resulting fractal is called Sierpinski-type Triangle as given in Fig. 2



(a) Initiator (b) Iteration 1 (c) Iteration 2

(d) Iteration 3 (e) Iteration 4 (f) Iteration 5

Figure 2. Construction of Sierpinski-type Triangle in CSbMS with Control Factor $\alpha(a,b) = 1$

Iteration	**Scaling ratio**	**Similarity dimension**
$N_1 = 3$	$\epsilon_1 = \frac{1}{2}$	$D_1 = \frac{\log(3)}{\log(2)} = 1.5849$
$N_2 = 9$	$\epsilon_2 = \frac{1}{4}$	$D_2 = \frac{\log(9)}{\log(4)} = 1.5849$
$N_3 = 27$	$\epsilon_3 = \frac{1}{8}$	$D_3 = \frac{\log(27)}{\log(8)} = 1.5849$
$N_4 = 81$	$\epsilon_4 = \frac{1}{16}$	$D_4 = \frac{\log(81)}{\log(16)} = 1.5849$
$N_5 = 243$	$\epsilon_5 = \frac{1}{32}$	$D_5 = \frac{\log(243)}{\log(32)} = 1.5849$

Using this similarity dimension theory, we can consider the Sierpinski-type triangle as a fractal in CSbMS for the case of $\alpha(a,b) = 1$.

Example 4.3. *Consider $Z = \{0,1,2\}$. Define $d : Z \times Z \to [0,\infty)$ as $d(a,a) = 0$ and $d(a,b) = d(b,a)$ where $a,b \in Z$ and*

$$d(1,2) = 2,\ d(2,0) = 4,\ d(0,1) = 9.$$

Consider a self mapping $g : Z \to Z$

$$g(1) = g(2) = 2,\ g(0) = 1.$$

Define $\alpha : Z \times Z \to [1, \infty)$ *as*

$$\alpha(a,b) = a + b + 2,\ \alpha(a,b) = \alpha(b,a),\ \alpha(a,a) = 1 \textit{ where } a, b \in Z.$$

Thus (Z,d) *is a CSbMS, follows all the conditions of Definition 2.2.*
If $\alpha(a,b) = t$*, then the CSbMS turns to SbMS where* t *is a constant in SbMS triangular inequality where* $t \geq 1$.
For $t = 2$*, and* $a = 0,\ b = 1,$
$d(0,1) = 9 > 8 = d(0,2) + t.d(2,1)$.
Thus, (Z,d) *is not a SbMS, this example shows that every controlled strong* $b-$*metric space is not a strong* $b-$*metric space. For all* $a, b \in Z$*, we consider the following four cases.*
Case 1) If $a = b$*. Then*

$$d(g(a), g(b)) = 0$$
$$d(a,b) = 0.$$

This implies that for all $\phi \in \left[\frac{1}{2}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 2) If $a = 0$ *and* $b = 1$*. Then*

$$d(1,2) = 2$$
$$d(0,1) = 9.$$

This implies that for all $\phi \in \left[\frac{1}{2}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 3) If $a = 1$ *and* $b = 2$*. Then*

$$d(2,2) = 0$$
$$d(1,2) = 2.$$

This implies that for all $\phi \in \left[\frac{1}{2}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 4) If $a = 0$ *and* $b = 2$*. Then*

$$d(1,2) = 2$$
$$d(0,2) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{2}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Thus, g *is a Banach contraction mapping on CSbMS* (Z,d) *for any* $\phi \in \left[\frac{1}{2}, 1\right)$.

Example 4.4. *Consider $Z = \{0,1,2\}$ and $d : Z \times Z \to [0,\infty)$. From Example (4.3), d is a CSbMS on Z. Assume $g_n : Z \to Z$ for $n = 1,2$.*

$$g_1(a) = \begin{cases} 2; & \text{if} \quad a = 0 \\ 1; & \text{if} \quad a \in \{1,2\} \end{cases}$$

$$g_2(a) = \begin{cases} 1; & \text{if} \quad a = 0 \\ 2; & \text{if} \quad a \in \{1,2\} \end{cases}$$

Thus $\{Z; g_1, g_2\}$ is the system of contraction mappings is a CSbB-IFS.

Example 4.5. *Consider $Z = \{0,1,2\}$. Define $d : Z \times Z \to [0,\infty)$ as $d(a,a) = 0$ and $d(a,b) = d(b,a)$ where $a,b \in Z$ and*

$$d(1,2) = 4,\ d(2,0) = 5,\ d(0,1) = 1.$$

Define $\alpha : Z \times Z \to [1,\infty)$ as

$$\alpha(a,b) = a + b + 2,\ \alpha(a,b) = \alpha(b,a),\ \alpha(a,a) = 1.$$

Consider a self mapping $g : Z \to Z$

$$g(0) = g(1) = 0, g(2) = 1.$$

Thus (Z,d) is a CSbMS, follows all the conditions of Definition 2.2.
For all $a,b \in Z$, we consider the following four cases.
Case 1) If $a = b$. Then

$$d(g(a), g(b)) = 0$$
$$d(a,b) = 0.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 2) If $a = 0$ and $b = 1$. Then

$$d(0,0) = 0$$
$$d(0,1) = 1.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 3) If $a = 1$ and $b = 2$. Then

$$d(0,1) = 1$$
$$d(1,2) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Case 4) If $a = 0$ and $b = 2$. Then

$$d(0,1) = 1$$

$$d(0,2) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$

$$d(g(a), g(b)) \leq \phi d(a,b).$$

Thus, g is a Banach contraction mapping on CSbMS (Z,d) for any $\phi \in \left[\frac{1}{4}, 1\right)$.

For $E = \{0,1\}$ and $F = \{0\}$, then

$$H_d(\{0,0\},\{0\}) = 0$$

$$H_d(\{0,1\},\{0\}) = 1.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$, *we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E,F).$$

For $E = \{0,1\}$ and $F = \{1\}$, then

$$H_d(\{0,0\},\{0\}) = 0$$

$$H_d(\{0,1\},\{0\}) = 1.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$, *we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E,F).$$

For $E = \{0,1\}$ and $F = \{2\}$, then

$$H_d(\{0,0\},\{1\}) = 1$$

$$H_d(\{0,1\},\{2\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$, *we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E,F).$$

For $E = \{1,2\}$ and $F = \{0\}$, then

$$H_d(\{0,1\},\{0\}) = 1$$

$$H_d(\{1,2\},\{0\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$, *we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E,F).$$

For $E = \{1,2\}$ and $F = \{1\}$, then

$$H_d(\{0,1\},\{0\}) = 1$$

$$H_d(\{1,2\},\{1\}) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{1, 2\}$ *and* $F = \{2\}$*, then*

$$H_d(\{0, 1\}, \{1\}) = 1$$
$$H_d(\{1, 2\}, \{2\}) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 2\}$ *and* $F = \{0\}$*, then*

$$H_d(\{0, 1\}, \{0\}) = 1$$
$$H_d(\{0, 2\}, \{0\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 2\}$ *and* $F = \{1\}$*, then*

$$H_d(\{0, 1\}, \{0\}) = 1$$
$$H_d(\{0, 2\}, \{1\}) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 2\}$ *and* $F = \{2\}$*, then*

$$H_d(\{0, 1\}, \{1\}) = 1$$
$$H_d(\{0, 2\}, \{1\}) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{0\}$*, then*

$$H_d(\{0, 0, 1\}, \{0\}) = 1$$
$$H_d(\{0, 1, 2\}, \{0\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{1\}$*, then*

$$H_d(\{0, 0, 1\}, \{0\}) = 1$$

$$H_d(\{0, 1, 2\}, \{1\}) = 4.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{2\}$*, then*

$$H_d(\{0, 0, 1\}, \{1\}) = 1$$

$$H_d(\{0, 1, 2\}, \{2\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{0, 1\}$*, then*

$$H_d(\{0, 0, 1\}, \{0, 0\}) = 1$$

$$H_d(\{0, 1, 2\}, \{0, 1\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{1, 2\}$*, then*

$$H_d(\{0, 0, 1\}, \{0, 1\}) = 1$$

$$H_d(\{0, 1, 2\}, \{1, 2\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

For $E = \{0, 1, 2\}$ *and* $F = \{0, 2\}$*, then*

$$H_d(\{0, 0, 1\}, \{0, 1\}) = 1$$

$$H_d(\{0, 1, 2\}, \{2, 0\}) = 5.$$

This implies that for all $\phi \in \left[\frac{1}{4}, 1\right)$*, we have*

$$H_d(G(E), G(F)) \leq \phi H_d(E, F).$$

Thus, G is a Banach contraction map in HCSbMS $(K_0(Z), H_d)$ *for any* $\phi \in \left[\frac{1}{4}, 1\right)$.

4.2. **Controlled Strong $b-$Reich Fractal.**

Theorem 4.4. *Given CCSbMS (Z, d), suppose $g : Z \to Z$ is a continuous Reich mapping with the contraction ratios ϕ, ω, δ. Then $g : K_0(Z) \to K_0(Z)$ is defined by $g(E) = \{g(a) : a \in E\}$, $\forall E \in K_0(Z)$, is a Reich contraction in $(K_0(Z), H_d)$ with the contraction ratios ϕ, ω, δ.*

Proof. Let $E, F \in K_0(Z)$. Then

$$
\begin{aligned}
H_d(g(E), g(F)) &= \max\{d(g(E), g(F)), d(g(F), g(E))\} \\
&\leq \max\{\phi d(E, g(E)) + \omega d(F, g(F)) + \delta d(E, F), \phi d(F, g(F)) + \omega d(E, g(E)) + \delta d(F, E)\} \\
&\leq \phi\left[\max\{d(E, g(E)), d(F, g(F))\}\right] + \omega\left[\max\{d(F, g(F)), d(E, g(E))\}\right] \\
&\quad + \delta\left[\max\{d(E, F), d(F, E)\}\right] \\
&= \phi d(E, g(E)) + \omega d(F, g(F)) + \delta d(E, F) \\
H_d(g(E), g(F)) &\leq \phi H_d(E, g(E)) + \omega H_d(F, g(F)) + \delta H_d(E, F).
\end{aligned}
$$

Then, g is a Reich contraction on $(K_0(Z), H_d)$. □

Definition 4.4. *(Reich IFS on CSbMS). Consider (Z, d) is CCSbMS and a family of Reich contractions g_r $(r = 1, 2, \ldots, K;\ K \in \mathbb{N})$ on Z having the corresponding contractivity ratios $\phi_r, \omega_r, \delta_r$. Then, $\{Z; g_r\}_{r=1}^{K}$ is said to be a controlled strong $b-$Reich iterated function system (CSbR-IFS), where the contractivity factor of this system is $\phi = \max_{r=1}^{K}\{\phi_r\}, \omega = \max_{r=1}^{K}\{\omega_r\}, \delta = \max_{r=1}^{K}\{\delta_r\}$.*

Definition 4.5. *(HB Operator of Reich Contractions on CSbMS). Given CCSbMS (Z, d), suppose $\{Z; g_r, r = 1, 2, \ldots, K\}$ is an CSbR-IFS with the contraction ratios $\phi = \max_{r=1}^{K}\{\phi_r\}, \omega = \max_{r=1}^{K}\{\omega_r\}, \delta = \max_{r=1}^{K}\{\delta_r\}$. Suppose $G : K_0(Z) \to K_0(Z)$ as $G(E) = \bigcup_{r=1}^{K} g_r(E), \forall E \in K_0(Z)$. Then G is called as HB operator for CSbR-IFS on CSbMS.*

Theorem 4.5. *(Reich Contractivity of HB Operator). Given CCSbMS (Z, d), suppose $\{Z; g_r, r = 1, 2, ..., K\}$ is an CSbRT-IFS. Then the HB operator G is a Reich map in $(K_0(Z), H_d)$ with the contraction ratios $\phi = \max_{r=1}^{K}(\phi_r)$, $\omega = \max_{r=1}^{K}(\omega_r)$ and $\delta = \max_{r=1}^{K}(\delta_r)$.*

Proof.

$$
\begin{aligned}
H_d(G(E), G(F)) &= H_d\Big(g_1(E) \cup g_2(E) \cup \ldots \cup g_K(E), g_1(F) \cup g_2(F) \cup \ldots \cup g_K(F)\Big) \\
&\leq \max\Big\{H_d(g_1(E), g_1(F)), H_d(g_2(E), g_2(F)), \ldots, H_d(g_K(E), g_K(F))\} \\
&\leq \max\Big\{\phi_1 H_d(E, g_1(E)) + \omega_1 H_d(F, g_1(F)) + \delta_1 H_d(E, F), \\
&\qquad \phi_2 H_d(E, g_2(E)) + \omega_2 H_d(F, g_2(F)) + \delta_2 H_d(E, F), \\
&\qquad \ldots
\end{aligned}
$$

$$
\begin{aligned}
&\phi_K H_d(E, g_K(E)) + \omega_K H_d(F, g_K(F)) + \delta_K H_d(E,F)\Big\} \\
&\leq \max_{1\leq r\leq K}\Big\{\phi_r\Big\}\Big[\max\Big\{H_d(E, g_1(E)), H_d(E, g_2(E)), \ldots, H_d(E, g_K(E))\Big\}\Big] \\
&+ \max_{1\leq r\leq K}\Big\{\omega_r\Big\}\Big[\max\Big\{H_d(F, g_1(F)), H_d(F, g_2(F)), \ldots, H_d(F, g_K(F))\Big\}\Big] \\
&+ \max_{1\leq r\leq K}\Big\{\delta_r\Big\}\Big[\max\Big\{H_d(E,F)\Big\}\Big] \\
&\leq \max_{1\leq r\leq K}\Big\{\phi_r\Big\} H_d\Big(E, g_1(E)\cup g_2(E)\cup\ldots\cup g_K(E)\Big) \\
&+ \max_{1\leq r\leq K}\Big\{\omega_r\Big\} H_d\Big(F, g_1(F)\cup g_2(F)\cup\ldots\cup g_K(F)\Big) + \max_{1\leq r\leq K}\Big\{\delta_r\Big\} H_d(E,F), \\
&\leq \phi H_d(E, G(E)) + \omega H_d(F, G(F)) + \delta H_d(E,F)
\end{aligned}
$$

Thus, G is a Reich map, where $\phi = \max\limits_{1\leq r\leq K}\Big\{\phi_r\Big\}$, $\omega = \max\limits_{1\leq r\leq K}\Big\{\omega_r\Big\}$, $\delta = \max\limits_{1\leq r\leq K}\Big\{\delta_r\Big\}$. □

Theorem 4.6. *(HB Theorem for CSbR-IFS on CSbMS). Given CCSbMS (Z,d), suppose $\{Z; (g_1, g_2, \ldots, g_K)\}$, is an CSbR-IFS with $\phi = \max\limits_{i=1}^{K}\{\phi_i\}$, $\omega = \max\limits_{i=1}^{K}\{\omega_i\}$ and $\delta = \max\limits_{i=1}^{K}\{\delta_i\}$, where $\phi_i, \omega_i, \delta_i$ are the contraction ratios of g_i and G is the HB operator of the given IFS. For $a_0 \in Z$, choose $a_r = G^r(a_0)$. Assume that $\sup\limits_{k\geq 1}\lim\limits_{i\to\infty}\alpha(a_{i+1}, a_k) < \left(\frac{1-\omega}{\phi+\delta}\right)$, and also consider that $\lim\limits_{r\to\infty}\alpha(a, a_r)$ and $\lim\limits_{r\to\infty}\alpha(a_r, a)$ exist and finite, for each $a_0 \in Z$. Then, the HB transformation $G : K_0(Z) \to K_0(Z)$, is a continuous Reich map in CCSbMS $(K_0(Z), H_d)$ with the contraction ratios ϕ, ω, δ. Moreover, G has a fixed point (unique) $E \in K_0(Z)$, such that*

$$
G(E) = \bigcup_{i=1}^{K} g_i(E) = E,
$$

given by $E = \lim\limits_{r\to\infty} G^r(F)$, $\forall\, F \in K_0(Z)$.

Proof. The HCSbMS $(K_0(Z), H_d)$ is Complete by Theorem 2.2, because (Z,d) is CCSbMS. From Theorem 4.5, the HB operator, G is a Reich contraction mapping in HCSbMS. Since $G : K_0(Z) \to K_0(Z)$, given by $G(E) = \bigcup\limits_{i=1}^{K} g_i(E), \forall E \in K_0(Z)$ is satisfied all assumptions of Theorem 3.1. Hence, we can conclude that G possesses a unique fixed point say E. Therefore $E = \lim\limits_{r\to\infty} G^r(F)$, $\forall\, F \in K_0(Z)$. Hence proved. □

Based on the fixed point generated in Theorem 4.6, we shall define the fractal set in HCSbMS through CSbR-IFS.

Definition 4.6 (Controlled Strong $b-$Reich Fractals)**.** The set $E \in K_0(Z)$, obtained in Theorem 4.6 as a fixed point of G (HB Operator) on $K_0(Z)$ for CSbR-IFS, is called as controlled strong $b-$Reich fractal on CSbMS. Then $E \in K_0(Z)$ is called as a fractal generated by a CSbR-IFS in CSbMS.

The fractal attractors generated with Banach contraction and Reich contraction in the proposed metric space are generalized versions of fractals in the strong $b-$metric space and metric space. If the control variable factor in the triangular inequality in controlled strong $b-$metric space is constant, then it reduces to strong $b-$metric space and if the control variable factor in the triangular inequality is one, then it turns to metric space. But the converse of those types is not always true. The controlled strong $b-$metric space is the larger class than strong $b-$metric space and metric space. Moreover, the distance function in this space is continuous. In contrast, the distance function in a $b-$metric space need not be continuous. The generation of fractal attractors with Reich contraction in strong $b-$metric space can be deduced from the controlled strong $b-$metric space, when the control factor in the triangular inequality is constant.

The controlled strong b-Reich fractal attractor is generated in this manuscript as a generalized case of the controlled strong $b-$Kannan fractal [29], if all the contraction ratios are considered as $\phi_i = \omega_i$ and $\delta_i = 0$. In this case, the controlled strong b-Reich IFS reduces to the controlled strong $b-$Kannan IFS.

The fractal attractor in the controlled metric space constructed by using Fisher type contraction mappings [15] is a particular case of a fractal generated by Reich contraction mappings, when $\phi_i = \omega_i$, since the Reich contraction becomes the Fisher contraction as a particular case. The same case can also be employed in the CSbMS to introduce Fisher type IFS and fractals.

The fractal sets can be generated in controlled extended form of the rectangular $b-$metric space, by using Banach contractions [28], as a particular case of a fractal generated by Reich contractions, since the Banach contraction is a particular case of the Reich contraction, when $\phi_i = \omega_i = 0$. The same argument can also be employed in the controlled strong $b-$metric space to generate Banach contractions-based IFS and fractals.

Furthermore, if all the contraction ratios are chosen as $\phi_i = \omega_i$ and $\delta_i = 0$ and the control factor in the triangular inequality of CSbMS is considered as $\alpha(a, b) = 1$, then the controlled Reich IFS reduces to the Kannan IFS. If $\phi_i + \omega_i = 0$ and $\alpha(a, b) = 1$, then the controlled Reich IFS reduces to the Banach IFS. If $\phi_i = \omega_i$ and $\alpha(a, b) = 1$, then the controlled Reich IFS reduces to Fisher-type IFS. The converse for all the above cases need not be true always.

5. Application

In this section, the existence of the solution for the Fredholm integral equation [31] in CSbMS is presented as an application for the proposed generalization.

Consider the space $Z = C([p, q], \mathbb{R})$, which is the collection of all continuous functions from $[p, q]$ to $\mathbb{R}$, where p and q are real numbers. Note that Z is CCSbMS by considering

$$d(a, b) = \sup_{s \in [p,q]} |a(s) - b(s)|^2, \quad \text{with} \quad \alpha(a, b) = |a(s)| + |b(s)| + 2,$$

where $\alpha : Z \times Z \to [1, \infty)$. Suppose the expression is the Fredholm integral equation,

$$a(s) = \int_p^q X(s,t,a(t))dt + b(s), \quad s,t \in [p,q], \tag{5.1}$$

where $b : [p,q] \to \mathbb{R}$ and $X : [p,q] \times [p,q] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $g : Z \to Z$ be the operator given by

$$g(a(s)) = \int_p^q X(s,t,a(t))dt + b(s) \quad \text{for} \quad s,t \in [p,q],$$

where $b : [p,q] \to \mathbb{R}$ and $X : [p,q] \times [p,q] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In addition, suppose that the following inequality is true

$$|X(s,t,a(t)) - X(s,t,b(t))| \leq \frac{1}{q-p}\Big[\phi|a(t) - g(a(t))| + \omega|b(t) - g(b(t))| + \delta|a(t) - b(t)|\Big]$$

for each $s,t \in [p,q]$ and $a \in Z$.

Thus, the integral Eqn. (5.1) admits a solution. We need to verify that the operator g satisfies all the hypotheses of Theorem 3.1. Given any $a \in Z$, we obtain:

$$\begin{aligned}
|g(a(s)) - g(b(s))|^2 &\leq \int_p^q |X(s,t,a(t)) - X(s,t,b(t)|^2 dt \\
&\leq \frac{1}{q-p}\Big[\int_p^q \phi|a(t) - g(a(t))| + \omega|b(t) - g(b(t))| + \delta|a(t) - b(t)|dt\Big]^2 \\
\sup_{s\in[p,q]} |g(a(s)) - g(b(s))|^2 &\leq \sup_{s\in[p,q]} \Big[\phi|a(t) - g(a(t))| + \omega|b(t) - g(b(t))| + \delta|a(t) - b(t)|\Big]^2 \\
d(g(a), g(b)) &\leq \phi d(a, g(a)) + \omega d(b, g(b)) + \delta d(a,b) \textit{ where } \phi + \omega + \delta < 1 \textit{ and } a,b \in Z.
\end{aligned}$$

Consequently, the operator g admits a fixed Point. The solution exists for the given Fredholm integral Eqn. (5.1).

The significance of this paper is to construct the fractal attractor on CSbMS. The fractal structures of Reich contraction on CSbMS has been explored at first time in the literature. The CSbMS is generalized from SbMS, where the control variable function is involved as a constant in the triangular inequality. This theory can be extended to develop the notion of multivalued controlled strong $b-$fractals and controlled strong $b-$interpolation in the generalized space.

6. Conclusion

In this research article, we established a fixed point theorem with respect to Reich contraction on a controlled strong version of the $b-$metric space and discussed with illustrative examples. This research aimed to construct a new type fractal attractor on the CSbMS, such as a controlled strong $b-$Banach fractal through CSbB-IFS and the controlled strong $b-$Reich fractal through CSbR-IFS. As the future research works in generalized spaces, the proposed theory will guide us along a new path for constructing a novel type of fractal attractors.

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